Differential Equations: Differentiation

Lecture Notes by Chris Wojtan

1 Goals

The goals of this lesson are to understand the following:

• Definition of derivative
• Geometric intuition
• Review methods for taking derivatives

2 Definition of Derivative

We start with a one-dimensional function \( f(x) \) that varies depending on the value of \( x \). The derivative of a function describes how the function changes. The derivative is written as \( \frac{df}{dx}(x) \), and here is its formal definition:

\[
\frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]  

(1)

This equation consists of many parts:

• The numerator \( df \) indicates that we are taking the derivative of the function \( f \).
• The denominator \( dx \) indicates that we are taking the derivative with respect to the variable \( x \).
• The symbol \( h \) is a new variable that has a very small value.
• The limit symbol \( \lim_{h \to 0} \) tells us that we want to consider an \( h \) that is so small that it is as close to zero as possible.
• The numerator \( f(x + h) - f(x) \) tells us that we are taking the difference between two nearby points in \( f \).

This equation basically says that the derivative is the difference between two nearby points on the function, divided by their distance in \( x \). These two points are extremely close to each other, because \( h \) is extremely small. The derivative \( \frac{df}{dx}(x) \) tells us what happens to the value of \( f \) if we perturb the value \( x \). For example, if moving the value of \( x \) by a tiny amount does not change the value of \( f \), then the numerator of the right hand side is zero, so the derivative is zero. If increasing \( x \) a little bit also causes \( f \) to increase, then the derivative is positive. If increasing \( x \) causes \( f \) to decrease, then the derivative is negative.

Different authors use different notation for the derivative. In calculus textbooks, the notation \( f'(x) \) is often used to mean \( \frac{df}{dx}(x) \), and physicists often use \( \dot{f}(x) \) to mean the derivative with respect to time \( \frac{df}{dt}(x) \).
3 Some Examples

Linear function

For \( f(x) = Cx \), with \( C \) constant,

\[
\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{C(x + h) - Cx}{h} = \lim_{h \to 0} \frac{Cx + Ch - Cx}{h} = \lim_{h \to 0} \frac{Ch}{h} = \lim_{h \to 0} C = C
\]

Cubic function

For \( f(x) = x^3 \)

\[
\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^3 - x^3}{h} = \lim_{h \to 0} \frac{(x + h)(x + h)(x + h) - x^3}{h} = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2)(x + h) - x^3}{h} = \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2
\]

Note that all terms except the first one get killed by the limit as \( h \) approaches zero. Every \( h^n \) term in the numerator where \( n > 1 \) is ignored. Basically we pretend that the function is linear in \( h \), and that all higher order terms are unimportant, because they drop to zero too quickly. Small values get extremely small when you raise them to some power.
4 Geometry

The numerator of the right hand side is the change in the function $f$ due to perturbations in its input, and the denominator is the amount that we perturbed the input. This might remind you of the definition for the slope of a line. Given a line:

$$y = mx + b$$

where $x$ is the horizontal coordinate in the $xy$ plane, $y$ is the vertical coordinate in the $xy$ plane, $m$ is the slope, and $b$ is the vertical offset. The slope $m$ is defined as “rise over run,” or $\frac{\Delta y}{\Delta x}$. The derivative $\frac{dy}{dx}$ is essentially what you get if you find a line tangent to the curve $y(x)$ at every point $x$, and then calculate its slope.

Another way to view this geometrically is that, by ignoring higher order terms, the derivative treats everything as if it was just a linear function. You can see this graphically if you plot a function and zoom in. Try to draw some squiggly line and keep zooming in on it. As you consider smaller and smaller pieces of the curve (as $h \to 0$), the function looks more and more like a straight line. See Figure 1 for an example of this.

![Figure 1: Starting from the top curve, we repeatedly zoom into a section of the curve. As we zoom in, the curve appears to flatten out, and it eventually looks identical to a line. This happens no matter where we choose to zoom in.](image-url)
5 Higher order derivatives

The second derivative is the derivative of the first derivative: \( \frac{d^2 f}{dx^2} = \frac{d}{dx} \left( \frac{df}{dx} \right) \). It measures the rate of change of the first derivative, or how the change in the function is changing.

