Wolbachia spreads when above a threshold frequency

Caspari-Watson model allows cytoplasmic incompatibility, $\sigma_b$, and reduced fecundity, $\sigma_f$:

$$p_{t+1} = f(p_t) = \frac{p_t(1-\sigma_f)}{1-\sigma_f p_t - \sigma_b p_t (1-p_t)}$$

$$p_{t+1} - p_t = f(p_t) - p_t = \sigma_b g(p_t) = \frac{\sigma_b p_t(1-p_t)(p_t - \hat{p})}{1-\sigma_f p_t - \sigma_b p_t (1-p_t)}$$

where $\hat{p} = \frac{\sigma_f}{\sigma_b}$

$$\frac{dp_t}{dt} = \sigma_b p_t (1-p_t) (p_t - \hat{p})$$

This compares the discrete CW recursion with the simplified differential equation. $\sigma_b = 1$, $\sigma_f = 0.2$; initially, $p_0 \sim 0.22$
Continuous time model

Follow the frequency of infected adults $\rho[t]$

Mosquitoes lay eggs in water; they emerge as adults after $\tau \sim 10$ days

Numbers emerging stay constant, regardless of infection

New adults emerge at $\theta$ per day; adults die at rate $\lambda \sim 0.1$ per day

The adult population is constant, at $N = \theta / \lambda$

The frequency of infections in eggs laid at time $t$ is $f[\rho[t]]$

The number of infected adults emerging per day is $\theta f[\rho[t - \tau]]$, and the number of infected adults lost per day is $-\lambda N \rho[t]$

So, the rate of increase of the infection frequency is:

$$\frac{\rho}{dt} = \frac{1}{N} \left( \theta f[\rho[t - \tau]] - \lambda N \rho[t] \right) = \lambda (f[\rho[t - \tau]] - \rho[t])$$

This is a delay differential equation
Spatial spread

Will Wolbachia spread from some initial \( \rho[0, x] \)?
What is the optimal \( \rho[0, x] \), which will allow spread for minimum \( \int \rho[0, x] \, dx \) ?
What is the optimal \( \rho[0, x] \), if we want the infection to spread over some area by time \( T \)?
Will spread be stopped by barriers?
Edge Hill/Whitfield: proportion of infected mosquitoes in vacuum traps

These are infection frequencies at times $\{90, 110, 120, 150, 250, 300, 400, 600\}$, plotted against distance from the edge of the release area:

The wave spreads at constant speed ($0.59 \text{m day}^{-1}$) and width (400m)
How to model spatial spread?

Assume one dimension: \( \ldots, p_{-1}, p_0, p_1, \ldots \) and population size constant in time & space

Migration between neighbours:

\[
p^*_i = (1 - m) f[p_i] + \frac{m}{2} (f[p_{i-1}] + f[p_{i+1}])
\]

Migration over longer distances:

\[
p^*_i = \sum_{k=-\infty}^{\infty} m_k f[p_{i-k}]
\]

where \( \sum_{k=-\infty}^{\infty} m_k = 1 \)

In continuous space:

\[
p^*[x] = \int_{-\infty}^{\infty} m[y] f[p[x - y]] dy \approx \int_{-\infty}^{\infty} m[y] (p[x - y] + \sigma_b g(p[x - y]) dy
\]

Assuming \( \sigma = 0 \) (no bias in migration direction)

\[
\frac{\partial p}{\partial t} = \sigma_b g(p) + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2} \approx \sigma_b p_t (1 - p_t) (p_t - \hat{p}) + \frac{\sigma^2}{2} \frac{\partial^2 p}{\partial x^2}
\]
How fast will the infection spread? What shape will the spread take?

A “travelling wave” with constant speed \( c \) will form: \( p(t, x) = u(x - ct) \)

\[
\frac{\partial u}{\partial t} = -c \frac{\partial u}{\partial x} + \sigma_b u (1 - u) (u - \hat{p}) + \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2}
\]

On dimensional grounds, the speed is proportional to \( \sigma \sqrt{\sigma_b} \), and the width to \( \sigma \sqrt{\sigma_b} \)

There is an exact solution, with \( w = 4 \sigma \sqrt{\sigma_b} \) and \( c = (\frac{1}{2} - \hat{p}) \sigma \sqrt{\sigma_b} \) (width being defined as \( 1/\max(\partial u/\partial x) \))

\( c \sim 0.59 \text{m day}^{-1}, w \sim 400 \text{m} \)

If \( \hat{p} = 0.2, \sigma^2 = (14 \text{m})^2 \) per day, \( \sigma_b \sim 0.02 \) per day

Since \( \sigma_b \sim 1 \) per generation, generation time is \( \sim 50 \) days, and \( \sigma^2 \sim (100 \text{m})^2 \) per generation
References