Binary Search Trees*

October 14, 2013

One of the purposes of sorting is to facilitate fast searching. However, while a sorted sequence stored in a linear array is good for searching, it is expensive to add and delete items. Binary search trees give you the best of both worlds: fast search and fast update.

Definitions and terminology. We begin with a recursive definition of the most common type of tree used in algorithms. A (rooted) binary tree is either empty or a node (the root) with a binary tree as left subtree and binary tree as right subtree. We store items in the nodes of the tree. It is often convenient to say the items are the nodes. A binary tree is sorted if each item is between the smaller or equal items in the left subtree and the larger or equal items in the right subtree. For example, the tree illustrated in Figure 1 is sorted assuming the usual ordering of English characters. Terms for relations between family members such as child, parent, sibling are also used for nodes in a tree. Every node has one parent, except the root which has no parent. A leaf or external node is one without children; all other nodes are internal. A node \( \nu \) is a descendant of \( \mu \) if \( \nu = \mu \) or \( \nu \) is a descendant of a child of \( \mu \). Symmetrically, \( \mu \) is an ancestor of \( \nu \) if \( \nu \) is a descendent of \( \mu \). The subtree of \( \mu \) consists of all descendents of \( \mu \). An edge is a parent-child pair.

![Diagram of a binary tree](image)

Figure 1: The parent, sibling and two children of the dark node are shaded. The internal nodes are drawn as circles while the leaves are drawn as squares.

The size of the tree is the number of nodes. A binary tree is full if every internal node has two children. Every full binary tree has one more leaf than internal node. To

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*These notes are extracted from the lecture notes on “Design and Analysis of Algorithms”, Fall 2005 by Herbert Edelsbrunner.
count its edges we can either count 2 for each internal node or 1 for every node other
than the root. Either way, the total number of edges is one less than the size of the
tree. A path is a sequence of contiguous edges without repetitions. Usually we only
consider paths that descend or paths that ascend. The length of a path is the number of
edges. For every node \( \mu \) there is a unique path from the root to \( \mu \). The length of that
path is the depth of \( \mu \). The height of the tree is the maximum depth of any node. The
path length is the sum of depths over all nodes, and the external path length is the same
sum restricted to the leaves in the tree.

**Searching.** A binary search tree is a sorted binary tree. We assume each node is a
record storing an item and pointers to two children:

```cpp
struct Node {
    item info;
    Node *l, *r;
};
typedef Node *Tree.
```

Sometimes it is convenient to also store a pointer to the parent, but for now we will do
without. We can search in a binary search tree by tracing a path starting at the root.

```cpp
Node *SEARCH(Tree \( \varrho \), item x)
    case \( \varrho = \) NULL: return NULL;
    x < \( \varrho \rightarrow info \): return SEARCH(\( \varrho \rightarrow \ell \), x);
    x = \( \varrho \rightarrow info \): return \( \varrho \);
    x > \( \varrho \rightarrow info \): return SEARCH(\( \varrho \rightarrow r \), x)
endcase.
```

The running time depends on the length of the path, which is at most the height of the
tree. Let \( n \) be the size. In the worst case the tree is a linked list and searching takes
time \( O(n) \). In the best case the tree is perfectly balanced and searching takes only time
\( O(\log n) \).

**Insert.** To add a new item is similarly straightforward: follow a path from the root
to a leaf and replace that leaf by a new node storing the item. Figure 2 shows the tree
obtained after adding \( w \) to the tree in Figure 1. Again the running time depends on

![Figure 2](image)

Figure 2: The shaded nodes indicate the path from the root we traverse when we insert \( w \) into
the sorted tree.

the length of the path. If the insertions come in a random order then the tree is usually
close to being perfectly balanced. Indeed, the tree is the same as the one that arises in the analysis of quicksort. The expected number of comparisons for a (successful) search is one \( n \)-th of the expected running time of quicksort, which is roughly \( 2 \ln n \).

**Delete.** The main idea for deleting an item is the same as for inserting: follow the path from the root to the node \( \nu \) that stores the item.

**Case 1.** \( \nu \) has no children: Remove \( \nu \).

**Case 2.** \( \nu \) has one child \( \mu \): Remove \( \nu \) and make \( \mu \) the child of \( \nu \)'s parent instead.

**Case 3.** \( \nu \) has two children, this case is illustrated in Figure 3: Delete the node \( \mu \) in the left subtree which stores the largest value \( J \) (\( \mu \) has no right child, and thus can be deleted using case 1 or 2). Replace the value in \( \nu \) with \( J \).

![Figure 3: Store \( J \) in \( \nu \) and delete the node \( \mu \) that used to store \( J \).](image)

The analysis of the expected search time in a binary search tree constructed by a random sequence of insertions and deletions is considerably more challenging than if no deletions are present. Even the definition of a random sequence is ambiguous in this case.

