3 Induction

In philosophy, deduction is the process of taking a general statement and applying it to a specific instance. For example: all students must do homework, and I am a student; therefore, I must do homework. In contrast, induction is the process of creating a general statement from observations. For example: all cars I have owned needed repair; therefore, all cars need repair. A similar concept is used in mathematics to prove that a statement is true for all integers. To distinguish it from the less precise philosophical notion, we call it mathematical induction of which we will introduce two forms.

Sum of integers. We begin with an example, namely the familiar problem of summing the first \( n \) positive integers.

CLAIM. For all \( n \geq 0 \), we have \( \sum_{i=0}^{n} i = \left( \frac{n+1}{2} \right) \).

PROOF. First, we note that \( \sum_{i=0}^{0} i = 0 = \left( \frac{1}{2} \right) \). Now, we assume inductively that for some \( n > 0 \), we have
\[ \sum_{i=0}^{n-1} i = \left( \frac{n}{2} \right). \]

If we add \( n \) on both sides, we obtain
\[ \sum_{i=0}^{n} i = \left( \frac{n}{2} \right) + n, \]
which is \( \frac{1}{2}(n-1)n + 2n = \frac{1}{2}(n+1)n = \left( \frac{n+1}{2} \right) \). Thus, by the Principle of Mathematical Induction,
\[ \sum_{i=0}^{n} i = \left( \frac{n+1}{2} \right) \]
for all non-negative integers \( n \).

To analyze why this proof is indeed a proof, we let \( p(k) \) be the statement that the claim is true for \( n = k \). For \( n = 0 \), we have \( p(0) \land [p(0) \Rightarrow p(1)] \). Hence, we get \( p(1) \). We can see that this continues:
\[ p(1) \land [p(1) \Rightarrow p(2)] \text{ hence } p(2); \]
\[ p(2) \land [p(2) \Rightarrow p(3)] \text{ hence } p(3); \]
\[ \ldots \text{ \ldots \ldots } \]
\[ p(n-1) \land [p(n-1) \Rightarrow p(n)] \text{ hence } p(n); \]
\[ \ldots \text{ \ldots \ldots } \]
We can think of induction as a game of dominoes. In this case, we let \( p(n) \) be the statement that the \( n \)-th domino falls over. If we push over the first domino, then \( p(1) \) is true. The first domino hits the second domino, and so \( p(2) \) is true. The second domino hits the third, which means that \( p(3) \) is true. In general, if the \( n \)-th domino falls over, then it knocks over the \( (n+1) \)-st domino. Hence, \( p(n) \Rightarrow p(n+1) \). Thus, all dominoes will fall over.

The weak form. We formalize the proof technique into the first, weak form of the principle. The majority of applications of Mathematical Induction use this particular form.

MATHEMATICAL INDUCTION (WEAK FORM). If the statement \( p(n_0) \) is true, and the statement \( p(n-1) \Rightarrow p(n) \) is true for all \( n > n_0 \), then \( p(n) \) is true for all integers \( n \geq n_0 \).

To write a proof using the weak form of Mathematical Induction, we thus take the following four steps:

Base Case: \( p(n_0) \) is true.
Inductive Hypothesis: \( p(n-1) \) is true.
Inductive Step: \( p(n-1) \Rightarrow p(n) \).
Inductive Conclusion: \( p(n) \) for all \( n \geq n_0 \).

Very often, but not always, the inductive step is the most difficult part of the proof. In practice, we usually sketch the inductive proof, only spelling out the portions that are not obvious. If we can guess the correct closed form expression for a finite sum, it is often easy to use induction to prove that it is correct.

CLAIM. For all integers \( n \geq 1 \), we have \( \sum_{i=1}^{n} 2^{i-1} = 2^n - 1 \).

PROOF. We prove the claim by the weak form of the Principle of Mathematical Induction. We observe that the equality holds when \( n = 1 \), because \( \sum_{i=1}^{1} 2^{i-1} = 1 = 2^1 - 1 \). Assume inductively that the claim holds for \( n-1 \). We get to \( n \) by adding \( 2^{n-1} \) on both sides:
\[ \sum_{i=1}^{n} 2^{i-1} = \sum_{i=1}^{n-1} 2^{i-1} + 2^{n-1} \]
\[ = 2^n - 1 + 2^{n-1} \]
\[ = 2^n - 1. \]

Here, we use the inductive assumption to go from the first to the second line. Thus, by the Principle of Mathematical Induction, \( \sum_{i=1}^{n} 2^{i-1} = 2^n - 1 \) for all \( n \geq 1 \).
Fibonacci numbers. We use the desire to give a closed-form description of Fibonacci numbers as the motivation for another use of mathematical induction. They are defined by the same principle, but that is beside the point. Set $F(1) = F(2) = 1$, and define $F(j) = F(j-1) + F(j-2)$, for all integers $j \geq 3$. The first few Fibonacci numbers are therefore

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

After a slow start, the sequence picks up considerable speed. We claim that we can write the $n$-th Fibonacci number as the sum of two simple powers. To write down the formula, we define $\phi = \frac{1}{2}(1 + \sqrt{5})$ and $\psi = \frac{1}{2}(1 - \sqrt{5})$.

