Differential Equations: Homework 2

Due Friday, 30 March 2012

Name of student:

Complete the following exercises. Be sure to show all of your intermediate work. You will not get full credit if you only submit solutions (unless the solution requires no intermediate steps). If you require more space than the space provided on these pages, feel free to use additional sheets of paper. You are allowed to use a computer or calculator for arithmetic calculations (like 237+12 or 56²) but all symbolic manipulation must be done by hand.

1 Taylor Series

As introduced in class, the Taylor series transforms a function $f(x)$ into a polynomial with an infinite amount of terms:

$$f(x_0 + \Delta x) = \sum_{n=0}^{\infty} \frac{d^n f(x_0)}{dx^n} \frac{\Delta x^n}{n!} \quad (1)$$

The Taylor series expansion starts at a point $x_0$, then each term uses information about higher derivatives to replace the original function with a polynomial. The more higher order terms that are included in the sum, the more accurate the result. The first three terms of the Taylor series are:

$$f(x_0 + \Delta x) \approx f(x_0) + \frac{df(x_0)}{dx} \Delta x + \frac{d^2 f(x_0)}{dx^2} \frac{\Delta x^2}{2} + \ldots$$

Write out the next five terms of this series below:
1.1 Application to arbitrary functions

Write out the first five terms of the Taylor series for the function $f(x) = 10$.

$$f(x_0 + \Delta x) \approx$$

Write out the first five terms of the Taylor series for the function $f(x) = x^2$.

$$f(x_0 + \Delta x) \approx$$

Now plug in $x_0 = 0$ and simplify to expand about 0.

Now plug in $\Delta x = 10$ and simplify down to a single number.

Write out the first five terms of the Taylor series for the function $f(x) = x^3 + x^2 + x + 1$.

$$f(x_0 + \Delta x) \approx$$

Now plug in $x_0 = 0$ and simplify to expand about 0.

Now plug in $\Delta x = 10$ and simplify. Use a calculator to get a single number.
Write out the first five terms of the Taylor series for the function $f(x) = \cos(x)$.

$$f(x_0 + \Delta x) \approx$$

Now plug in $x_0 = 0$ and simplify to expand about 0.

Now plug in $\Delta x = \pi$ and simplify to get an approximation for $\cos(\pi)$.

Write out the first five terms of the Taylor series for the function $f(x) = \sin(x)$.

$$f(x_0 + \Delta x) \approx$$

Now plug in $x_0 = 0$ and simplify to expand about 0:

Now plug in $\Delta x = \pi$ and simplify to get an approximation for $\sin(\pi)$. 
Write out the first five terms of the Taylor series for the function \( f(x) = e^x \).

\[ f(x_0 + \Delta x) \approx \]

Now plug in \( x_0 = 0 \) and simplify to expand about 0:

Now plug in \( \Delta x = 1 \) and simplify to approximate the value of \( e \).

1.2 Euler’s formula

Remember that the imaginary number \( i \) is defined as \( i = \sqrt{-1} \). Therefore, \( i^2 = -1, \ i^3 = -i, i^4 = 1, i^5 = i \), etc. Use the Taylor series expansions about \( x_0 = 0 \) for \( \cos(x) \) and \( \sin(x) \) to write out the first five terms of the Taylor series for the function:

\[ \cos(x) + i \sin(x) \approx \]

Next, use the Taylor series expansion of \( e^x \) to write out first five terms of the Taylor series for:

\[ e^{ix} \approx \]
It should be clear from your answers that the Taylor series of $e^{ix}$ is equal to that of $\cos(x) + i\sin(x)$ (If not, check your work for mistakes). In fact you can use a similar argument to prove that they are exactly equal:

$$e^{ix} = \cos(x) + i\sin(x).$$

Use this identity to derive a famous mathematical equation called “Euler’s formula.”
Plug in the value of $x = \pi$ below and simplify:

$$e^{i\pi} =$$

1.3 Approximating Integrals

There is no simple analytical formula for calculating the integral of $e^{-x^2}dx$. However, we can approximate this function to arbitrary accuracy using a Taylor series. First, compute the integral of the first 5 terms of the Taylor series of $f(x) = e^{-x^2}$:

$$f(x_0 + \Delta x) \approx$$

Now plug in $x_0 = 0$ to expand about 0:

$$e^{-\Delta x^2} \approx$$

Now you should have 5 terms of a polynomial in terms of $\Delta x$. To clean up your notation, substitute $y = \Delta x$ below, in order to get a polynomial in $y$:

$$e^{-y^2} \approx$$

Integrate this polynomial with respect to $y$:

$$\int e^{-y^2}dy \approx$$
2 Fourier Series

The Fourier series is described by the following transformation:

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(bx)]
\]

with the values of \(a_n\) and \(b_n\) defined in the following way:

\[
a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx
\]

\[
b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx
\]

2.1 Average value of a function

The first term of the Fourier series, \(a_0/2\), is actually the average of the function \(f(x)\). Find the average of the following functions over the interval \(-\pi\) to \(\pi\) by computing the value of \(a_0/2\):

\[
f(x) = 23
\]

\[
\frac{a_0}{2} =
\]

\[
f(x) = 3x + 1
\]

\[
\frac{a_0}{2} =
\]

\[
f(x) = \cos(x) + 2
\]

\[
\frac{a_0}{2} =
\]
2.2 Thinking about frequencies

The Fourier series allows us to transform a function into a sum of waves. This transformation gives us the freedom to describe the shape of any function in terms of frequencies and amplitudes. Graphically, this means that I can plot the function in a number of ways. The following examples show the original function, the standard $x$ vs $f(x)$ plot, and a frequency vs amplitude histogram where $\omega$ is the frequency and $A$ is the amplitude. Observe how the two plots are related to each other:

