9 Random Walk

The Bernoulli trial process of repeatedly flipping a coin can be interpreted as a random walk on the 1-dimensional grid. We study related questions and generalize the setting to two and higher dimensions.

Walking on the line. The 1-dimensional integer grid consists of all integers, \( \mathbb{Z} \). For every \( i \geq 1 \), we have a random variable that determines whether the \( i \)-th move is a step to the left or a step to the right:

\[
P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.
\]

Starting at the origin, the position after \( n \) moves is

\[
S_n = X_1 + X_2 + \ldots + X_n;
\]

see Figure 12. For example, we could repeatedly flip a coin and count head as +1 and tail as -1. With this interpretation, we have \( S_n = \# \text{heads} - \# \text{tails} \). The main questions we ask about the random walk are as follows:

1. What is the average distance of the random walker from the origin?
2. What is the probability that after \( n \) moves the walker is back at the origin?
3. More generally, what is the probability distribution of \( S_n \)?
4. How often does the random walker return to the origin?

\[\text{Figure 12: The 1-dimensional integer grid and a path encoding the steps } 1, 1, -1, -1, 1, -1, -1, 1, 1, \ldots \text{ of a random walk.}\]

Average distance. The expected value of \( X_i \) is zero, for every \( i \). This implies

\[
E(S_n) = E(X_1) + E(X_2) + \ldots + E(X_n) = 0,
\]

for all \( n \geq 0 \). It is more difficult to compute the expected distance from the origin, but it is easy to compute the expected squared distance:

\[
E(S_n^2) = E \left( \left( \sum_{i=1}^{n} X_i \right)^2 \right)
\]

\[
= E \left( \sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k \right)
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\[
= \sum_{j=1}^{n} \sum_{k=1}^{n} E(X_j X_k),
\]

which evaluates to \( n \) because \( E(X_j X_k) = 0 \), unless \( j = k \), in which case we have \( E(X_j X_k) = 1 \). In words, the average squared distance of \( S_n \) from the origin is \( n \), suggesting that the average distance is about \( \sqrt{n} \). There are a constant times \( \sqrt{n} \) integers at distance at most \( \sqrt{n} \) from the origin. We would therefore guess that the probability of the random walker to be back at the origin after \( n \) moves is some constant times \( 1/\sqrt{n} \).

Note that \( E(S_n^2) \) is the variance of \( S_n \), and we have learned that for independent random variables, the variance is additive:

\[
V(S_n) = V(X_1) + V(X_2) + \ldots + V(X_n).
\]

The variance of \( X_i \) is 1, for each \( i \), which gives the same result about the average squared distance from the origin.

Probability distribution. The position of the random walker is odd when \( n \) is odd. It follows that it can be at the origin only after an even number of moves. We therefore limit ourselves to even numbers. The probability for the random walker to be at position \( 2j \) after \( 2n \) moves is

\[
P(S_{2n} = 2j) = \binom{2n}{n+j} \cdot 2^{-2n} = \frac{(2n)!}{(n+j)!(n-j)!} \cdot 2^{-2n}. \quad (9)
\]

To better understand this expression, it is useful to have a good approximation. Here, we use

\[
\text{Stirling’s Formula. } n! \sim \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n},
\]

\[
\text{Average distance. } E(S_n) = E(X_1) + E(X_2) + \ldots + E(X_n) = 0,
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where \( \sim \) means that the limit of the ratio of the left over the right hand side, as \( n \) goes to infinity, is 1. Proving this formula is a bit too much effort than we are willing to spend, but a weaker estimate is easy to get. Taking the logarithm of the factorial, we get \( \log n! = \sum_{i=1}^n \ln i \). Using
\[
\int \ln x \, dx = x \ln x - x,
\]
we get
\[
[x \ln x - x]^n = n \ln n - n + 1,
\]
as lower and upper bounds of \( \ln n! \), and therefore
\[
n^n \cdot e^{-n+1} \leq n! \leq (n+1)^n \cdot e^{-n}.
\]

**Central limit theorem.** We are interested in the limit distribution, when \( n \) goes to infinity. Not surprisingly, this is the Gaussian normal distribution, defined as
\[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

We can prove this starting from (9), in a sequence of transformations and approximations. To simplify the notation, we write \( P_{2j} = P(S_{2n} = 2j) \).
\[
P_{2j} = \frac{(2n)^{2j+\frac{1}{2}}}{\sqrt{2\pi(n+j)^{n+j+\frac{1}{2}}(n-j)^{n-j+\frac{1}{2}}} \cdot 2^{-2n}}
\]
\[
= \frac{\sqrt{n}}{\pi(n^2-j^2)^{n^2}(n+j)^{n}} \cdot \frac{n^3}{n^3} \cdot \frac{n-j}{n-j}
\]
\[
= \frac{\sqrt{n}}{\pi(n^2-j^2)^{n}} \cdot \left(1 - \frac{j^2}{n^2}\right)^{-n}
\]
\[
\cdot \left(1 + \frac{j^2}{n}ight)^{j^2} \cdot \left(1 - \frac{j^2}{n^2}\right)^{j^2}
\]
\[
= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.
\]

