3 Induction

In philosophy, deduction is the process of taking a general statement and applying it to a specific instance. For example: all students must do homework, and I am a student; therefore, I must do homework. In contrast, induction is the process of creating a general statement from observations. For example: all cars I have owned needed repair; therefore, all cars need repair. A similar concept is used in mathematics to prove that a statement is true for all integers. To distinguish it from the less precise philosophical notion, we call it mathematical induction of which we will introduce two forms.

Sum of integers. We begin with an example, namely the familiar problem of taking the sum the first $n + 1$ non-negative integers.

**Claim.** For all $n \geq 0$, we have $\sum_{i=0}^{n} i = \binom{n+1}{2}$.

**Proof.** First, we note that $\sum_{i=0}^{0} i = 0 = \binom{1}{2}$. Now, we assume inductively that for some $n > 0$, we have

$$\sum_{i=0}^{n-1} i = \binom{n}{2}.$$

If we add $n$ on both sides, we obtain

$$\sum_{i=0}^{n} i = \binom{n}{2} + n,$$

which is $\frac{1}{2}[(n-1)n + 2n] = \frac{1}{2}(n+1)n = \binom{n+1}{2}$. Thus, by the Principle of Mathematical Induction,

$$\sum_{i=0}^{n} i = \binom{n+1}{2}$$

for all non-negative integers $n$.

To analyze why this proof is indeed a proof, we let $p(k)$ be the statement that the claim is true for $n = k$. For $n = 0$, we have $p(0) \land [p(0) \Rightarrow p(1)]$. Hence, we get $p(1)$. We can see that this continues:

$$p(1) \land [p(1) \Rightarrow p(2)] \quad \text{hence} \quad p(2);$$

$$p(2) \land [p(2) \Rightarrow p(3)] \quad \text{hence} \quad p(3);$$

$$\ldots \quad \ldots \quad \ldots$$

$$p(n-1) \land [p(n-1) \Rightarrow p(n)] \quad \text{hence} \quad p(n);$$

$$\ldots \quad \ldots \quad \ldots$$

We can think of induction as a game of dominoes. In this case, we let $p(n)$ be the statement that the $n$-th domino falls over. If we push over the first domino, then $p(1)$ is true. The first domino hits the second domino, and so $p(2)$ is true. The second domino hits the third, and so $p(3)$ is true. In general, if the $n$-th domino falls over, then it knocks over the $(n+1)$-st domino. Hence, $p(n) \Rightarrow p(n+1)$. Thus, all dominoes will fall over.

The weak form. We formalize the proof technique into the first, weak form of the principle. The majority of applications of Mathematical Induction use this form.

**Math. Induction (Weak Form).** If $p(n_0)$ is true, and $p(n-1) \Rightarrow p(n)$ is true, for all $n > n_0$, then $p(n)$ is true for all $n \geq n_0$.

To write a proof using the weak form of Mathematical Induction, we thus take the following four steps:

**Base Case:** $p(n_0)$ is true.

**Inductive Hypothesis:** $p(n-1)$ is true.

**Inductive Step:** $p(n-1) \Rightarrow p(n)$.

**Inductive Conclusion:** $p(n)$ for all $n \geq n_0$.

Very often, but not always, the inductive step is the most difficult part of the proof. In practice, we usually sketch the inductive proof, only spelling out the portions that are not obvious. If we can guess the correct closed form expression for a finite sum, it is often easy to use induction to prove that it is correct.

**Claim.** For all $n \geq 1$, we have $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$.

**Proof.** We prove the claim using the weak form of the Principle of Mathematical Induction. We observe that the equation holds for $n = 1$, because $\sum_{i=1}^{1} 2^{i-1} = 1 = 2^1 - 1$. Assume inductively that the claim holds for $n - 1$. We get to $n$ by adding $2^{n-1}$ on both sides:

$$\sum_{i=1}^{n} 2^{i-1} = \sum_{i=1}^{n-1} 2^{i-1} + 2^{n-1} = (2^{n-1} - 1) + 2^{n-1} = 2^n - 1.$$

Here, we use the inductive assumption to go from the first to the second line. Thus, by the Principle of Mathematical Induction, $\sum_{i=1}^{n} 2^{i-1} = 2^n - 1$ for all $n \geq 1$. 


Fibonacci numbers. We use the desire to give a closed-form description of Fibonacci numbers as the motivation for another use of mathematical induction. They are defined by the same principle, but that is beside the point. Set \( F(1) = F(2) = 1 \), and define \( F(j) = F(j-1)+F(j-2) \), for all integers \( j \geq 3 \). The first few Fibonacci numbers are therefore

\[ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots \]

After a slow start, the sequence picks up considerable speed. We claim that we can write the \( n \)-th Fibonacci number as the sum of two simple powers. To write down the formula, we define \( \varphi = \frac{1}{2} (1 + \sqrt{5}) \) and \( \psi = \frac{1}{2} (1 - \sqrt{5}) \).

