4 Inference

Logic is generally considered to lie in the intersection between Philosophy and Mathematics. It studies the meaning of statements and the relationship between them.

Boolean operations. A logical statement is either true (T) or false (F). We call this the truth value of the statement. We will frequently represent the statement by a variable, which can be either true or false. A boolean operation takes one or more truth values as input and produces a new output truth value. For example, negation is a unary operation. It maps a truth value to the opposite; see Table 1. Much more common are binary operations; such as and, or, and exclusive or. We use a truth table to specify the values for all possible combinations of inputs; see Table 2. Binary operations have two input variables, each in one of two states. The number of different inputs is therefore only four. We have seen the use of these particular boolean operations before, namely in the definition of the common set operations. The boolean operations behave very much like ordinary arithmetic operations. For example, they commute: $p \land q \iff q \land p$, etc. Also, the conjunction distributes over the disjunction: $p \land (q \lor r) \iff (p \land q) \lor (p \land r)$, and, symmetrically, the disjunction distributes over the conjunction. Similarly, negation distributes over the conjunction and the disjunction, but it changes one into the other as it does so.

De Morgan’s Law. Letting $p$ and $q$ be two boolean variables, we have

\[
\neg(p \land q) \iff \neg p \lor \neg q; \\
\neg(p \lor q) \iff \neg p \land \neg q.
\]

PROOF. We construct the truth table, with a row for each combination of truth values for $p$ and $q$; see Table 3. Since

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$\neg(p \land q)$</th>
<th>$\neg p \lor \neg q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Table 3: The truth table for the expressions on the left and the right of the first de Morgan Law.

the two relations are symmetric, we restrict our attention to the first. We see that the truth values of the two expressions are the same in each row, as required.

Implications and equivalences. The implication is another binary boolean operation. It frequently occurs in statements of lemmas and theorems. The operation is defined in Table 4. We see the contrapositive in the second column on the right, which is equivalent, as expected. We also note that $q$ is forced to be true if $p$ is true and that $q$ can be anything if $p$ is false. This is expressed in the third column on the right. Applying de Morgan’s Law to it, we get the last column.

We recall that a logical statement is either true or false. This is referred to as the law of the excluded middle. In other words, a statement is true precisely when it is not false. There is no allowance for ambiguities or paradoxes. An example is the sometimes counter-intuitive definition that false implies true is true. Write $p$ for the statement “it is raining”, $q$ for “I use my umbrella”, and consider $p \Rightarrow q$. Hence, if it is raining then I use my umbrella. This does not preclude me from using the umbrella if it is not raining. In other words, the implication is not false if I use my umbrella without rain. Hence, it is true. If implications go both ways, we have an equivalence. The operation is defined in Table 5. The last column shows that equivalence is the opposite of the exclusive or operation.

Direct inference. Next, we discuss the application of logic to proving theorems. In principle, every proof should be reducible to a sequence of simple logical deductions.
While this is not practical for human consumption, there have been major strides toward that goal in computerized proof systems. The cornerstone of logical arguments is the direct inference, whose most common version is the *modus ponens*.

**Principle of Modus Ponens.** From \( p \) and \( p \implies q \), we conclude \( q \).

We read this as a recipe to prove \( q \). First we prove \( p \), then we prove that \( p \) implies \( q \), and finally we conclude \( q \). Let us take a look at Table 6 to be sure. We see that modus ponens is indeed a tautology, that is, it is always true. Every theorem is this way, namely always true. We have seen modus ponens before, namely in the construction of the mathematical induction as a general proof technique. There, we have an infinite chain of steps, each justified by modus ponens.

**Indirect inference.** The first version of an indirect reasoning method is the contrapositive of an implication.

**Principle of Contraposition.** The logical statements \( p \implies q \) and \( \neg q \implies \neg p \) are equivalent, and so a proof of one is a proof of the other.

We have seen a truth table that shows the equivalence of the two statements earlier. Using this inside modus ponens, we get \( p \land (\neg q \implies \neg p) \) implies \( q \).

A second example of indirect reasoning is the *reduction to absurdity*, also known as *proof by contradiction*. We begin with an example, namely the proof that irrational numbers exist. A real number \( u \) is *rational*, if there are integers \( m \) and \( n \) such that \( u = \frac{m}{n} \), and it is *irrational*, otherwise. The set of rational numbers is denoted as \( \mathbb{Q} \). For any two different rational numbers, \( u < w \), we can always find a third that lies strictly between them. For example, if \( w = \frac{1}{2} \) then

\[
v = \frac{u + w}{2} = \frac{m\ell + nk}{2n\ell}
\]

lies halfway between \( u \) and \( w \). This property is sometimes expressed by saying the rational numbers are dense in the set of real numbers. How do we know that not all real numbers are rational?

**Claim.** \( \sqrt{5} \) is irrational.

**Proof.** Assume the square root of \( 5 \) is rational, that is, there exist integers \( m \) and \( n \) such that \( \sqrt{5} = \frac{m}{n} \). Squaring the two sides, we get \( 5 = \frac{m^2}{n^2} \), or, equivalently, \( 5n^2 = m^2 \). But \( m^2 \) has an even number of prime factors, namely each factor twice, while \( 5n^2 \) has an odd number of prime factors, namely 5 together with an even number of prime factors for \( n^2 \). Since the prime factor decompositions are unique, this implies \( 5n^2 \neq m^2 \), a contradiction.

We take a look at the logic structure of this proof. Let \( p \) be the statement that \( \sqrt{5} = 5 \), and let \( q \) be the statement that \( \sqrt{5} \) is irrational. Thus \( \neg q \) is the statement that \( \sqrt{5} = \frac{m}{n} \). From assuming \( p \) and \( \neg q \), we derive \( r \), that is: the statement \( 5n^2 = m^2 \). But we also have \( \neg r \), because each integer has a unique decomposition into prime factors. We thus derived \( r \) and \( \neg r \). But this cannot be true. We conclude that \( p \) implies \( q \). Finally, we make sure that the proof by contradiction is logically justified.

**Principle of Reduction to Absurdity.** If from \( p \) and \( \neg q \) we can derive \( r \) as well as \( \neg r \) then \( p \implies q \).

Here \( r \) can be any statement. Often we use an \( r \) that is always true (or always false) so that we only need to derive \( \neg r \) (or \( r \)). Inspecting Table 7, we see that the principle of reduction to absurdity is indeed a tautology.

![Table 5: The truth table for the equivalence (⇔).](image)

![Table 6: The truth table for modus ponens.](image)

![Table 7: The truth table for the reduction to absurdity.](image)