**Physical intuition**  
In physics, the first derivative (with respect to time) of an object’s position is its velocity:

\[
\frac{dx}{dt} = v
\]

The second derivative with respect to time is the object’s acceleration.

\[
\frac{d^2 x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{dv}{dt} = a
\]

Consider an ball flying through the air. Its position in the air is described by the time-dependent function \( y(t) \). It starts at a height \( y(0) \), then I throw it straight up. The ball shoots upward, gradually slows down, eventually stops its motion, and then gradually starts moving faster and faster downward (See Figure 2).

![Figure 2](image_url)

Figure 2: The path of a ball through time. The purple curve is the ball’s height \( y(t) \), the red curve is the first derivative, and the blue curve is the second derivative. Notice that the first derivative is zero when the ball stopped moving upward.

The first derivative \( dy/dt \) encodes the velocity of the ball. Its velocity is positive when the ball moves upward, and it is negative when the ball moves downward. The
magnitude of the velocity indicates the ball’s speed. It has a large positive first derivative when the ball is moving upward most quickly, and it has a very negative first derivative when the ball moves fastest downward. The first derivative is zero at the exact time when the ball stopped moving upward. The actual value \( \frac{dy}{dt} \) is exactly equal to the slope of the line tangent to the curve \( y \) at each time \( t \).

The second derivative of the ball describes the change in velocity. The velocity \( \frac{dy}{dt} \) moves downward at a steady, apparently linear rate. The velocity changes by the same amount every time we move to the right on the \( t \) axis; in other words, the change is constant. Because the slope of the line representing \( \frac{dy}{dt} \) is negative, the change in velocity is negative. The second derivative curve appears to be a negative constant value. We know that the second derivative of position (the first derivative of velocity) is equal to the ball’s acceleration. Thus, the acceleration of the ball is a constant negative value. This agrees with our intuition that gravity is constant pulls things downward.

**Geometric intuition** Geometrically, the first derivative of a line represents the slope of the tangent line at that point. If the curve is angled upward, then its first derivative is positive. The steeper the angle, the larger the first derivative. The second derivative of the curve roughly represents the curvature. If the slope is steadily decreasing, then the shape curves downward. If the slope rapidly turns upward, then the shape has a large positive curvature. See Figure 3.

![Figure 3](image)

Figure 3: The purple curve represents the function \( f(x) \), which varies over space. The red curve is the first derivative, and the blue curve is the second derivative. Notice that the second derivative positive when \( f(x) \) curves upwards and negative when \( f(x) \) curves downwards.
6 Derivative rules

We can learn many properties of derivatives by applying equation 1 to different scenarios.

Derivative of a constant

Consider \( f(x) = C \), where \( C \) is a constant.

\[
\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{C - C}{h} = \lim_{h \to 0} \frac{0}{h} = 0
\]

The derivative of a constant is always zero. This is intuitively true, because a constant function never changes.

Adding two functions

Now we consider two functions \( f(x) \) and \( g(x) \). We want to know what happens when we take the derivative of the sum of these functions.

\[
\frac{d}{dx} (f(x) + g(x)) = \lim_{h \to 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h} = \lim_{h \to 0} \frac{f(x + h) - f(x) + g(x + h) - g(x)}{h} = \lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} + \frac{g(x + h) - g(x)}{h} \right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

This basically says that we can conveniently treat each term independently when we take the derivative. The derivative of a sum is the sum of the derivatives.

Multiplying two functions

What if we wish to multiply two functions \( f(x) \) and \( g(x) \)? Let’s consider a geometric example. We are given a rectangle with sides of length \( f(t) \) and \( g(t) \). The side lengths \( f \) and \( g \) are functions of time, so they may get longer or shorter as time changes. The shape of the rectangle will change as time goes on, and we are particularly interested in
Figure 4: What happens to the area of rectangle W if f and g change over time?

describing how the area of this rectangle changes over time. We can compute the area by multiplying the length of the rectangle by its width:

\[ A(t) = f(t)g(t). \]  \hspace{1cm} (2)

After an extremely short amount of time, the side lengths change to \( f(t+h) \) and \( g(t+h) \). Similarly, the area changes to:

\[ A(t+h) = f(t+h)g(t+h). \]  \hspace{1cm} (3)

Now take a look at Figure 4. The area of rectangle W is given by Equation 2. Similarly, Equation 3 describes the areas of rectangles W, X, Y, and Z combined. The derivative of this area with respect to time is

\[ \frac{d}{dt}A(t) = \lim_{h \to 0} \frac{A(t+h) - A(t)}{h}. \]  \hspace{1cm} (4)