**Optimal binary search trees.** Instead of hoping the incremental construction yields a shallow tree, we can construct the tree that minimizes the search time. We consider the common problem in which items have different probabilities to be the target of a search. For example, some words in the English dictionary are more commonly searched than others and are therefore assigned a higher probability. Let \( a_1 < a_2 < \ldots < a_n \) be the items and \( p_i \) the corresponding probabilities. To simplify the discussion, we only consider successful searches and thus assume \( \sum_{i=1}^{n} p_i = 1 \). The expected number of comparisons for a successful search in a binary search tree \( T \) storing the \( n \) items is

\[
1 + C(T) = \sum_{i=1}^{n} p_i \cdot (\delta_i + 1)
\]

\[
= 1 + \sum_{i=1}^{n} p_i \cdot \delta_i,
\]

where \( \delta_i \) is the depth of the node that stores \( a_i \). \( C(T) \) is the *weighted path length* or the *cost* of \( T \). We study the problem of constructing a tree that minimizes the cost. To
develop an example, let \( n = 3 \) and \( p_1 = \frac{1}{2}, p_2 = \frac{1}{3}, p_3 = \frac{1}{6} \). Figure 4 shows the five binary trees with three nodes and states their costs. It can be shown that the number of different binary trees with \( n \) nodes is \( \frac{1}{n+1} \binom{2n}{n} \), which is exponential in \( n \). This is far too large to try all possibilities, so we need to look for a more efficient way to construct an optimum tree.

**Dynamic programming.** We write \( T_i^j \) for the optimum weighted binary search tree of \( a_i, a_{i+1}, \ldots, a_j \), \( C_i^j \) for its cost, and \( p_i^j = \sum_{k=i}^{j} p_k \) for the total probability of the items in \( T_i^j \). Suppose we know that the optimum tree stores item \( a_k \) in its root. Then the left subtree is \( T_i^{k-1} \) and the right subtree is \( T_{k+1}^j \). The cost of the optimum tree is therefore \( C_i^j = C_i^{k-1} + C_{k+1}^j + p_i^j - p_k \). Since we do not know which item is in the root we try all possibilities and find the minimum:

\[
C_i^j = \min_{i \leq k \leq j} \{C_i^{k-1} + C_{k+1}^j + p_i^j - p_k\}.
\]

This formula can be translated directly into a dynamic programming algorithm. We use three two-dimensional arrays, one for the sums of probabilities, \( p_i^j \), one for the costs of optimum trees, \( C_i^j \), and one for the indices of the items stored in their roots, \( R_i^j \).

We assume that the first array has already been computed. We initialize the other two arrays along the main diagonal and add one dummy diagonal for the cost.

\[
\text{for } k = 1 \text{ to } n \text{ do}
C[k, k - 1] = C[k, k] = 0; \quad R[k, k] = k
\text{endfor;}
C[n + 1, n] = 0.
\]

We fill the rest of the two arrays one diagonal at a time.
for $\ell = 2$ to $n$ do
for $i = 1$ to $n - \ell + 1$ do
  $j = i + \ell - 1$; $C[i, j] = \infty$;
  for $k = i$ to $j$ do
    cost = $C[i, k - 1] + C[k + 1, j]$
    + $p[i, j] - p[k, k]$;
    if cost < $C[i, j]$ then
      $C[i, j] = \text{cost}$; $R[i, j] = k$
    endif
  endfor
endfor
endfor.

The main part of the algorithm consists of three nested loops each iterating through at most $n$ values. The running time is therefore in $O(n^3)$.

Example. Table 1 shows the partial sums of probabilities for the data in the earlier example. Table 2 shows the costs and the indices of the roots of the optimum trees computed for all contiguous subsequences. The optimum tree can be constructed from

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<td></td>
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<tr>
<td>3</td>
<td></td>
<td>1</td>
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</table>

Table 1: Six times the partial sums of probabilities used by the dynamic programming algorithm.

$R$ as follows. The root stores the item with index $R[1, 3] = 1$. The left subtree is therefore empty and the right subtree stores $a_2, a_3$. The root of the optimum right subtree stores the item with index $R[2, 3] = 2$. Again the left subtree is empty and the right subtree consists of a single node storing $a_3$.

<table>
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<table>
<thead>
<tr>
<th>$R$</th>
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<tr>
<td>3</td>
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</tr>
</tbody>
</table>

Table 2: Six times the costs and the roots of the optimum trees.

Improved running time. Notice that the array $R$ in Table 2 is monotonic, both along rows and along columns. Indeed it is possible to prove $R_{i-1}^j \leq R_i^j$ in every row and $R_i^l \leq R_i^{l+1}$ in every column. We omit the proof and show how the two inequalities can be used to improve the dynamic programming algorithm. Instead of trying all roots from $i$ through $j$ we restrict the innermost for-loop to

for $k = R[i, j - 1]$ to $R[i + 1, j]$ do
The monotonicity property implies that this change does not alter the result of the algorithm. The running time of a single iteration of the outer for-loop is now

\[ U_\ell(n) = \sum_{i=1}^{n-\ell+1} (R^i_{i+1} - R^i_{i-1} + 1). \]

Recall that \( j = i + \ell - 1 \) and note that most terms cancel, giving

\[ U_\ell(n) = R^n_{n-\ell+2} - R^\ell_1 + (n - \ell + 1) \leq 2n. \]

In words, each iteration of the outer for-loop takes only time \( O(n) \), which implies that the entire algorithm takes only time \( O(n^2) \).