Reminder. We call a function \( f : X \to Y \) linear, if and only if

1) \( \forall x, x' \in X : f(x + x') = f(x) + f(x') \),

2) \( \forall x \in X, \alpha \in \mathbb{R} : f(\alpha x) = \alpha f(x) \).

Reminder. Affine functions \( f(x) = ax + b \) are not linear, unless \( b = 0 \).

We now want to study functions \( f(x_1, x_2, ..., x_n) \) that have multiple input variables, e.g.

- \( \text{Area}(w, h) = wh \),
- \( \text{Perimeter}(w, h) = 2(w + h) \).

where \( w, h \) are the width and height of a rectangle.

**Definition 3.1.**

- We can add two vectors componentwise: for any

  \[
  x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \ x' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} \in \mathbb{R}^n, \text{ we have } \ x + x' = \begin{pmatrix} x_1 + x'_1 \\ x_2 + x'_2 \\ \vdots \\ x_n + x'_n \end{pmatrix} \in \mathbb{R}^n.
  \]

- We can multiply a vector componentwise with a real number:

  for any \( x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \alpha \in \mathbb{R}, \text{ we have } \alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix} \in \mathbb{R}^n.\]
Remark. This is the same as vector calculus in school (see blackboard drawing). For \( n = 1 \), it is also the same as for ordinary real numbers.

Remark. We will not define a way to multiply two \( n \)-vectors with a \( n \)-vector as output. In principle, one could define this, but it turns out it's rarely useful. We will see the scalar product between vectors later in the lecture, but there the result is a number, not an \( n \)-vector.

**Theorem 3.2.** A function \( f : \mathbb{R}^n \to \mathbb{R} \) is linear, if and only if there exist values \( c_1 \in \mathbb{R}, \ldots, c_n \in \mathbb{R} \), such that

\[
f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n
\]

where \( x_1, \ldots, x_n \) are the components of \( x \).

Remark. Before we try to prove this, let’s see if it make sense with respect to what we already know, namely the special case \( n = 1 \). Then the theorem says: \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) is linear if and only if there is a constant \( c_1 \in \mathbb{R} \) such \( f(x_1) = c_1 x_1 \) for all \( x_1 \in \mathbb{R} \). Good, so there is no contradiction, but Theorem 2.3 is a special case of Theorem 2.3.

*Proof of Theorem 3.2.* The theorem states an equivalence, so we again show both directions: "⇐" and "⇒".

"⇐". We show: if there are values \( c_1 \in \mathbb{R}, \ldots, c_n \in \mathbb{R} \), such that for all \( x \in \mathbb{R}^n \): \( f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \), then \( f \) is linear.

We check the defining properties of linearity. For any \( x, x' \in \mathbb{R}^n \) we have:

\[
f(x + x') = f \left( \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix} \right) = f \left( \begin{pmatrix} x_1 + x'_1 \\ x_2 + x'_2 \\ \vdots \\ x_n + x'_n \end{pmatrix} \right)
\]

\[
= c_1 (x_1 + x'_1) + c_2 (x_2 + x'_2) + \cdots + c_n (x_n + x'_n)
\]

\[
= c_1 x_1 + c_1 x'_1 + c_2 x_2 + c_2 x'_2 + \cdots + c_n x_n + c_n x'_n
\]

\[
= c_1 x_1 + c_2 x_2 + \ldots c_n x_n + c_1 x'_1 + c_2 x'_2 + \ldots c_n x'_n
\]

\[
= f(x) + f(x').
\]
For any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$:

$$f(\alpha x) = f\left(\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}\right)$$

(3.7)

$$= c_1 \alpha x_1 + c_2 \alpha x_2 + \cdots + c_n \alpha x_n$$

(3.8)

$$= \alpha (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)$$

(3.9)

$$= \alpha f(x)$$

(3.10)

So $f$ is indeed linear.

