B. Space-filling Tetrahedra

The regular (equilateral) triangle tiles the Euclidean plane, but the regular tetrahedron does not tile the 3-dimensional Euclidean space. We ask whether there are other types of tetrahedra that do. Before going any further, we introduce some definitions. Two sets in Euclidean space are congruent if one can be obtained from the other by a sequence of translations, rotations, and reflections. A tetrahedron fills space if \( \mathbb{R}^3 \) can be covered with congruent copies whose interiors are disjoint. For example, any one tetrahedron in the decompositions of the cube illustrated in Figure 44 fills space. These three types are referred to as Sommerville tetrahedra. There is a fourth space-filling tetrahedron in this family, which needs two neighboring cubes for its construction; see Figure 45. It is the only one of the four that is centrally symmetric. Why?

But the Sommerville tetrahedra are not the only ones that fill space. Indeed, there are infinite families of such tetrahedra. Still, we do not know all such tetrahedra or, if we do, we do not know how to prove that there are no others. For example, we do not even know whether any of the space-filling tetrahedra has all dihedral angles less than 90°. We call such a tetrahedron acute.


QUESTION. Does there exist an acute space-filling tetrahedron?

It is not even easy to decompose \( \mathbb{R}^3 \) into acute tetrahedra if we do not require that they are all congruent copies of each other. However, such decompositions are known to exist.