10 Gambler’s Ruin

Deepening our understanding of random walks, we study the expected number of steps it takes to leave a given interval on the line, or a finite region in two or higher dimensions. It will be helpful to relate the random process of walking with the deterministic process of diffusion.

Escaping an interval. Consider a finite interval of the 1-dimensional integer grid, \{0, 1, \ldots, N\}, and let the starting position of a random walk be a point \(x\) within this interval. As before, \(X_i = \pm 1\) with probability \(\frac{1}{2}\) for each choice, and \(S_n = x + \sum_{i=1}^{n} X_i\). The assumption now is that the walker stops when she reaches either endpoint of the interval. More formally, the position of the walker at time \(n\) is \(S_{\min(n,T)}\), where \(T = \min\{n \mid S_n = 0\text{ or } S_n = N\}\). To study this problem, we focus on the case \(S_T = N\). Specifically, we introduce the function \(F\) from the interval \{0, 1, \ldots, N\} to \(\mathbb{R}\) defined by

\[
F(x) = P(S_T = N \mid S_0 = x).
\]

In words, the gambler begins with \(x\) chips and plays until she has \(N\) chips in her possession or she is bankrupt, whatever occurs first. \(F(x)\) is the probability that she reaches her goal of \(N\) chips. Clearly, \(F(0) = 0\) and \(F(N) = 1\). Assume now that \(0 < x < N\). After the first round, the gambler has either \(x - 1\) or \(x + 1\) chips, each with probability one half, which implies

\[
F(x) = \frac{1}{2}F(x - 1) + \frac{1}{2}F(x + 1). \quad (10)
\]

One function that satisfies (10) together with the boundary conditions is \(F(x) = x/N\). We prove that it is the only solution.

**Theorem.** The only function \(F : \{0, 1, \ldots, N\} \to \mathbb{R}\) that satisfies (10) together with \(F(0) = a\) and \(F(N) = b\) is \(F_0(x) = (1 - \frac{a}{N})a + \frac{x}{N}b\).

**Proof.** Every function \(f\) that satisfies (10) has

\[
f(x) \geq \min\{f(x - 1), f(x + 1)\},
\]

\[
f(x) \leq \max\{f(x - 1), f(x + 1)\},
\]

for every \(0 < x < N\). Hence, the minimum is at 0 or at \(N\), and so is the maximum. If \(a = b = 0\), then this is only possible if \(f\) is identically zero. Suppose you have \(F\) satisfying (10) with \(F(0) = a\) and \(F(N) = b\), and let \(F_0\) be the linear function with this property. Then \(g = F - F_0\) satisfies (10) with \(g(0) = g(N) = 0\). Hence, \(g\) is identically zero, and \(F = F_0\). \(\Box\)

**Beyond one dimension.** The same problem in two and higher dimensions is more interesting, as we now explain. Let \(A\) be a finite subset of \(\mathbb{Z}^d\), let \(\partial A\) be the set of points in \(\mathbb{Z}^d - A\) at distance 1 from \(A\), and set \(\bar{A} = A \cup \partial A\); see Figure 14. We may think of \(A\) is the discrete interior of a

![Figure 14](image-url)

region, \(\partial A\) as its boundary, and \(\bar{A}\) as its closure. Let now \(F : \bar{A} \to \mathbb{R}\). We consider two linear operators, which map \(F\) to another function on \(A\):

\[
QF(x) = \frac{1}{2d} \sum_{||y-x||=1} F(y),
\]

\[
\mathcal{L}F(x) = (QF(x) - F(x)) = \frac{1}{2d} \sum_{||y-x||=1} (F(y) - F(x)).
\]

The first operator averages the values at the distance 1 neighbors and assigns this average to the point. Writing the function values at the points in \(\bar{A}\) as a vector, \(F\), we can write \(Q\) as a matrix, and get \(QF\) by multiplying the matrix with the vector. In this notation, the **discrete Laplacian** is \(\mathcal{L} = Q - I\), and we have \(\mathcal{L}F = (Q - I)F\), where \(I\) is the identity matrix. Equivalently, the discrete Laplacian averages the differences between \(x\) and its neighbors and assigns this average to the point. There is an interesting connection to random walks:

\[
\mathcal{L}F(x) = E(F(S_1) - F(S_0) \mid S_0 = x).
\]

Indeed, the expected difference between \(S_0\) and \(S_1\) is the average difference between \(x = S_0\) and its neighbors. We say \(F\) is **discrete harmonic** at \(x\) if \(\mathcal{L}F(x) = 0\).

**The Dirichlet problem.** Given a finite set \(A \subseteq \mathbb{Z}^d\), and a function \(F : \partial A \to \mathbb{R}\), find a harmonic extension \(F_0 : \bar{A} \to \mathbb{R}\); that is:

\[
F_0(z) = F(z), \quad \forall z \in \partial A, \quad (11)
\]

\[
\mathcal{L}F_0(x) = 0, \quad \forall x \in A. \quad (12)
\]
In dimension $d = 1$, we were able to guess the solution, but in higher dimensions it is not quite as obvious. Let $T$ be the smallest $n$ such that $S_n \notin A$.