$\Rightarrow$. We show: if $f$ is linear, then there exist values $c_1, \ldots, c_n \in \mathbb{R}$, such that $f(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ for all $x \in \mathbb{R}^n$.

We start with a helping definition and lemma:

**Definition 3.3.** For any $i \in \{1, 2, \ldots, n\}$, let $e^{(i)} \in \mathbb{R}^n$ be the vector with all components 0, except the $i$th, which is 1. So $e^{(i)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \overset{\text{i-th}}{\leftarrow}$ We call $e^{(i)}$ the $i$-th unit vector.

**Lemma 3.4.** For any $x \in \mathbb{R}^n$ with coefficient $x_1, \ldots, x_n$ one has

$$x = x_1 e^{(1)} + x_2 e^{(2)} + \cdots + x_n e^{(n)} = \sum_{i=1}^{n} x_i e^{(i)}.$$  

(3.12)
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**Proof.**

\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_2 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ x_n \end{pmatrix} (3.13)
\]

\[
= x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} (3.14)
\]

\[
= x_1 e^{(1)} + x_2 e^{(2)} + \ldots x_n e^{(n)} = \sum_{i=1}^{n} x_i e^{(i)} (3.15)
\]

**Proof of Theorem 3.2 (cont.)** We can now prove the $\Rightarrow$ direction of Theorem 3.2 analogously to the one-dimensional case. We can assume that $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is linear. Let $c_1 := f(e^{(1)}), c_2 := f(e^{(2)}), \ldots, c_n := f(e^{(n)})$. Then

\[
f(x) = f\left(\sum_{i=1}^{n} x_i e^{(i)}\right) = \sum_{i=1}^{n} x_i f(e^{(i)}) = \sum_{i=1}^{n} x_i c_i = c_1 x_1 + \ldots c_n x_n (3.16)
\]

**Example 3.5.**

- A weather station measures the temperature every day of a month: $T_1, \ldots, T_{30}$. The average temperature of the month is a linear function of the vector of measurements:

  \[
f_{\text{avg.temp.}}(T_1, \ldots, T_{30}) = \frac{T_1 + T_2 + \ldots + T_{30}}{30} = \frac{1}{30} \sum_{i=1}^{30} T_i (3.17)
\]

- Let $\Omega = \{1, \ldots, K\}$ be a set of random events (e.g. outcomes of rolling a dice: $K = 6$). Let $P(\omega)$ for $\omega = 1, \ldots, K$, be probability for each of the events to happen. Assume for each event that you have a pay-off, $T_\omega$ (could be negative, then we’d rather call it a cost). The expected value of the pay-off is a linear function of the vector of pay-offs:

  \[
  E(T) = \sum_{\omega \in \Omega} P(\omega) T_\omega (3.18)
  \]
Let $R, G, B$ be the strength of red, green and blue pixels on a color screen. The total intensity that the human eye sees from this is (approximately) given by

$$\text{intensity}(R, G, B) = 0.3 \cdot R + 0.59 \cdot G + 0.11 \cdot B$$

which is linear function of the vector $\begin{pmatrix} R \\ G \\ B \end{pmatrix} \in \mathbb{R}^3$.

**Linear functions:** $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Vectors can not just be inputs to linear functions but also outputs. We now look at function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ (note: $n \neq m$ is possible, so the dimensions of input and output can differ).

**Example 3.6.** For $n = 2$ and $m = 2$, we are quite familiar with such functions: many geometric operations are just functions $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, e.g. rotating a vector, or mirroring it at one of the axes.

\[
\begin{align*}
\text{rotate 90 deg c.c.w.} & \quad f(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \\
\text{mirror at } x_2\text{-axis} & \quad f(x) = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}
\end{align*}
\]

**Definition 3.7.** For $n, m \in \mathbb{N}$, a rectangular arrangement of numbers,

\[
\begin{pmatrix}
a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\
a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m,1} & a_{m,2} & \ldots & a_{m,n}
\end{pmatrix}
\]

is called a matrix with $m$ rows and $n$ columns, or short $m \times n$-matrix. For the set of all $m \times n$-matrices with real-valued entries we write $\mathbb{R}^{m \times n}$. 

