Differential Equations: Homework 4

Due Wednesday, 25 June 2014

Name of student:

Complete the following exercises. Be sure to show all of your intermediate work. You will not get full credit if you only submit solutions (unless the solution requires no intermediate steps). If you require more space than the space provided on these pages, feel free to use additional sheets of paper. You are allowed to use a computer or calculator for arithmetic calculations (like $237 + 12$ or $56^2$) but all symbolic manipulation must be done by hand.

1 2D Dynamical Systems

For the following dynamical systems, do the following:

- Find the location of any fixed points
- Classify each fixed point (source, sink, spiral, etc) and discuss its stability
- Draw the fixed points
- Draw arrows on the graph representing the direction field
- Trace some trajectories to give an idea of the behavior of this system.

See the following page for an example.
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x
\end{align*}
\]

The plot below marks the fixed point with a bold black dot, draws the direction field with small gray arrows, and traces a few trajectories with long black lines:

This system is linear, so it has only one fixed point. Solving the system for \( y = 0, -x = 0 \) yields \( x = 0, y = 0 \).

\textbf{The system has only one fixed point, which is located at \((x,y) = (0,0)\).}

The Jacobian matrix is \( J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \), which has trace \( T = 0 + 0 = 0 \) and determinant \( 0 \cdot 0 + 1 \cdot 1 = 1 \), which means \( \textbf{the point} \ (0,0) \ \textbf{is a center} \).

Perform a similar analysis on the following problems, and be sure to show your work.
\[
\begin{align*}
\dot{x} &= x + y \\
\dot{y} &= y - x
\end{align*}
\]
\dot{x} = -x + 2y
\dot{y} = 3x - 2y
\[
\begin{align*}
\dot{x} &= -y \\
\dot{y} &= -y - x^2 + 1
\end{align*}
\]
Consider the equation for a pendulum:

\[ ml\frac{d^2\theta}{dt^2} = -bl\frac{d\theta}{dt} - mg\sin \theta \]

which describes the motion of a swinging ball on the end of a stick of length \( l \). \( m \) is the mass of the ball, \( b \) is a damping parameter, and \( g \) is gravity. \( \theta \) represents the angle that the pendulum makes with a vertical line, so \( \theta = 0 \) means that the pendulum is hanging straight down, \( \theta = 90^\circ \) means the stick is pointing horizontally to the right, and \( \theta = 180^\circ \) means that the pendulum is pointing straight upward. For simplicity, let’s assume that \( m = l = g = 1 \), so we get:

\[ \ddot{\theta} = -b\dot{\theta} - \sin \theta \]

- What are the fixed points of this system?

- What are the classifications of the fixed points if the damping term \( b \) is equal to zero?

- What are the classifications of the fixed points if the damping term \( b \) is slightly greater than zero?
• Plot the system when \( b = 0 \) and when \( b > 0 \) below.

• Compare this system to the one that would arise if we used the “small angle approximation” \( \sin \theta \approx \theta \). How do the fixed points differ between these systems? Which behaviors stay the same?

• Plot the “small angle approximation” system below.
2 Bifurcations

Near a bifurcation point, the first few terms of a function’s Taylor series expansion look identical to the forms we saw in class:

$$\frac{dx}{dt} \approx (r + c_0) - c_1 x^2, \quad \frac{dx}{dt} \approx (r + c_0) + c_1 x^2$$  \quad \text{Saddle node bifurcations}

$$\frac{dx}{dt} \approx (r + c_0)x - c_1 x^2, \quad \frac{dx}{dt} \approx (r + c_0)x + c_1 x^2$$  \quad \text{Transcritical bifurcations}

$$\frac{dx}{dt} \approx (r + c_0)x - c_1 x^3, \quad \frac{dx}{dt} \approx (r + c_0)x + c_1 x^3$$  \quad \text{Pitchfork bifurcations}

where $r$ is a variable parameter, $c_0$ is a constant offset, and $c_1$ is just a scaling factor. The important thing to notice is that the saddle node bifurcation is a quadratic function with a variable constant term, the transcritical bifurcation is a quadratic function with a variable linear term, and the pitchfork bifurcation is a cubic function with a variable linear term (and no quadratic term). A bifurcation occurs when the first few terms of a function’s Taylor series behave like the equations above.

