Probabilistic Graphical Models

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Exercise Sheet 1 v1.1

1 Refresher I

Prove or disprove that the following statements hold for all random variables X, Y:

a) $p(X = x, Y = y) \leq \frac{1}{2}(p(X = x) + p(Y = y))$

b) $p(X = x|Y = y) \geq p(X = x)$

c) $p(X = x, Y = y) \leq p(X = x|Y = y)$

d) $p(X = x, Y = y) \geq p(X = x) + p(Y = y) - 1$

2 Refresher II

Prove/check the following results from the lecture for any random variables X, Y:

a) $\text{Var}(X) = \mathbb{E}_x[x^2] - (\mathbb{E}_x[x])^2$

b) $\text{Var}(\lambda X) = \lambda^2 \text{Var}(X) \quad \text{for } \lambda \in \mathbb{R}$

c) $\text{Std}(\lambda X) = |\lambda| \text{Std}(X) \quad \text{for } \lambda \in \mathbb{R}$

d) $\text{Corr}(\lambda X, Y) = \text{Corr}(X, Y)$ and $\text{Corr}(-\lambda X, Y) = -\text{Corr}(X, Y) \quad \text{for } \lambda > 0$ ← typo fixed, 29/11/16, 14:00

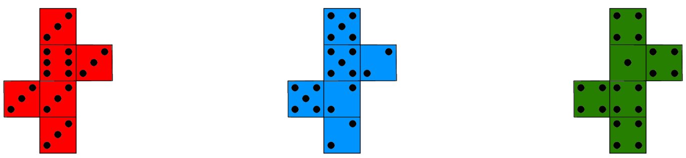
e) $\text{Corr}(X, Y) \in [-1, 1]$

f) $X \perp\!\!\!\perp Y \quad \Rightarrow \quad \text{Cov}(X, Y) = 0$

g) There exist random variables X, Y with $\text{Cov}(X, Y) = 0$, but $X \not\perp\!\!\!\perp Y$ ← typo fixed, 29/11/16, 15:25

3 Fancy Dice

(by James Grime). There are three dice with the following faces:



RED BLUE OLIVE

In a game, each player picks one of the dice and rolls it. The player who rolls the higher values wins.

a) What is the *expected value* of output for each of the dice?

b) What are the probabilities that 1) *red* beats *blue*, 2) *blue* beats *olive*, 3) *olive* beats *red*?

c) Instead of the above game, the players play "best of 3" to decide who wins. How do the above probabilities change?

d) Instead of the rolling one die, each player rolls two dice of their color and sums the numbers. Again the player with the higher values wins. What changes in the above probabilities?

4 Terrorist detector

(adapted from the Barber book based on [Lass. Elements of Pure and Applied Mathematics. McGraw-Hill, 1957])
A secret government agency has developed a scanner which determines whether a person is a terrorist. The scanner is fairly reliable; 95% of all scanned terrorists are identified as terrorists (called "sensitivity"), and 95% of all upstanding citizens are identified as such (called "specificity"). An informant tells the agency that exactly one passenger of 100 aboard an aeroplane in which you are seated is a terrorist.

a) The scanner tests positive for the person in the seat next to you. What is the probability that this person is a terrorist?

b) What's the expected number of positive tests if everyone on the plane is tested?

c) What is the probability that the actual terrorist is one of the positive tests?

d) Discuss: as an engineer developing on the next generation of scanner, which value is more important to be improved, the sensitivity or the specificity?

5 Dependence/Independence

Come up with examples for the following situations (or prove that they are impossible)

a) three random variables X, Y, Z such that $X \perp\!\!\!\perp Y$ and $Y \perp\!\!\!\perp Z$ but $X \not\perp\!\!\!\perp Z$

b) three random variables X, Y, Z such that $X \not\perp\!\!\!\perp Y$ and $Y \not\perp\!\!\!\perp Z$ but $X \perp\!\!\!\perp Z$

c) three random variables X, Y, Z such that $X \perp\!\!\!\perp Y$, $Y \perp\!\!\!\perp Z$ and $X \perp\!\!\!\perp Z$, but $p(X, Y, Z) \neq p(X)p(Y)p(Z)$

d) three random variables X, Y, Z such that $X \perp\!\!\!\perp Y|Z$, but $Y \not\perp\!\!\!\perp X|Z$

6 Discussion item: Simpson's Paradox

This exercise is not about computing anything (much), but about interpretation.
A new drug is given to a group of people in order to find out if it improves the recovery from a disease. The following table shows the result of the study.

Combined	Recovered	Not Recovered	Rec. Rate
Given Drug	20	20	50%
Not Given Drug	16	24	40%

To better understand this results, the outcome is also analyzed separately for male and female participants.

Males	Recovered	Not Recovered	Rec. Rate
Given Drug	18	12	60%
Not Given Drug	7	3	70%

Females	Recovered	Not Recovered	Rec. Rate
Given Drug	2	8	20%
Not Given Drug	9	21	30%

Now the doctors are very confused: overall, patients recover better if given the drug, but both men and women recover worse if given the drug??? Can you help?