**THEOREM.** Let $A \subseteq \mathbb{Z}^d$ be finite. Then every function $F : \partial A \to \mathbb{R}$ has a unique harmonic extension $F_0 : A \to \mathbb{R}$ given by

$$F_0(x) = E(F(S_T) \mid S_0 = x) = \sum_{z \in \partial A} P(S_T = z \mid S_0 = x) \cdot F(z).$$

We only prove the easy part, namely that $F_0$ as specified satisfies (12). For this, we observe that the probability of $S_T = z$ given $S_0 = x$ is the average over the $2d$ neighbors of $x$. Plugging this into the formula for $F_0$, we get

$$F_0(x) = \sum_{z \in \partial A} \sum_{\|y-z\|=1} P(S_T = z \mid S_1 = y) \cdot \frac{F(z)}{2d} = \frac{1}{2d} \sum_{\|y-z\|=1} F_0(y).$$

Hence, $\sum_{\|y-z\|=1} (F_0(y) - F_0(x)) = 0$, which means that $F_0$ is indeed harmonic at $x$, as required. The proof that $F_0$ is the only harmonic extension of $F$ is more difficult and omitted.

**Discrete heat equation.** We get another interpretation of the process by modeling it as the flow of heat. Remarkably, this interpretation will be deterministic, devoid of any probabilities.

Set the temperature at $z \in \partial A$ to zero, at all times, and at $x \in A$ to its initial value, $p_0(x)$. At each time $n$, the heat at $x$ spreads evenly to the $2d$ neighbors. The heat that goes to a boundary point is lost forever. The connection to the earlier context is that we can think of the heat as a large collection of particles, each behaving randomly — performing a random walk in $\mathbb{Z}^d$ — until it leaves $A$. The temperature at time $n+1$ is

$$p_{n+1}(x) = \frac{1}{2d} \sum_{\|y-z\|=1} p_n(y).$$ (13)

We write $\partial p_n(x) = p_{n+1}(x) - p_n(x)$ to get the discrete heat equation:

$$\partial p_n(x) = \mathcal{L}p_n(x),$$ (14)

for every $x \in A$. Consider for example the case $p_0(x) = 1$ for some $x \in A$, and $p_0(y) = 0$ for all other points. Then

$$p_n(y) = P(S_{\min\{n,T\}} = y \mid S_0 = x).$$

In words, $p_n$ is the restriction of the probability distribution of a random walker after $n$ steps. As $n$ goes to infinity, the probabilities within $A$ go to zero. We can compute $p_{n+1}$ iteratively, using (13). In vector and matrix notation, we write this as

$$p_{n+1} = Qp_n = Q^{n+1}p_0.$$ 

As $n$ goes to infinity, the function approaches the zero function. If we set the boundary to fixed but not necessarily zero values, we approach the unique harmonic extension of the boundary function. Instead of iterating toward this solution, it is also possible to compute it directly, using the eigenvalues and eigenvectors of $Q$.

**Time to escape.** A natural question in this context is the average number of steps it takes before $S_n$ exits $A$. In dimension $d = 1$, we define

$$e(x) = E(T \mid S_0 = x).$$

Clearly, $e(0) = e(N) = 0$. Suppose $0 < x < N$, and note that the first step takes the random walker from $x$ to either $x-1$ or $x+1$, each with probability one half. Therefore,

$$e(x) = 1 + \frac{1}{2}e(x-1) + \frac{1}{2}e(x+1).$$ (15)

We see that $e$ satisfies $Le(x) = -1$ for all $0 < x < N$. To get a feeling for the solution, consider $f(x) = x^2$ and note that

$$\mathcal{L}f(x) = \frac{1}{2}f(x-1) + \frac{1}{2}f(x+1) - f(x) = \frac{1}{2}(2x^2 + 2) - x^2,$$

which evaluates to $1$. Similarly, if $g(x) = x$, then $\mathcal{L}g(x) = 0$. Adding a linear term to $-x^2$ would thus give zero Laplacian. We conclude that

$$e(x) = x(N-x)$$ (16)

satisfies the boundary conditions as well as (15). Not surprisingly, it is the unique solution.

While (16) seems rather benign, it has some surprising consequences. Setting $N = 2m$, we get $e(m) = m^2$. In words, the expected number of steps for a random walker to reach a distance $m$ from its initial position is precisely $m^2$. This is consistent with our earlier result that the average square distance after $n$ steps is some constant $\sqrt{n}$, but it is not the same. In contrast, for $x = 1$, we get $e(1) = N - 1$. While it is the expected value, it is not
the typical value since most of the time, the walker exits quickly at 0. However, there is a small probability that she takes many steps, namely for example when it reaches $m$ before exiting, which happens with probability $1/m$.

In two and higher dimensions, the result about the escape time generalizes to

$$E(T) = E(\|S_T\|^2) - \|S_0\|^2.$$ (17)

Indeed, if we evaluate this formula for $S_0 = x$ in dimension $d = 1$, we get

$$E(\|S_T\|^2) = \frac{N - x}{N} \cdot 0^2 + \frac{x}{N} \cdot N^2 = xN,$

and therefore $E(T) = xN - x^2$, as before. Suppose, for example, that $A$ consists of all points $x \in \mathbb{Z}^d$ with $\|x\| < r$. Then $r \leq \|z\| < r + 1$ for every $z \in \partial A$. Starting the walk at the origin, we get $r^2 \leq E(T) < (r + 1)^2$ from (17). The expected number of steps the walker reaches a distance $r$ from the starting position is again $r^2$, as in dimension $d = 1$. But this is really just a consequence of Pythagoras’ Theorem about the sum of squares.