We have
\[
P_{[a,b]} = \lim_{n \to \infty} \sum_{j=a \sqrt{2n}}^{b \sqrt{2n}} \frac{1}{\sqrt{\pi n}} \cdot e^{-j^2/n},
\]
where the sum is over all \( a \leq j \sqrt{2/n} \leq b \). Noting that this sum is a Riemann integral with intervals of length \( \sqrt{2/n} \), we define \( x = j \sqrt{2/n} \) to get
\[
P_{[a,b]} = \lim_{n \to \infty} \left( \sum_{j=a \sqrt{2n}}^{b \sqrt{2n}} \frac{1}{\sqrt{\pi n}} \cdot e^{-j^2/n} \right) / \sqrt{2/n}
\]
\[
= \int_a^b \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \, dx.
\]

This proves that the probability distribution in the limit is \( f \), as claimed above. What we proved is just a special case of the Central Limit Theorem, which holds more generally for sums of independent random variables drawn from a common distribution with finite average and variance.

**Returning to the origin.** We now come back to the fourth question about 1-dimensional random walks. To count the number of times the random walker returns to the origin, we add up the indicator function for that event:
\[
J_n = \begin{cases} 1 & \text{if } S_n = 0, \\ 0 & \text{otherwise}, \end{cases}
\]
\[
\#\text{Returns} = \sum_{n=0}^{\infty} J_{2n}.
\]

The expected number of visits to the origin can now be computed by adding the expectations of the indicator functions:
\[
E(\#\text{Returns}) = \sum_{n=0}^{\infty} E(J_{2n})
\]
\[
= \sum_{n=0}^{\infty} P(S_{2n} = 0)
\]
\[
\sim \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}}
\]
which is infinity. In words, the random walker is expected to visit the origin infinitely often. It is even true that with probability 1, the random walker returns to the origin infinitely often, but this does not follow from the infinite expectation. The gap in the argument results from the possibility to have a random variable with finite values but infinite expectation. An example is the St. Petersburg Paradox. You flip a coin until getting tail the first time. If you
get $k$ heads before the tail, your payoff is $2^{k+1}$ euros. The expected value is

$$E(\text{payoff}) = \frac{1}{2} \cdot 2^0 + \frac{1}{4} \cdot 2^1 + \frac{1}{8} \cdot 2^2 + \ldots$$

which is infinitely many euros. You should therefore be ready to pay any finite amount to play this game, which is apparently not true. Herein lies the paradox.

**Beyond one dimension.** The four questions for a random walker in one dimension can also be asked for the $d$-dimensional integer grid, $\mathbb{Z}^d$; see Figure 13. Its elements are the $d$-vectors with integer coordinates. As before, we define

$$S_n = X_1 + X_2 + \ldots + X_n,$$

where each $X_i$ is $\pm 1$ times one of the $d$ unit coordinate vectors. We have $E(X_i) = 0$, for every $i$, and therefore $E(S_n) = 0$, for every $n$. We have $X_i X_j = 1$, for all $i$, where multiplication means the scalar product. This is a useful operation because $X_i X_i = \|X_i\|^2$ is the squared distance of $X_i$ from the origin. For $i \neq j$, we have

$$P(X_i X_j = 1) = P(X_i X_j = -1) = \frac{1}{2d},$$

$$P(X_i X_j = 0) = 1 - \frac{1}{d}.$$

The expected value of $X_i X_j$ is therefore 1, if $i = j$, and 0, if $i \neq j$. Hence,

$$E(S_n S_n) = E \left( \left( \sum_{i=1}^n X_i \right) \left( \sum_{j=1}^n X_j \right) \right)$$

evaluates to $n$, as in one dimension. This suggests that the average distance to the origin in $\mathbb{Z}^d$ is some constant times $\sqrt{n}$. In contrast to one dimension, there are many integer points at distance at most $\sqrt{n}$ from the origin, namely some constant times $n^{d/2}$. This makes a difference when we calculate the expected number of times the random walker returns to the origin. Doing the computations rigorously would take too much time, but we get qualitatively the correct result even when we take short-cuts. Specifically, for $S_{2n}$ to be 0, it must be zero in each coordinate. For this, it is necessary that the walker takes an even number of steps in each coordinate. Assuming this even number is $2n/d$, for each coordinate, the probability that the walker is at 0 is

$$\left( \frac{1}{\sqrt{n/d}} \right)^d = \text{const} \cdot n^{-d/2}.$$

Taking the sum, for $n$ from 0 to $\infty$, we get infinity, for $d = 1, 2$, and a finite value, for $d \geq 3$. In words, there is a striking difference between 2 and 3 dimensions, namely that it is much easier to find a place in 2 dimensions even if the search has neither the benefit of knowledge nor of a strategy beyond randomness.