Example 1: $f(x) = 2\cos(4x)$

For this single wave, an amplitude of 2 indicates the maximum height of the wave is equal to 2, and a frequency of 4 means the wave repeats itself 4 times in $2\pi$ (6.28) units along the x-axis.
Example 2: \( f(x) = \sin(x) + 0.5 \cos(10x) \)

![Graph showing amplitude and frequency for Example 2](image)

Example 3: \( f(x) = 1 + \cos(2x) + \cos(4x) + \cos(6x) \)

![Graph showing amplitude and frequency for Example 3](image)
Given the frequency vs. histogram plots below, draw your interpretation of the $x$ vs $f(x)$ plot of the function. Be sure to put numbers on the $x$ and $f(x)$ axes in order to indicate the scale of the function.
3 Optimization

3.1 Graphs

Draw the first and second derivatives of the following functions. Mark the critical points of the original function by circling them. Write X near all local maxima, write N near all local minima, and write S near all saddle points.

3.2 Analytical Roots and Critical Points

Find the roots of the equation $0 = x^3 - x$.

For each critical point of the equation $0 = x^3 - x$, give the $x$-value and classify it as min, max, or saddle.
Write an example of a function with a root at x=2.

Draw an example of a function with a root at x=2.

Write an example of a function with a critical point at x=0. Classify the point as local min/max/saddle. Show your work.

Write an example of a function with an inflection point at x=0.

What is the first derivative of this function?

What is the second derivative of this function?

3.3 Newton’s Method

*Newton’s method* is an algorithm for finding the root of a function using repeated iteration. The method starts with an initial guess $x_0$ and then repeatedly computes new guesses that are closer and closer to the root of the function. The method modifies the
previous guess \( x_n \) by subtracting the function value at \( x_n \) divided by its first derivative at \( x_n \). In order to simplify notation, let us say that an apostrophe signifies a derivative with respect to \( x \). So \( f'(x) = \frac{df}{dx}(x) \). The update rule is then:

\[
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}
\]

Newton’s method is derived from a first-order Taylor series expansion. At each iteration, it approximates the function with a line. Write out the first order Taylor expansion of a function \( f(x) \) below (ignore all terms of order \( \Delta x^2 \) and higher):

\[
f(x + \Delta x) \approx \]

To make notation easier, rewrite your answer by replacing \( x \) with \( a \) and replacing \( \Delta x \) with \( b - a \):

\[
f(a + (b - a)) = f(b) \approx
\]

Finally, we want to find the root of this approximate function. Derive Newton’s method by setting \( f(b) = 0 \) and then solving for \( b \):

\[
b =
\]

What if we use a second order Taylor series instead? Each step would then approximate the function with a parabola instead of a line. Write out the second order Taylor expansion of a function \( f(x) \) below (ignore all terms of order \( \Delta x^3 \) and higher):

\[
f(x + \Delta x) \approx
\]
To make notation easier, rewrite your answer by replacing \( x \) with \( a \) and replacing \( \Delta x \) with \( b - a \):

\[
f(a + (b - a)) = f(b) \approx
\]

Now, if we want to find the root of this approximate function, then we must set \( f(b) = 0 \) and then solve for the value of \( b \). Once we know the value of \( b \), we can repeat this operation over and over to quickly find the root of any function; we will have a root-finding method that converges even faster than Newton’s method! Derive this “improved” method below: set \( f(b) = 0 \), and solve for \( b \). Show your work.

\[
b =
\]

A critical point is any point in a function where its first derivative is equal to zero. Modify Newton’s method so that it finds critical points of the function \( f(x) \).

An inflection point is any point in a function where its second derivative is equal to zero. Modify Newton’s method so that it finds inflection points of the function \( f(x) \).

When \( x_n \) is close to a root, the update rule will provide an \( x_{n+1} \) that is even closer to the root. When \( x_n \) is exactly equal to a root, then the value of \( x \) should not change, and \( x_{n+1} \) should equal \( x_n \). Given the function \( f(x) = \sin(x) \), with \( x_n = 1 \), what is \( x_{n+1} \)? Show your work.
Given the function \( f(x) = \sin(x) \), with \( x_n = 0 \), what is \( x_{n+1} \)? Show your work.

Given the function \( f(x) = x^3 - x^2 + x - 1 \), with \( x_n = 3 \), what is \( x_{n+1} \)? Show your work.

Given the function \( f(x) = x^3 - x^2 + x - 1 \), with \( x_n = 1 \), what is \( x_{n+1} \)? Show your work.

Newton’s method typically converges extremely quickly (it only requires a small number of iterations to find a good guess). However, the method converges more slowly for certain functions, like \( f(x) = x^3 \).

What is the root of \( f(x) = x^3 \)?

Given the function \( f(x) = x^3 \), with \( x_n = 1 \), what is \( x_{n+1} \)? Show your work.

Given the function \( f(x) = x^3 \), with \( x_n = 0 \), what is \( x_{n+1} \)? Show your work.

Hopefully, you discovered that it is difficult to compute the value of \( x_{n+1} \) at the root. What is the problem?

The function \( f(x) = x^3 \) is just an example of the type of functions that have this problem. What types of functions will exhibit this problem in general?