**Definition 3.8.** For any matrix \( A \in \mathbb{R}^{m \times n} \) and any vector \( x \in \mathbb{R}^n \), we define the product of \( A \) with \( x \) as

\[
Ax = \begin{pmatrix}
a_{1,1} & \cdots & a_{1,n} \\
a_{2,1} & \cdots & a_{2,n} \\
\vdots & \ddots & \vdots \\
a_{m,1} & \cdots & a_{m,n}
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix} = \begin{pmatrix}
a_{1,1}x_1 + \cdots + a_{1,n}x_n \\
a_{2,1}x_1 + \cdots + a_{2,n}x_n \\
\vdots \\
a_{m,1}x_1 + \cdots + a_{m,n}x_n
\end{pmatrix}.
\] (3.21)

**Remark.** Note: we can only multiply a matrix with a vector if the number of columns of the matrix is the same as the number of the rows of the vector.

**Theorem 3.9.** A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is linear, if and only if there exists a matrix \( A \in \mathbb{R}^{m \times n} \) such that \( f(x) = Ax \).

**Proof.** Because of Definition 3.8, Theorem 3.9 is equivalent to the following statement:

A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is linear if and only if there are \( n \cdot m \) values, \( a_{i,j} \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, n \), such that

\[
f(x) = \begin{pmatrix}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\
a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\
\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n
\end{pmatrix}.
\] (3.22)

We prove this by showing both implications: "\( \Leftarrow \)" and "\( \Rightarrow \)":

"\( \Leftarrow \)" we show: if \( f \) is of form \((*)\) it fulfills the properties of linearity. We leave this proof as an exercise. Idea: use the same steps as the one for \( f : \mathbb{R}^n \to \mathbb{R} \) case, separately in each component of the output.

"\( \Rightarrow \)" we assume that \( f : \mathbb{R}^n \to \mathbb{R}^m \) is linear and show \((*)\).

**Lemma 3.10.** For any \( i = 1, \ldots, n \), the function \( \pi_i : \mathbb{R}^n \to \mathbb{R} \), defined as \( \pi_i(x) = x_i \) (i.e. keep only the \( i \)-th component of \( x \in \mathbb{R}^n \)), is linear.

**Proof.** This follows from Theorem 3.2, because we can write \( \pi_i(x) = 0 \cdot x_1 + \cdots + 0 \cdot x_{i-1} + 1 \cdot x_i + 0 \cdot x_{i+1} + \cdots + 0 \cdot x_n \).
Proof of Theorem 3.9 (cont.) For any $x \in \mathbb{R}^n$, $f(x)$ is a vector of $m$ components, which can write as

$$f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \quad (3.23)$$

where $f_i = \pi_i \circ f$ for $i = 1, \ldots, m$. Each $f_i : \mathbb{R}^n \to \mathbb{R}$ is linear, as follows from Proposition 2.7 (*the concatenation of two linear function is linear*), because $f$ is linear (by assumption), and $\pi_i$ is linear (by Lemma 3.10). Because $f_i$ is linear, there are constants $a_{i,1}, \ldots, a_{i,n}$ such that $f_i(x) = a_{i,1}x_1 + \cdots + a_{i,n}x_n$ (by Theorem 3.2). Plugging this representation into (**) we see that $f$ can be written in the form (*).

Example 3.11. Both geometric examples above are linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$:

- $f\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \cdot x_1 - 1 \cdot x_2 \\ 1 \cdot x_1 + 0 \cdot x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$
- $f\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right) = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 - 1 \cdot x_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Example of linear functions and how to write them as matrices-vector products

Example 3.12. The function $f_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates a vector $x$ by $\theta$ counterclockwise is linear and can be written as

$$f(x) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = R_{\theta} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (3.24)$$

A matrix of the form $R_{\theta}$ is called a rotation matrix.