First, we define two new variables \( a \) and \( b \) as shown in Fig. 4: \( a := f(t+h) - f(t) \) and \( b := g(t+h) - g(t) \). Now we can replace \( f(t+h) \) with \( f(t) + a \) and \( g(t+h) \) with \( g(t) + b \) in Eq. (3) and plug it into Eq. (4) to get

\[ \frac{d}{dt}A(t) = \frac{d}{dt}[f(t)g(t)] = \lim_{h \to 0} \frac{[f(t) + a][g(t) + b] - f(t)g(t)}{h}. \]  \hspace{1cm} (5)
The nominator on the right-hand side in Eq. (4) simplifies to \( f(t)b + g(t)a + ab \), and we’ll compute the limits separately, also note that \( f(t) \) and \( g(t) \) are now independent of \( h \) so we can pull them out of the limit:

\[
\frac{d}{dt}[f(t)g(t)] = f(t) \lim_{h \to 0} \frac{b}{h} + g(t) \lim_{h \to 0} \frac{a}{h} + \lim_{h \to 0} \frac{ab}{h}.
\]

According to the definitions of \( a \) and \( b \), we see immediately that

\[
\lim_{h \to 0} \frac{a}{h} = \lim_{h \to 0} \frac{f(t + h) - f(t)}{h} = \frac{df}{dt}
\]

and similarly \( \lim_{h \to 0} \frac{b}{h} = \frac{dg}{dt} \). Finally, we’ll show that \( \lim_{h \to 0} \frac{ab}{h} = 0 \): by expanding the fraction with \((h/h)\) and then splitting the limit again we get

\[
\lim_{h \to 0} \frac{ab}{h} = \lim_{h \to 0} \frac{a}{h} \lim_{h \to 0} \frac{b}{h} \lim_{h \to 0} h = 0.
\]

This is easy to see since the first two limits are the derivatives of \( f \) and \( g \) but the last one \( \lim_{h \to 0} h \) is zero. Consequently, Eq. (6) turns into the “product rule”:

\[
\frac{d}{dt}[f(t)g(t)] = f(t) \frac{dg}{dt} + g(t) \frac{df}{dt}.
\]

When taking the derivative of a product of two terms, you can remember this rule as “the first times the derivative of the second plus the second times the derivative of the first.”

Additional derivative rules

The following rules are also important, but we will not derive them here.

**Derivative of a monomial**

\[
\frac{d}{dx} x^n = nx^{n-1}
\]

**Derivative of the exponential function**

\[
\frac{d}{dx} e^x = e^x
\]

Note that the derivative of \( e^x \) is itself! This function is extremely special because it is the only one which is equal to its derivative. Please familiarize yourself with this rule, because it will be an essential building block for understanding differential equations later in this course.

**Derivative of the natural log**

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]
Derivative of trigonometric functions

\[
\frac{d}{dx} \sin x = \cos x \quad \frac{d}{dx} \cos x = -\sin x
\]

What happens when we take the second derivative of these functions?

\[
\frac{d^2}{dx^2} \sin x = -\sin x \quad \frac{d}{dx} \cos x = -\cos x
\]

These functions are special, because they are equal to the negative of their second derivatives. What’s more, they are exactly equal to their fourth derivatives (try it out). You may presume that these functions may be somehow related to the exponential function because of their relationships to their own derivatives. In fact, they are very intimately related through complex numbers. We will not go into this relationship here, but we will see later that the exponential function and these trigonometric functions often show up together when solving differential equations.

Chain rule

\[
\frac{d}{dx} f(u) = \frac{df(u)}{du} \cdot \frac{du}{dx}
\]

This is an important one which we will use extremely often. It allows us to take the derivative of several nested functions. Please become familiar with this idea and its application. We will see it again very soon when we discuss the difference between partial derivatives and total derivatives.

Quotient rule

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{g \frac{df}{dx} - f \frac{dg}{dx}}{g^2}
\]

Some people find this rule useful, but I personally prefer not to memorize it. You can easily sidestep the use of this rule by rewriting \( f(x)/g(x) \) as \( f(x)(g(x))^{-1} \) and then using the product rule.

Other rules  There are a few other derivative rules that you may find useful, but we will not list them here. Please consult the table of derivative rules on the course website instead.