Differential Equations: Homework 5

Due Wednesday, 2 July 2014

Name of student:

Complete the following exercises. Be sure to show all of your intermediate work. You will not get full credit if you only submit solutions (unless the solution requires no intermediate steps). If you require more space than the space provided on these pages, feel free to use additional sheets of paper. You are allowed to use a computer or calculator for arithmetic calculations (like $237+12$ or $56^2$) but all symbolic manipulation must be done by hand.

1 Classification of PDEs

1.1 Determine whether the following 2d second order linear partial differential equations are elliptic, parabolic or hyperbolic. Assume that $u(x,y)$ is sufficiently differentiable such that $\partial_x \partial_y u = \partial_y \partial_x u$, to bring the equations into a suitable form.

- $\partial_x \partial_y u + \partial_y^2 u = 2\partial_x^2 u$

- $\frac{5}{2} \partial_x^2 u + \frac{1}{2} \partial_x \partial_y u + \frac{1}{2} \partial_y \partial_x u + \partial_y^2 u = 0$

- $\nabla \cdot \left( \begin{pmatrix} 9 & 6 \\ 6 & 4 \end{pmatrix} \nabla u \right) = 0$
For which values of the parameter $r$ are the following PDEs elliptic, parabolic, and hyperbolic equations?

- $r\partial_x\partial_y u + 2\partial_x^2 u + \partial_y^2 u = 0$

- $r\partial_x\partial_y u + \partial_x^2 u - \partial_y^2 u = 0$

- $r\partial_y^2 u + 3\partial_x^2 u - \partial_x\partial_y u - \partial_y\partial_x u = 0$
2 Standing wave equation

Consider the domain \((x, y) \in [-1, 1]^2 = \Omega\) with homogeneous Dirichlet boundary conditions \(f = 0\) on \(\partial \Omega\). For which values of \(\alpha\) is the following function an Eigenfunction of the spatial Laplace operator (i.e. \(-\nabla^2 f = \lambda f\), with \(\nabla^2 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\))?

\[
f(x, y) = \frac{1}{2} \cos(\alpha x) \cos(\alpha y)
\]

Choose \(f_1(x, y)\) to be the function with the smallest possible positive value for \(\alpha\) such that the boundary conditions hold. What is the eigenvalue \(\lambda_1\) corresponding to \(f_1\)?

Now consider the undamped wave equation on the same domain over time

\[
u(x, y, t) : \Omega \times [0, \infty) \mapsto \mathbb{R}
\]

\[
\frac{\partial^2 u}{\partial t^2} = \nabla^2 u
\]

with the initial conditions \(u(x, y, t = 0) = 3f_1(x, y), \frac{\partial u}{\partial t}(t = 0) = 0\) and boundary conditions as before \(u = 0\) on \(\partial \Omega \times t\). Use \(u(x, y, t) = f_1(x, y)g(t)\) and then solve for \(g\)!

Describe in (a few) words how the solution \(u\) would change (both in space and time) if we chose higher (allowed) values for \(\alpha\) and modified the initial condition accordingly!
3 Numerics

Assume that (ignoring air resistance) the acceleration of a ball in flight is \( \frac{d^2x}{dt^2} = 0 \), \( \frac{d^2y}{dt^2} = -g \approx -9.81 \text{m/s}^2 \), so its horizontal velocity will not change, but it will drop under gravity.

At \( t = 0 \) you throw the ball with a horizontal velocity of \( \frac{dx}{dt}(t = 0) = 10 \text{ m/s} \) from a height of \( y(t = 0) = 2 \text{ m} \). At some time \( t_{\text{end}} \) you want to hit a target that is 25m away at a height of 5m, so \( x(t = t_{\text{end}}) = 25 \text{ m}, y(t = t_{\text{end}}) = 5 \text{ m} \).

Since \( x(t) \) and \( y(t) \) can be treated separately in this example, start by calculating \( x(t) \) and \( t_{\text{end}} \) (this is easily done by hand), then you have a 1d boundary value problem to solve for \( y(t) \): \( \frac{d^2y}{dt^2} = -g \) with boundary conditions \( y(t = 0) = 2 \text{ m}, y(t = t_{\text{end}}) = 5 \text{ m} \).

This example also has an analytic solution - calculate it and compare the numerical results to it later on.

Use a computer and a finite difference discretization to solve this problem as follows:

- Chop up the time interval \([0, t_{\text{end}}]\) into \( N \) equal timesteps \( h = \frac{t_{\text{end}}}{N} \), so \( t_i = h \cdot i \).
- Write down the finite difference discretization of the second derivative in terms of \( y(t_i), y(t_{i+1}) \) and \( y(t_{i-1}) \).
- The discretized solution \( y_i = y(t_i) \) can be written as a vector of unknowns \( \mathbf{y} = (y_i) \), and the FD second derivative as a matrix-vector multiplication \( \mathbf{L} \mathbf{y} \), since we wrote the derivative using only three values per timestep, the matrix \( \mathbf{L} \) will be sparse (i.e. many entries will be zero). Write a script which constructs this matrix.
- Applying the boundary conditions \( y_0 = 2 \) and \( y_N = 5 \) we can solve the linear system \( \mathbf{L} \mathbf{y}_{\text{unknown}} = \mathbf{b} \) with \( \mathbf{b} = -g - \mathbf{L} \mathbf{y}_{\text{known}} \) to find the height of the ball in each timestep. Here \( \mathbf{L} \) and \( \mathbf{b} \) indicate that the system is reduced by removing the rows and columns corresponding to the known values from the matrix as well as the corresponding entries from the right hand side vector. (In MatLab/Octave you can use the command ‘\( \mathbf{y} = \mathbf{L} \backslash \mathbf{b} \)’ to solve a linear system. You can also use subscripting like ‘\( \mathbf{y}(3 : \text{end} - 2) \)’ to access certain entries of vectors and matrices.)
- Use \( N = 10 \) at first (while testing your implementation) and once you are confident that it works, try \( N = 200 \). Plot your results and compare to the exact solution.
- Once you have a solution for all \( y_i \), what is (approximately) the initial vertical velocity, and consequently the initial angle at which you need to throw the ball in order to hit the target?