CLAIM. $F(n) = \phi^n - \psi^n$, for all $n \geq 1$.

PROOF. We first test the claim for small values of $n$. For $n = 1$, we have

$$F(1) = \frac{1}{\sqrt{5}}(\phi + \psi),$$

which evaluates to 1, as it should. To prepare the next step, we note that $\phi^2 = \frac{1}{4}(6 + 2\sqrt{5}) = \phi + 1$, and similarly $\psi^2 = \psi + 1$. For $n = 2$, we therefore have

$$F(2) = \frac{1}{\sqrt{5}}(\phi^2 - \psi^2)$$

$$= \frac{1}{\sqrt{5}}(\phi + 1 - \psi - 1),$$

which is again 1. Since the claimed formula is correct twice, perhaps it is correct always. Let us assume the claimed formula for $n - 1$ and for $n - 2$. Then

$$F(n) = F(n - 1) + F(n - 2)$$

$$= \frac{1}{\sqrt{5}} \left[ \phi^{n-2}(\phi + 1) + \psi^{n-2}(\psi + 1) \right]$$

$$= \frac{1}{\sqrt{5}}(\phi^n - \psi^n),$$

as claimed.

We note that this proof goes beyond the weak form of the principle, because we need the hypothesis for $n - 1$ as well as for $n - 2$.

The strong form. As we have seen in a mild form for the Fibonacci numbers, it is sometimes not enough to use the validity of $p(n - 1)$ to derive $p(n)$. Indeed, we have $p(n - 2)$ available and $p(n - 3)$ and so on. Why not use them?

Mathematical Induction (Strong Form). If the statement $p(n_0)$ is true and the statement $p(n_0) \land p(n_0 + 1) \land \cdots \land p(n - 1) \Rightarrow p(n)$ is true for all $n > n_0$, then $p(n)$ is true for all integers $n \geq n_0$.

We use the strong form to prove that every integer has a decomposition into prime factors.

CLAIM. Every integer $n \geq 2$ is the product of prime numbers.

PROOF. We know that 2 is a prime number and thus also a product of prime numbers. Suppose now that we know that every positive number less than $n$ is a product of prime numbers. Then, if $n$ is a prime number we are done. Otherwise, $n$ is not a prime number. By definition of prime number, we can write it as the product of two smaller positive integers, $n = a \cdot b$. By our supposition, both $a$ and $b$ are products of prime numbers. The product, $a \cdot b$, is obtained by merging the two products, which is again a product of prime numbers. Therefore, by the strong form of the Principle of Mathematical Induction, every integer $n \geq 2$ is a product of prime numbers.

Multisets revisited. We use induction to give an alternative proof of the number of size-$k$ multisets we can form using $n$ distinct elements. We will need the strong form of the paradigm.

Base Case. $k = 0$. There is exactly one size-$0$ multiset, namely the empty multiset, no matter how many distinct elements we have at our disposal. This agrees with the formula we want to prove, namely $\binom{n - 1}{0} = 1$, for all $n \geq 1$.

$n = 1$. There is exactly one multiset we can form with one element. This also agrees with the formula, namely $\binom{1}{0} = 1$, for all $k \geq 0$.

Inductive Hypothesis. The number of size-$j$ multisets that can be formed with $m$ distinct elements is $\binom{m + j - 1}{j}$, for every $0 \leq j \leq k$ and every $1 \leq m < n$.

Inductive Step. Assume $k \geq 1$ and $n \geq 2$. To count the size-$k$ multisets, we assume the first element is chosen $i$ times, where $i$ can be any integer between 0 and $k$. This reduces the problem to counting the size-$(k - i)$ multisets that can be formed with $n - 1$ distinct elements. By inductive hypothesis, this number is $\binom{n + k - i - 2}{k - i - 1}$. Hence, the number we seek can be
written as
\[ \sum_{i=0}^{k} \binom{n+k-i-2}{k-i} = \sum_{j=0}^{k} \binom{n+j-2}{n-2}, \]
where \( j = k - i \). Now recall the Pascal triangle, mark the items in the sum, and note that you get the initial segment of a decreasing line. This implies that the sum is equal to \( \binom{n+k-1}{n-1} = \binom{n+k-1}{k} \), as required.

**Inductive Conclusion.** The number of size-\( k \) multisets of \( n \) distinct elements is \( \binom{n+k-1}{n-1} \), for all \( k \geq 0 \) and \( n \geq 1 \).

Note that we did not use the full power of the strong form. Specifically, we used the hypothesis only for \( m = n - 1 \), but we did use it for all \( 0 \leq j \leq k \). Indeed, we can visualize the inductive step to fill the matrix of values column by column, always using an initial segment of the previous column to make the next step; see Table 1.

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<th>3</th>
<th>4</th>
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<td>15</td>
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<td>70</td>
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</table>

Table 1: The number of size-\( k \) multisets we can form with \( n \) different elements. To prove the number for \( n = 5 \) and \( k = 4 \), the last entry in the matrix, we use the inductive hypothesis for the boldface cases.