Problem 1: Consider the game of tic-tac-toe with player cross (X) and circle (O). Show a board configuration where it is X-player’s turn to play such that the following condition holds: X-player cannot win in the next move, but there is a move of player X, such that no matter the next move of player O, in the following turn there is a move of player X to win (i.e., player X does not win immediately but can play so that after one more move player O player X can win). Show the game configuration tree, and the moves of player X to illustrate the above process.

Problem 2: Denote with $K_n$ the graph of $n$ nodes and all possible edges. The following is $K_5$.

Given $K_n$, you and I play a game of coloring as follows. I pick red, and you pick blue, and we take turns in coloring the edges of $K_n$ (assume that I go first). The game ends when a triangle is formed with only one color on its edges (called a monochrome triangle, as opposed to a bichrome triangle), or no further moves are possible (i.e., all edges are colored). In the latter case the game ends in a draw, otherwise the winner is the player whose color formed the monochrome triangle.

1. Describe the game on $K_6$. Define the state space, identify my and your vertices in the state space, and describe the edge relation and our target sets.

2. Show that the game on $K_6$ cannot end in a draw. Hint: Consider 5 edges meeting in a node of $K_6$. At least 3 must have the same color. Can $K_7$ end in a draw?

3. (Bonus). A winner gets two points if he manages to form two monochrome triangles in one move. Show that on $K_6$, if the game has not end after 14 turns, I win two points. Hint: Consider a complete coloring of $K_6$. There are $\binom{6}{3}$ triangles, and $\binom{5}{2}$ pairs of edges meeting in each node. Call such a pair bichrome, if the edges have different color. Draw a relation between the number of biochrome pairs and the number of bichrome triangles. Establish an upper bound on the number of such biochrome pairs.