Homework Assignment 1 (min-cut/max-flow; LP duality) 4 March, 2013

- Write the solution to each problem on a single page.

- The discussion of questions and solutions before the due date is not discouraged, but you must formulate your own solution.

- The deadline for handing in solutions is Monday 11 March before the lecture

**Problem 1.** (20 points). Let $G = (V, E)$ be an undirected graph with vertices $s, t \in V$ and non-negative node weights $c : V \to \mathbb{R}_{\geq 0}$. A set $U \subseteq V$ is called an $s$-$t$ node-cut if every path from $s$ to $t$ in $G$ passes through $U$. Give a polynomial-time algorithm for computing such set $U$ of minimum cost $c(U) = \sum_{i \in U} c_i$. Argue why your solution is correct.

**HINT:** use a reduction to an $s$-$t$ min cut problem in a graph $G'$ with twice as many nodes. You can use edges with infinite capacities.

**Problem 2.** (20 points). Consider the following LP for a directed weighted graph $G = (V, E, w)$ with nodes $s, t \in V$ and weights $w_{ij} \geq 0$ for $(i, j) \in E$.

$$ p^* := \max \pi \quad \pi_t - \pi_s $$

subject to

$$ \pi_j - \pi_i \leq w_{ij} \quad \forall (i, j) \in E $$

(a) Write the dual for this LP.

(b) Write complementary slackness conditions.

(c) Give an interpretation of the dual problem under the assumption that dual variables take values in $\{0, 1\}$.

(d) Prove that there exists an optimal dual solution with components in $\{0, 1\}$.

**HINT:** define a primal vector as follows: $\pi_i$ is the shortest distance from $s$ to $i$ in $G$. Define an integral dual vector. Prove then both vectors are feasible, and they satisfy complementary slackness conditions.

The due date for problem 2 is Wednesday 13 March, since it was released later.