Differential Equations: Partial Derivatives

Lecture Notes by Chris Wojtan

1 Goals

The topics of this lesson are the following:

• Partial derivative
• Total derivative
• Review of vector operations (Linear Algebra)
• Vector calculus operations
  – Gradient
  – Divergence
  – Laplacian
  – Jacobian
  – Hessian

2 Partial derivative

Previously, we discussed functions of a single variable, like $f(x)$. We have to be a bit more precise with derivatives when we have functions of several variables, like $f(x, y, z, t)$. Sometimes, we will only only want to take the derivative with respect to one of the variables, while holding all of the other variables constant. This is called a partial derivative, and it is written like this:

$$\frac{\partial f}{\partial x}$$

with special curly $\partial$ symbols instead of the standard $d$.

The partial derivative is useful when we only really care about studying how a single variable changes the function, without getting confused by all of the variables simultaneously changing in their own way. For example, consider a guitar string. We can draw the shape of the string with some function $f(x, t)$. The string has a particular shape which varies along the length of the string, so the function depends on position $x$. The string also wobbles and vibrates as time goes on, so it depends on time $t$ as well. If we only want to know how about the slope or curvature of the string at a particular time, then we can hold $t$ constant and only consider partial derivatives with respect to position, $\partial f/\partial x$. Alternatively, if we really want to know how quickly a point on the string is vibrating up and down, we consider the partial derivative with respect to time, $\partial f/\partial t$. 
2.1 Example

Let’s consider a hot air balloon that is rapidly inflating and heating up. The air inside the balloon is governed by the ideal gas law:

\[ V = \frac{nRT}{p} \]  

(1)

where \( V \) is the volume of the balloon, \( n \) is the number of air molecules in the balloon, \( R \) is some constant called the “gas constant”, \( T \) is the temperature of the air in the balloon, and \( p \) is the pressure of the air in the balloon.

The balloon is inflating, so the number of air molecules in the balloon increases with time. The number of air molecules at time \( t = 0 \) is \( n(t = 0) = n_0 \). Let’s say \( n \) is a simple linear function, so as time increases, the number of molecules in the balloon also goes up:

\[ n(t) = n_0 + t \]  

(2)

The balloon is also being heated by a flame, so the temperature of the air molecules begins at zero and then rapidly approaches the temperature of the fire \( T_f \) as time goes on:

\[ T(t) = T_f (1 - e^{-t}) \]  

(3)

The pressure inside the balloon is always equal to the outside air temperature, so it is constant in time:

\[ p = p_{\text{outside}} \]  

(4)

Lastly, the rate constant \( R \) is also constant in time.

If we want to know how quickly the volume of the balloon changes, we can take the derivative with respect to time. The partial derivative will tell us how quickly the volume changes with respect to a single variable. For example, if we only wish to know how the volume is affected by inflating the balloon, we can take \( \partial V/\partial n \), or if we want to isolate how the volume of the balloon changes with temperature, we can take \( \partial V/\partial T \).

If we want to know how the balloon changes when considering all possible variables, we need the total derivative.

3 Total derivative

Continuing with the above example, let’s say we want to know how the volume of the balloon changes with time \( t \). If we use equation ?? to calculate volume, then we find that there is no explicit dependence on time — the variable \( t \) does not appear anywhere in equation ???. Taking the partial derivative of this equation with respect to \( t \) will give us zero, because the partial derivative holds all other variables constant. Essentially, the partial derivative is will ignore any implicit dependence on the variable \( t \).

One way to ensure that we actually take the dependence on \( t \) into account is to plug in equations ??, ??, and ?? into equation ??, giving us

\[ V = \frac{(n_0 + t)RT_f(1 - e^{-t})}{p_{\text{outside}}} \]
Now we have removed all hidden dependences on $t$ by making $V$ explicitly depend on $t$. We can take the derivative to get

$$\frac{dV}{dt} = \frac{RT_f e^{-t}(n_0 + e^t + t - 1)}{p_{\text{outside}}}$$  \hspace{1cm} (5)

This is possible because we knew ahead of time what all of the exact relationships between $n, T, p$, and $t$. Sometimes this is not possible. We may know that something explicitly depends on $t$, but we don’t know exactly in what way. Other times, we may know the exact relationships but want a more concise and general way to relate them.

The **total derivative** gives us the ability to take derivatives of functions even when they only implicitly depend on a variable. Given a function $f(x, y, t)$, the total derivative is given by:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial t}$$  \hspace{1cm} (6)

Note how the total derivative consists of partial derivatives with respect to each variable, as well as additional total derivatives. This recursive nature of the total derivative allows us to uncover many nested layers of dependencies. The total derivative is identical to the derivative that you saw in the first lecture for a single variable. In fact, it can be thought of as a multi-dimensional version of the **chain rule**.

