Problem 1: Exponentiation

Algorithm 1

Input: Two integers $M > 0$ and $N \geq 0$
Output: An integer $k$ such that $k = M^N$

\[
\begin{align*}
\hspace{1cm} a &\leftarrow M \\
\hspace{1cm} b &\leftarrow N \\
\hspace{1cm} k &\leftarrow 1 \\
\text{while } b > 0 &\text{ do} \\
\hspace{2cm} k &\leftarrow k \times a \\
\hspace{2cm} b &\leftarrow b - 1 \\
\text{end while}
\end{align*}
\]

Find an invariant that is strong enough to prove the correctness of the program.
Use Hoare logic to prove $\{\text{true}\} P \{k = M^N\}$

Problem 2: Fast Exponentiation

Algorithm 2

Input: Two integers $M > 0$ and $N \geq 0$
Output: An integer $k$ such that $k = M^N$

\[
\begin{align*}
\hspace{1cm} a &\leftarrow M \\
\hspace{1cm} b &\leftarrow N \\
\hspace{1cm} k &\leftarrow 1 \\
\text{while } b > 0 &\text{ do} \\
\hspace{2cm} \text{if even}(b) \text{ then } \\
\hspace{3cm} a &\leftarrow a \times a \\
\hspace{3cm} b &\leftarrow b/2 \\
\hspace{2cm} \text{else} \\
\hspace{3cm} k &\leftarrow k \times a \\
\hspace{3cm} b &\leftarrow b - 1 \\
\hspace{2cm} \text{end if} \\
\text{end while}
\end{align*}
\]

Note that $/$ is an integer division. For instance, the predicate even$(b)$ can be implemented as $b = 2 \times (b/2)$.
Find an invariant that is strong enough to prove the correctness of the program.
Use Hoare logic to prove $\{\text{true}\} P \{k = M^N\}$