Exercise 1. a) Show: A diagonal matrix $D = \text{diag}(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n \times n}$ with $\alpha_i \neq 0$ for all $i = 1, \ldots, n$ is invertible, and its inverse is $\text{diag}(\frac{1}{\alpha_1}, \ldots, \frac{1}{\alpha_n})$.

b) Show: If $\alpha_i = 0$ for any $i$, then the matrix $D$ is not invertible (hint: compute $[DA]_{ii}$ for an arbitrary matrix $A \in \mathbb{R}^{n \times n}$).

Exercise 2. a) Show: For a rotation matrix, $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, its inverse $R_\theta^{-1}$ is the same as its transpose, $R_\theta^\top$.

b) Can you come up with any matrix $A \in \mathbb{R}^{2 \times 2}$ that is not a rotation matrix and also has the property that $A^{-1} = A^\top$? (note: $\text{Id}_2$ does not count, it is a rotation matrix for $\theta = 0$).

Exercise 3.

a) Fill in the missing values in the graph on the right (marked by ?) such that the arrows define a Markov Chain with states $\{1, 2, 3, 4, 5\}$.

b) Construct the transition matrix $P \in \mathbb{R}^{5 \times 5}$.

c) Compute (manually) the probabilities after 4 steps of each state $i$, starting at each state $j$, i.e.

$$\Pr(X^{(4)} = i | X^{(0)} = j) \quad \text{for } i = 1, \ldots, n, \ j = 1, \ldots, n.$$  

Hint: multiplying matrices with fractional entries becomes easier if you first factor out the least common denominator.

d) What do you think would happen if you let the process run infinitely long (if you want, you can use a computer to get an 'intuition')? Justify your hypothesis.

e) Bonus: Can you prove the hypothesis you made in d)?

Exercise 4. The following process describes the population dynamics of Spotted Owls (Strix occidentalis): We distinguish three stages: \{juvenile (J), subadult (S), adult (A)\}. For each stage, $J, S, A$ we denote the average number of individuals at time $t$ in a certain isolated region as $J^{(t)}$, $S^{(t)}$ and $A^{(t)}$. The population follows a dynamical system with rule set:

$$J^{(t+1)} = F_J J^{(t)} + F_S S^{(t)} + F_A A^{(t)} \quad S^{(t+1)} = P_J J^{(t)} \quad A^{(t+1)} = P_S S^{(t)} + P_A A^{(t)}$$

where $F_J$, $F_S$ and $F_A$ are fecundities (reproduction rates), and $P_J$, $P_S$ and $P_A$ are survival rates.

a) Define a notation in terms of vectors and matrices that describes the above population dynamics in a compact way.

b) Does the above process describe a Markov Chain? If yes, why? If not, why not?

c) At different geographic locations, researchers determined different values for the fecundities and survival rates, e.g.

<table>
<thead>
<tr>
<th>Location</th>
<th>Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>i) Klamath Province, California</td>
<td>$F_J = 0.09, F_S = 0.20, F_A = 0.33$, $P_J = 0.325, P_S = 0.8677, P_A = 0.8667$</td>
</tr>
<tr>
<td>ii) Olympic Island, Washington</td>
<td>$F_J = 0, F_S = 0, F_A = 0.38, P_J = 0.24, P_S = 0.86, P_A = 0.87$</td>
</tr>
<tr>
<td>iii) Coast Range, Western Ontario</td>
<td>$F_J = 0, F_S = 0.071, F_A = 0.231, P_J = 0.24, P_S = 0.82, P_A = 0.81$</td>
</tr>
</tbody>
</table>

Construct the dynamics matrices for each case (these are called "Leslie-Lefkovitch" matrices).

d) Assume we start with a population of 300 individuals, 100 of each stage. How many individuals of each stage are there after one time step (for each of the situations i), ii) and iii) ).

e) Assume we start with a population of 1000 individuals per stage. What changes compared to e)?

f) What happens if we start with exactly a single adult owl in the region?

g) Derive an expression that computes the total number of owls after one time step (as a function of a given start stage). Is it linear? If yes, what’s its matrix? Can you do the same for $k$ time steps?

h) Bonus: What animal is depicted on the right?
Exercise 5. Identify the matrices corresponding to the following linear functions \( f : \mathbb{R}^n \to \mathbb{R}^m \) and their sizes (number of columns, number of rows).

- \( f_{20}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \)
- \( f_{21}(x) = \frac{1}{2} \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \\ x_2 - x_3 \\ \vdots \\ x_n - x_{n-1} \\ x_n + x_{n-1} \end{pmatrix} \)
- \( f_{22}(x) = a^\top x + b^\top x \) (for fixed \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^n \))
- \( f_{23}(x) = \pi \)
- \( f_{24}(x) = \sum_{i=1}^{n} (-2)^i x_i \)
- \( f_{25}(x) = x^\top a \) (for fixed \( a \in \mathbb{R}^n \))
- \( f_{26}(x) = \begin{cases} (x_1 \ x_3 \ \cdots \ x_n)^\top & \text{for } n \text{ odd.} \\ (x_1 \ x_3 \ \cdots \ x_{n-1})^\top & \text{for } n \text{ even.} \end{cases} \)

Exercise 6. Compute all matrix-matrix products \( B_i A_j \) (note the different order compared to last week’s sheet) that are possible for the following matrices \( B_1, \ldots, B_7 \) and \( A_1, \ldots, A_8 \). Mark combinations that cannot be multiplied by an \( \times \).