Note: there are many exercises on this sheet, but none of them require a long answer. Try to make it your goal to be as concise and correct as possible.

Except if indicated otherwise, you may use any results, theorems, etc. from the lecture in your answers. Do not use facts from elsewhere (Wikipedia, undergraduate courses, etc.), except elementary high school maths (real numbers, etc.).

1.1 Logics

Exercise 1.1. Which of the following statements are correct? (mark all correct answers directly on the sheet) (8 points)

a) □ All mammals have warm blood.
b) □ All birds have warm blood.
c) □ There exists a warm-blooded fish.
d) □ ∀X: X is a mammal ⇒ X has warm blood.
e) □ ∀X: X has warm blood ⇒ X is a mammal.
f) □ ∀X: X has warm blood ∧ X does not lay eggs ⇒ X is a bird.
g) ∀X: X is a fish ∧ X is a mammal ⇒ X is a bird.
h) □ ∃X: X has warm blood ∧ X lays eggs ∧ X is a mammal ∧ ¬(X is a platypus)

Exercise 1.2. Find the exact locations of the mistakes made in the following "proofs": (6 points)

1) We want to prove 1 = 2 by starting with a true statement and making only logically true derivations from it.

We assume a = b. Then

\[ a = b \] (1.1)
\[ a^2 = ab \] (1.2)
\[ 2a^2 = a^2 + ab \] (1.3)
\[ 2a^2 - 2ab = a^2 - ab \] (1.4)
\[ 2(a^2 - ab) = 1(a^2 - ab) \] (1.5)
\[ 1 = 2 \] (1.6)

2) We prove 0 = 1 by starting with a true statement and only logically true derivations from it.

For arbitrary \( n \in \mathbb{N} \), we have

\[ (n + 1)^2 = n^2 + 2n + 1 \] (1.7)
\[ (n + 1)^2 - (2n + 1) = n^2 \] (1.8)
\[ (n + 1)^2 - (2n + 1) - n(2n + 1) = n^2 - n(2n + 1) \] (1.9)
\[ (n + 1)^2 - (n + 1)(2n + 1) = n^2 - n(2n + 1) \] (1.10)
\[ (n + 1)^2 - (n + 1)(2n + 1) + (2n + 1)^2/4 = n^2 - n(2n + 1) + (2n + 1)^2/4 \] (1.11)
\[ \left( (n + 1) - 2(n + 1)/2 \right)^2 = \left( (n + 1) + 2(n + 1)/2 \right)^2 \] (1.12)
\[ (n + 1) - 2(n + 1)/2 = (n - 2(n + 1)/2) \] (1.13)
\[ n + 1 = n \] (1.14)
\[ 1 = 0 \] (1.15)

3) We prove that a sandwich is better than eternal happiness. This is really just three lines of argument:

- Nothing is better than eternal happiness.
- Eating a sandwich is better than nothing.
- Therefore, eating a sandwich is better than eternal happiness.

please turn page over...
1.2 Linearity

Reminder: to prove that a function is linear, show either that it fulfills the defining properties (i) and (ii), or use one of the results proved in the lecture. To prove that a function is not linear, it suffices to provide a counterexample, i.e.

a case in which property (i) or (ii) is violated.

Exercise 1.3. Which of these functions $f_i : \mathbb{R} \to \mathbb{R}$ are linear? Justify your answers. (8 points)

- $f_1(x) = 0$
- $f_2(x) = 1$
- $f_3(x) = 1x$
- $f_4(x) = 0x$
- $f_5(x) = \begin{cases} -|x| & \text{for } x \leq 0 \\ |x| & \text{otherwise.} \end{cases}$
- $f_6(x) = (x + 1)^2 - (x - 1)^2$
- $f_7(x) = \sqrt{x^2}$
- $f_8(x) = \sqrt[3]{x^3}$

Exercise 1.4. For which values of $a, b, c \in \mathbb{R}$ is the function $f(x) = ax^2 + bx + c$ linear? (3 points)

Exercise 1.5. Which of these functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are linear? (6 points)

- $f_9(x) = \sum_{j=2,4,6,...,n} x_j$ (for even $n \in \mathbb{N}$)
- $f_{10}(x) = \sum_{j=1,...,n} \left( \frac{1}{2} \right)^2 x_j$
- $f_{11}(x) = \max(x_1, \ldots, x_n) - \min(x_1, \ldots, x_n)$, where $\max$ and $\min$ pick the largest and smallest value from a list

Exercise 1.6. Let $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates a two-dimensional vector counterclockwise by the angle $\theta$. Prove geometrically, i.e. without using the example about rotation matrices from the lecture, that $f_\theta$ is linear. (4 points)

Exercise 1.7. Prove or disprove the following two statements: (4 points)

- For any linear function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, the function $h : \mathbb{R}^2 \to \mathbb{R}$ given by $h(x_1, x_2) := f(x_1) + g(x_2)$ is linear.
- For any linear function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, the function $h : \mathbb{R}^2 \to \mathbb{R}$ given by $h(x_1, x_2) := f(x_1) \cdot g(x_2)$ is linear.

Exercise 1.8. Identify the matrices corresponding to the following linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ and their sizes (number of columns, number of rows): (5 points)

- $f_{12}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
- $f_{13}(x) = (x_1 + x_2 + \cdots + x_n)$
- $f_{14}(x) = x$
- $f_{15}(x) = \pi x$
- $f_{16}(x) = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{pmatrix}$

Exercise 1.9. Compute the following matrix-vector multiplications if possible. (12 points)

- $\begin{pmatrix} 1 & 2 \\ \frac{3}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix}$
- $\begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{3} \\ \frac{1}{2} \end{pmatrix}$
- $\begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{pmatrix}$
- $\begin{pmatrix} -1 & -1 \\ \frac{\pi}{2} & \frac{\pi}{2} \end{pmatrix} \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}$
- $\begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$
- $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -5 \\ 0 \\ -4 \\ -3 \\ -1 \end{pmatrix}$
- $\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 9 \\ 8 \\ 6 \\ 3 \end{pmatrix}$
- $\begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} \in \mathbb{R}^n$.

Exercise 1.10. In the lecture, we covered least-squares parameter estimation. Show that for fixed data $x^{(1)}, \ldots, x^{(n)} \in \mathbb{R}$, the optimal coefficient $c^*$ is a linear function of the vector of observations $y = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix} \in \mathbb{R}^n$. (4 points)

Exercise 1.11. Use least-squares regression to estimate the equation of a linear function from the following samples:

<table>
<thead>
<tr>
<th>$x$</th>
<th>4</th>
<th>-3</th>
<th>3</th>
<th>2</th>
<th>5</th>
<th>-2</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>-5.5</td>
<td>4.9</td>
<td>-5.0</td>
<td>-3.9</td>
<td>-8.1</td>
<td>0.9</td>
<td>1.9</td>
</tr>
</tbody>
</table>

(4 points)