Differential Equations: Homework 3

Due Wednesday, 2 May 2012

Name of student:

Complete the following exercises. Be sure to show all of your intermediate work. You will not get full credit if you only submit solutions (unless the solution requires no intermediate steps). If you require more space than the space provided on these pages, feel free to use additional sheets of paper. You are allowed to use a computer or calculator for arithmetic calculations (like 237+12 or 56^2) but all symbolic manipulation must be done by hand.

1 Derivatives

Remember that the total derivative with respect to \( t \) of some function \( f(x, y, z, t) \) is defined as:

\[
\frac{df}{dt}(x, y, z, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt}.
\]

Given the following functions:

\[
\begin{align*}
  f(x, y) &= \cos(x) + \sin(y) \\
  g(x, y) &= x^2 y \\
  h(x, y) &= y^2 - e^x \\
  k(f, g, h) &= (f + g)^2 + h
\end{align*}
\]

Compute the following derivatives:

\[
\begin{align*}
  \frac{df}{dx}(x, y) &= \\
  \frac{df}{dy}(x, y) &= \\
  \frac{dg}{dx}(x, y) &= \\
  \frac{dg}{dy}(x, y) &=
\end{align*}
\]
\[
\frac{d}{dx} h(x, y) = \\
\frac{d}{dy} h(x, y) = \\
\frac{\partial}{\partial f} k(f, g, h) = \\
\frac{\partial}{\partial g} k(f, g, h) = \\
\frac{\partial}{\partial h} k(f, g, h) = \\
\frac{d}{dx} k(f, g, h) = \\
\frac{d}{dy} k(f, g, h) = \\
\]

2 Vector Calculus

The gradient, divergence, and Laplacian in 3 dimensions are defined as:

\[
\nabla f(x, y, z) = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right), \quad \nabla \cdot \mathbf{v}(x, y, z) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}, \quad \nabla^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\]

Compute the following:

\[
\nabla (xyz) = \\
\nabla (\sin(x + y) + \sin(x + z) + \sin(y + z) + 12) = \\
\n\nabla \cdot \begin{pmatrix}
2x + 3y + 4z \\
5xyz \\
6xy + 7xz + 8yz
\end{pmatrix} = \\
\n\nabla^2 (1 + 2x + 3y + 4z + 5xy + 6xz + 7yz) =
\]

2
\[ \nabla^2(e^{2x} + \sin(yz) + (xyz)^3) = \]

In the following function, find all of the points that are extremal, and classify them as local minima, local maxima, or saddle points.

\[ f(x, y, z) = x^2 - 6x + 2y^2 - 4y + 3z^2 + 6z + 14 \]

3 Ordinary Differential Equations (ODEs)
3.1 Separation of Variables

Solve each of the following ordinary differential equations by using the “separation of variables” technique. Solve for the integration constant by assigning \( y(x = 0) = y_0 \), and write the solution in the form:

\[ y = (\text{some function of } x \text{ and } y_0). \]

\[ \frac{dy}{dx} = \frac{x^3}{y^2} \]

\[ \frac{dy}{dx} = -\left(\frac{\tan(x)}{y^2}\right)\cos(x) \]
\[ \frac{dy}{dx} = 3yx + 2x \]

\[ \frac{dy}{dx} = e^{3x-2y+1} \]

### 3.2 Fixed points and one-dimensional flow

Find the fixed points of the following ODEs, and classify each fixed point as *stable*, *unstable*, or *half-stable*:

\[ \frac{dy}{dx} = y^3 - 5y^2 + 6y \]

\[ \frac{dx}{dt} = 3 - e^x \]

\[ \frac{dx}{dt} = 1 - e^{-x^2} \]
In each of the following graphs, circle all of the fixed points. Write $S$ near the stable fixed points, write $U$ near the unstable fixed points, and write $H$ near the half-stable fixed points. Near each section of the x-axis where the derivative is non-zero, draw a left or right arrow indicating the direction of the flow, as I did in class.
\[ \frac{df(x)}{dx} \]

\[ f(x) \]
3.3 Bifurcations

Near a bifurcation point, the first few terms of a function’s Taylor series expansion look identical to the forms we saw in class:

\[
\frac{dx}{dt} \approx (r + c_0) - c_1 x^2, \quad \frac{dx}{dt} \approx (r + c_0) + c_1 x^2 \quad \text{Saddle node bifurcations}
\]

\[
\frac{dx}{dt} \approx (r + c_0)x - c_1 x^2, \quad \frac{dx}{dt} \approx (r + c_0)x + c_1 x^2 \quad \text{Transcritical bifurcations}
\]

\[
\frac{dx}{dt} \approx (r + c_0)x - c_1 x^3, \quad \frac{dx}{dt} \approx (r + c_0)x + c_1 x^3 \quad \text{Pitchfork bifurcations}
\]

where \( r \) is a variable parameter, \( c_0 \) is a constant offset, and \( c_1 \) is just a scaling factor. The important thing to notice is that the saddle node bifurcation is a quadratic function with a variable constant term, the transcritical bifurcation is a quadratic function with a variable linear term, and the pitchfork bifurcation is a cubic function with a variable linear term (and no quadratic term). A bifurcation occurs when the first few terms of a function’s Taylor series behave like the equations above.

Each of the following functions undergoes a bifurcation at \( x = 0 \) for some value of the parameter \( r = r^* \). For each scenario, perform three actions:

1. Find the value of \( r^* \) that causes the bifurcation.
2. Classify the bifurcation type as saddle-node bifurcation, transcritical bifurcation, or pitchfork bifurcation.
3. Make two graphs of the function near \( x = 0 \). On the left, graph the function for \( r < r^* \), and on the right, graph the function for \( r > r^* \).
\[ \frac{dx}{dt} = x - e^x + r \]

(1) \( r^* = \)

(2) Classify the bifurcation:

(3) Graph the function for \( r < r^* \) (left) and \( r > r^* \) (right)
\[
\frac{dx}{dt} = (r - 1)x + \sin(x)
\]

(1) \( r^* = \)

(2) Classify the bifurcation:

(3) Graph the function for \( r < r^* \) (left) and \( r > r^* \) (right)
\[
\frac{dx}{dt} = \cos(x) + r \sin(x) - 1
\]

(1) \( r^* = \)

(2) Classify the bifurcation:

(3) Graph the function for \( r < r^* \) (left) and \( r > r^* \) (right)
\[ \frac{dx}{dt} = \frac{r + x^2 - rx}{1 - x} \]

(1) \( r^* = \)

(2) Classify the bifurcation:

(3) Graph the function for \( r < r^* \) (left) and \( r > r^* \) (right)