**Claim.** \( F(n) = \frac{\varphi^n - \psi^n}{\sqrt{5}}, \) for all \( n \geq 1 \).

**Proof.** We first test the claim for small values of \( n \). For \( n = 1 \), we have

\[ F(1) = \frac{1}{\sqrt{5}} (\varphi - \psi), \]

which evaluates to 1, as it should. To prepare the next step, we note that \( \varphi^2 = \frac{1}{4} (6 + 2\sqrt{5}) = \varphi + 1 \), and similarly \( \psi^2 = \psi + 1 \). For \( n = 2 \), we therefore have

\[ F(2) = \frac{1}{\sqrt{5}} (\varphi^2 - \psi^2) = \frac{1}{\sqrt{5}} (\varphi + 1 - \psi - 1), \]

which is again 1. Since the claimed formula is correct twice, perhaps it is correct always. Let us assume the claimed formula for \( n - 1 \) and for \( n - 2 \). Then

\[ F(n) = F(n-1) + F(n-2) = \frac{1}{\sqrt{5}} (\varphi^{n-2}(\varphi + 1) - \psi^{n-2}(\psi + 1)) = \frac{1}{\sqrt{5}} (\varphi^n - \psi^n), \]

as claimed.

We note that this proof goes beyond the weak form of the principle, because we need the hypothesis for \( n - 1 \) as well as for \( n - 2 \).

**The strong form.** As we have seen in a mild form for the Fibonacci numbers, it is sometimes not enough to use the validity of \( p(n-1) \) to derive \( p(n) \). Indeed, we have \( p(n-2) \) available and \( p(n-3) \) and so on. Why not use them?

**Math. Induction (Strong Form).** If \( p(n_0) \) is true and \( p(n_0) \land p(n_0 + 1) \land \cdots \land p(n-1) \Rightarrow p(n) \) is true, for all \( n > n_0 \), then \( p(n) \) is true for all \( n \geq n_0 \).

We use the strong form to prove that every integer has a decomposition into prime factors.

**Claim.** Every integer \( n \geq 2 \) is the product of prime numbers.

**Proof.** We know that 2 is a prime number and thus also a product of prime numbers. Suppose now that we know that every positive number less than \( n \) is a product of prime numbers. Then, if \( n \) is a prime number we are done. Otherwise, \( n \) is not a prime number. By definition of non-prime number, we can write it as the product of two smaller positive integers, \( n = a \cdot b \). By our supposition, both \( a \) and \( b \) are products of prime numbers. The product, \( a \cdot b \), is obtained by merging the two products, which is again a product of prime numbers. Therefore, by the strong form of the Principle of Mathematical Induction, every integer \( n \geq 2 \) is a product of prime numbers.

**Multisets revisited.** We use induction to give an alternative proof of the number of size-\( k \) multi-sets we can form using \( n \) distinct elements. Write \( f(k, n) \) for this number. We will need the strong form of the paradigm. There are two parameters, \( k \) for the size and \( n \) for the number of distinct elements, which requires a 2-dimensional table to record all numbers. Accordingly, the Base Case is more complicated. Instead of following the paradigm step by step, we start with the Inductive Step and then decide which Base Case we need to get the process going.

<table>
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<tr>
<th>( k )</th>
<th>( n )</th>
<th>( f(k, n) )</th>
</tr>
</thead>
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<td>1</td>
</tr>
<tr>
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<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 3 6</td>
<td>1 4 10 20</td>
</tr>
<tr>
<td>3</td>
<td>1 4 10 35</td>
<td>1 5 15 35 70</td>
</tr>
</tbody>
</table>

Table 1: The number of size-\( k \) multi-sets we can form with \( n \) different elements. According to (9), the entry in row \( k \) and column \( n \) is the sum of the numbers from row 0 to row \( k \) in column \( n - 1 \). For the last entry in the table, these numbers are shown in boldface.

Our first attempt to do the Inductive Step is to argue that

\[ f(k, n) = f(k, n-1) + \ldots + f(0, n-1); \quad (9) \]

see Table 1. Indeed, we can generate all size-\( k \) multi-sets by selecting the \( n \)-th element \( i \) times, for \( 0 \leq i \leq k \).
Equivalently, we choose all size-\((k-i)\) multi-sets of \(n-1\) elements and add \(i\) copies of the \(n\)-th element. In the table, we get the entry in row \(k\) and column \(n\) as the sum of the entries in rows \(k\) down to 0 of the previous column. To get the process started, we need the Base Case to fill the first column, which we do by observing that \(f(k, 1) = 1\) for all \(k \geq 0\).

Note that \(f(0, n-1)\) to \(f(k-1, n-1)\) add up to give \(f(k-1, n)\). Hence, we can simplify the Inductive Step to

\[
f(k, n) = f(k, n-1) + f(k-1, n). \quad (10)
\]

Indeed, we can generate all size-\(k\) multi-sets made of \(n\) distinct elements by taking all size-\(k\) multi-sets made of \(n-1\) distinct elements plus adding one copy of the \(n\)-th element to all size-\((k-1)\) multi-sets made of \(n\) distinct elements; see Table 2. To fill the table, we need the Base Case to provide the entries in the first column and the first row. We have seen above that \(f(k, 1) = 1\) for all \(k\). Furthermore, \(f(0, n) = 1\) for all \(n\) because there is only one empty set, independent on the number of distinct elements we have at our disposal.

<table>
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<th></th>
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<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
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<td>70</td>
</tr>
</tbody>
</table>

Table 2: Alternatively, we can get the last entry in the table by adding the entry to its left and the entry above.