Using $\sin \frac{\pi}{2} = 1$ and $\cos \frac{\pi}{2} = 0$ we can check that the first matrix of Example 3.11 is indeed a rotation matrix with $\theta = \frac{\pi}{2}$ (i.e. 90 deg).
Example 3.13 (Population Dynamics). We study the population size over multiple generations of spotted owls, where we distinguish between 3 stages: young, adult and dead.

From one generation to the next, the following happens:

- 30% of the young reach adulthood, 70% die.
- 50% of the adults survive the winter, 50% die.
- on average, each adult has 1.3 young as offspring.

We call the current state: \( x = \begin{pmatrix} x_{\text{young}} \\ x_{\text{adult}} \end{pmatrix} \) and the future state \( \ y = \begin{pmatrix} y_{\text{young}} \\ y_{\text{adult}} \\ y_{\text{dead}} \end{pmatrix} \).

Then we can describe the population dynamic by a linear function

\[
\begin{pmatrix} y_{\text{young}} \\ y_{\text{adult}} \\ y_{\text{dead}} \end{pmatrix} = \begin{pmatrix} 1 \cdot 3 & x_{\text{adult}} \\ 0 \cdot 3 \cdot x_{\text{young}} + 0 \cdot 5 \cdot x_{\text{adult}} \\ 0 \cdot 7 \cdot x_{\text{young}} + 0 \cdot 5 \cdot x_{\text{adult}} \end{pmatrix} \begin{pmatrix} x_{\text{young}} \\ x_{\text{adult}} \end{pmatrix} \in \mathbb{R}^3
\]

Definition 3.14. For any polynomial \( p = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \), we call \( x_p := \begin{pmatrix} a_n \\ a_{n-1} \\ \vdots \\ a_1 \\ a_0 \end{pmatrix} \in \mathbb{R}^{n+1} \) its coefficient vector.

Example 3.15 (Derivatives of polynomials). The function \( D : \mathbb{R}^{n+1} \to \mathbb{R}^n \) that maps the coefficient vector of a polynomial to the coefficient vector of its derivative is a linear function.

Check: The derivative of any polynomial

\[
p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0
\]

is

\[
p'(t) = n a_n t^{n-1} + (n - 1) a_{n-1} t^{n-2} + \cdots + 2 a_2 t + a_1,
\]

which has the coefficient vector \( x_{p'} = \begin{pmatrix} n a_n \\ (n - 1) a_{n-1} \\ \vdots \\ 2 a_2 \\ a_1 \end{pmatrix} \in \mathbb{R}^n \).
LECTURE 3. LINEAR FUNCTIONS: \( F : \mathbb{R}^N \rightarrow \mathbb{R}^M \)

We be write \( D \) as

\[
D(x_p) = x_p' = \begin{pmatrix}
  n a_n \\
  (n-1) a_{n-1} \\
  \vdots \\
  2 a_2 \\
  a_1 \\
\end{pmatrix}
\]

(3.28)

\[
= \begin{pmatrix}
  n a_n + 0 a_{n-1} + \ldots + 0 a_1 + 0 a_0 \\
  0 a_n + (n-1) a_{n-1} + \ldots + 0 a_1 + 0 a_0 \\
  \vdots \\
  0 a_n + 0 a_{n-1} + \ldots + 1 a_1 + 0 a_0 \\
\end{pmatrix}
\]

(3.29)

\[
= \begin{pmatrix}
  n & 0 & \ldots & 0 & 0 \\
  0 & (n-1) & \ldots & 0 & 0 \\
  \vdots \\
  0 & 0 & \ldots & 1 & 0 \\
\end{pmatrix}
\begin{pmatrix}
  a_n \\
  a_{n-1} \\
  \vdots \\
  a_1 \\
  a_0 \\
\end{pmatrix}
\in \mathbb{R}^n
\]

(3.30)

(3.31)

Example 3.16 (Slopes of discretized functions).