Each of the following functions undergoes a bifurcation at $x = 0$ for some value of the parameter $r = r^*$. For each scenario, perform three actions:

1. Find the value of $r^*$ that causes the bifurcation.
2. Classify the bifurcation type as saddle-node bifurcation, transcritical bifurcation, or pitchfork bifurcation.
3. Make two graphs of the function near $x = 0$. On the left, graph the function for $r < r^*$, and on the right, graph the function for $r > r^*$. 


\frac{dx}{dt} = rx + \sin(x)

(1) \ r^* =

(2) Classify the bifurcation:

(3) Graph the function for \( r < r^* \) (left) and \( r > r^* \) (right)
\[
\frac{dx}{dt} = \cos(x) + r \sin(x) - 1 - 2x
\]

(1) \(r^* = \)

(2) Classify the bifurcation:

(3) Graph the function for \(r < r^*\) (left) and \(r > r^*\) (right)
3 Limit Behavior

- The limit behavior of a fixed point in 2D can be classified by examining the eigenvalues of the Jacobian matrix. Each eigenvalue has two components (the real and imaginary parts), and there are two eigenvalues. Therefore, the behavior is completely described by four parameters. In class, we showed a method for classifying fixed points using only two parameters: the trace and determinant of the Jacobian matrix. Essentially, we are trying to describe a four-dimensional parameter space using only two parameters. Unfortunately, we will lose some descriptive power when we reduce the parameters like this.

Below, give an example of two different systems of 2D ODEs that have exactly the same trace and determinant, but still have qualitatively different behaviors. Describe in words how they are different, and plot them below.
• Give an example of a 2D ODE system that has a limit cycle, describe how the cycle behaves in words (including whether it should turn clockwise or counter-clockwise), and plot the system below.
4 Numerics

The harmonic oscillator (with unit mass) is described by the second-order ODE
\[ \frac{d^2 x}{dt^2} = -kx, \]
where \( k \) is a positive stiffness parameter. Using the notation \( v = \dot{x} = dx/dt \) and \( \ddot{v} = d^2x/dt^2 \), we can write this equation as a system of two first-order ODEs:
\[ \dot{x} = v \]
\[ \dot{v} = -kx \]

The system is of the form \( dy/dt = f(t,y) \), if we set \( y = (x,v)^T \), i.e. \( y \) is a vector containing both position and velocity. Now we can use numerical solvers in a similar fashion as in the previous homework to simulate this system. Recall that forward and backward Euler integration schemes are defined as
\[ y(t+h) = y(t) + hf(t,y(t)) \quad \text{and} \quad y(t+h) = y(t) + hf(t+h,y(t+h)) \]
respectively. Write down the update rules for these methods (and solve the backward Euler scheme for \( y(t+h) \))!

Now use your computer to simulate the harmonic oscillator using both schemes with a timestep of \( h = 0.3 \), stiffness \( k = 0.5 \), starting at \( x(t = 0) = 1 \) and \( v(t = 0) = 0 \). Run the simulation for \( 0 < t \leq 15 \).
Plot your results in phase-space (i.e. position on the x-axis and velocity on the y-axis) and interpret them! Are they behaving as expected?

Also try the following integration scheme, known as semi-implicit Euler, with the same time-step, stiffness, and initial conditions:
\[ \dot{v}(t+h) = v(t) - hkx(t) \]
\[ \dot{x}(t+h) = x(t) + hv(t+h) \]
Plot the results as before and compare them to the previous ones!
Finally, plot the mechanical energy of the system \( U = 0.5(kx^2 + mv^2) \) (with \( m = 1 \)) over time for all three solutions, and comment on your results!