To go back to our example with the hot air balloon, the total derivative is given by:

$$\frac{dV}{dt} = \frac{\partial V}{\partial n} \frac{dn}{dt} + \frac{\partial V}{\partial T} \frac{dT}{dt} + \frac{\partial V}{\partial R} \frac{dR}{dt} + \frac{\partial V}{\partial p} \frac{dp}{dt}$$

So now, instead of plugging everything into the original equation and taking the derivative, we have to compute several smaller individual derivatives and then multiply and add them together. The smaller pieces are:

$$\frac{\partial V}{\partial n} = \frac{RT}{p}, \quad \frac{dn}{dt} = 1, \quad \frac{\partial V}{\partial T} = \frac{nR}{p}, \quad \frac{dT}{dt} = T_f e^{-t}$$

and we know that $dR/dt$ and $dp/dt$ are zero, so the two rightmost terms disappear. Plugging these into the equation for the total derivative, we get:

$$\frac{dV}{dt} = \frac{RT}{p} + \frac{nR}{p} T_f e^{-t}$$

$$= \frac{RT_f (1 - e^{-t})}{p_{\text{outside}}} + \frac{n_0 + t}{p_{\text{outside}}} R T_f e^{-t}$$

$$= \frac{RT_f e^{-t} (n_0 + e^t + t - 1)}{p_{\text{outside}}}$$

which is the same as what we got earlier by explicitly plugging in the equations for $n(t)$, $p(t)$, and $T(t)$. 

3
4 Linear algebra review

4.1 Vectors

To review, a **scalar** is a real number that represents the magnitude, size, or a quantity of something. We often represent scalars using variables like $a$ or $x$ or $\alpha$. A **vector** is a list of scalars. It has a magnitude and direction. We denote vectors by either putting an arrow on top of the variable or making the notation bold. For example,

$$\vec{v} = \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

I may also use a subscript in 2 or 3 dimensions to indicate a specific component of a vector:

$$v_x = a, \quad v_y = b, \quad v_z = c$$

The magnitude of a vector can be calculated using the Pythagorean theorem. This works in any dimension.

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$$

You can add and subtract vectors

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a + d \\ b + e \\ c + f \end{pmatrix}$$

and you can multiply them by a scalar

$$a \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax \\ ay \\ az \end{pmatrix}$$

The **dot product** or **inner product** takes two vectors of the same length as input and outputs a scalar:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} d \\ e \\ f \end{pmatrix} = ad + be + cf$$

The dot product operation is equivalent to the following formula:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$$

4.2 Vector fields

For our purposes, we will refer to a function that exists over many points in space as a “field”. A **scalar field** is a function that outputs a scalar value for any point in space. A **vector field** is a function that outputs a vector for any point in space. For example, $f(x, y, z)$ is a 3-dimensional scalar field, and $\vec{f}(x, y)$ is a 2-dimensional vector field.
4.3 2 × 2 matrices

We will mainly deal with 2 × 2 matrices in this class, in order to avoid the complications that come with higher dimensions. A 2 × 2 matrix \( M \) is essentially a table of scalars, and it can be written as:

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

We can multiply a matrix times a vector in the following way:

\[
Mv = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}
\]

The transpose of a matrix is a reflection of the matrix values about the diagonal.

\[
M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}
\]

The trace of a matrix is the sum of its diagonal elements.

\[
\text{Tr}(M) = a + d
\]

The determinant of a matrix is computed in the following way:

\[
\det(M) = ad - bc
\]

5 Vector calculus operations

The symbol \( \nabla \) is pronounced “nabla” and is often used when taking partial derivatives involving many variables. While it is not a precise definition, you can think of the nabla operator as a vector of partial derivatives:

\[
\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}
\]

The gradient of a scalar field produces a vector field.

\[
\nabla f(x, y, z) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}
\]

(7)

The divergence of a vector field produces a scalar field.

\[
\nabla \cdot \mathbf{f}(x, y, z) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}
\]

(8)
The **Laplacian** of a scalar field produces another scalar field.

\[
\nabla^2 f(x, y, z) = \frac{\partial^2 f_x}{\partial x^2} + \frac{\partial^2 f_y}{\partial y^2} + \frac{\partial^2 f_z}{\partial z^2}
\]  

(9)

The **Jacobian** of a vector function produces a matrix that lists all first-order partial derivatives. For the vector-valued function

\[
v = \begin{pmatrix} v_1(x, y) \\ v_2(x, y) \end{pmatrix},
\]

the Jacobian is:

\[
J(v) = \begin{pmatrix}
\frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\
\frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y}
\end{pmatrix}
\]

(10)

Similarly, the **Hessian** matrix of a scalar-valued function lists the second derivatives:

\[
H(f(x, y)) = \begin{pmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}
\end{pmatrix}
\]

(11)