Let \( x^{(1)}, x^{(n)} \) be \( n \) increasing positions on the \( x \)-axis, and let \( y^{(1)}, \ldots, y^{(n)} \) be the values of a function at this position. We can form the piece-wise linear (actually: piece-wise affine) approximation to the unknown function by "connecting the dots" (see Figure).

For \( i = 1, \ldots, n-1 \), we can estimate the slope of the piecewise linear approximation by \( s_i = \frac{dy_i}{dx_i} = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \). The function \( D \) that maps the values
LECTURE 3. LINEAR FUNCTIONS: $F: \mathbb{R}^N \to \mathbb{R}^M$

$y_1, \ldots, y_n$ to the slopes $s_1, \ldots, s_{n-1}$ is linear:

$$D(y) = \begin{pmatrix} s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} = \begin{pmatrix} \frac{y_2-y_1}{x_2-x_1} \\ \vdots \\ \frac{y_n-y_{n-1}}{x_n-x_{n-1}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{x_2-x_1} & \frac{1}{x_2-x_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{-1}{x_3-x_2} & \frac{1}{x_3-x_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{-1}{x_n-x_{n-1}} & \frac{1}{x_n-x_{n-1}} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}$$ (3.32)

Remark. The rule how to multiply a matrix $A \in \mathbb{R}^{m \times n}$ with a vector $x \in \mathbb{R}^n$ is typically written as

$$[Ax]_i = \sum_{j=1}^{n} a_{ij} x_j \quad \text{(matrix–vector multiplication)} \quad (3.34)$$

for $i = 1, \ldots, m$. ($Ax$ is an $m$-vector, and this explains how to compute its $m$ components).

How to remember?

1) Each of the $m$ rows in matrix defines linear function on input, result of that goes into corresponding output row

$$\begin{pmatrix} * & * & * & * \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} * \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} * \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} \quad \text{each row/column pair: multiply componentwise and add up}$$

2) Alternative view: Think of $A$ as collection of $n$ columns: $(c^{(1)}, c^{(2)}, \ldots, c^{(n)})$, so for each $j = 1, \ldots, n$, $c^{(j)} \in \mathbb{R}^m$ and $[c^{(j)}]_i = a_{ij}$.

$$[Ax]_i = \sum_{j=1}^{n} a_{ij} x_j = \sum_{j=1}^{n} x_j [c^{(j)}]_i \quad \text{linear} \quad \left[ \sum_{j=1}^{n} x_j c^{(j)} \right]_i$$ (3.35)

$$Ax = \sum_{j=1}^{n} x_j c^{(j)}$$ (3.36)

The output vector is a weighted sum (called "linear combination") of the matrix columns with coefficients given by the entries of $x$. 
Note: Both views are equivalent, use whichever is more convenient. I use $\text{ii})$ if $x$ has very few non-zero entries, otherwise $\text{i})$.
Also note: Both versions only make sense if the number of columns of the matrix is the same as the number of entries (rows) of the vector.

**Example 3.17.**

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} = (1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3) = (1 + 4 + 9) = \begin{pmatrix} 14 \\ 10 \end{pmatrix}
\]

(3.37)

\[
\begin{pmatrix}
1 & 2 & 3 \\
3 & 2 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} = 1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix} + \begin{pmatrix} 9 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 10 \end{pmatrix}
\]

(3.38)

**Example 3.18.**

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
0 \\
-1 \\
0
\end{pmatrix} = (0 + 2 \cdot (-1) + 0) = \begin{pmatrix} -2 \\ -5 \\ -8 \end{pmatrix}
\]

(3.39)

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\begin{pmatrix}
0 \\
-1 \\
0
\end{pmatrix} = -1 \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} -2 \\ -5 \\ -8 \end{pmatrix}
\]

(3.40)