to be returned: Oct 10, 2013, before the lecture (in the lecture, or to Florian’s pigeon hole, 2nd floor, main building)

Note: there are many exercises on this sheet, but none of them should require a long answer. Try to make it your goal to be as concise and correct as possible.

Except if indicated otherwise, you may use any results, theorems, etc. from the lecture in your answers. Do not use facts from elsewhere (Wikipedia, undergraduate courses, etc.), except elementary high school maths (real numbers, etc.).

1.1 Vector Spaces

Exercise 1.1. Which of the following sets are vector spaces? If your answer is no, prepare give a counterexample, which properties are violated. (10 points)

- \( V_1 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 + x_2 = 0\} \) (read: "all vectors \((x_1, x_2)\) with coefficients that fulfill \(x_1 + x_2 = 0\)"
- \( V_2 = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq x_2\} \)
- \( V_3 = \{-4, -2, 0, 2, 4, \ldots\} \), all even integers
- \( V_4 = \{\text{all polynomials of degree at most } k, \text{ i.e. } p(t) = a_k t^k + \cdots + a_1 t + a_0\} \)
- \( V_5 = \{\text{all polynomials (of arbitrary degree) that have zeros (at least) at } -1, 0 \text{ and } +1\} \)
- \( V_6 = \{\text{all (at most) quadratic polynomials with non-negative coefficients,}\} \)
- \( V_7 = \{\text{all polynomials of degree exactly } k \text{ (i.e. like } V_4, \text{ but with } a_k \neq 0\} \)
- \( V_8 = \{\text{all normalized polynomials, i.e. polynomials where the coefficient in front of the highest power is } 1\} \)
- \( V_9 = \{\text{all function } f : \mathbb{R} \to \mathbb{R} \text{ with } f(1) = 0\} \),
- \( V_{10} = \{\text{all function } f : \mathbb{R} \to \mathbb{R} \text{ with } f(0) = 1\} \),

1.2 Linearity

Reminder: to prove that a function is linear, show either that it fulfills the defining properties \( i) \) and \( ii) \), or use one of the results proved in the lecture. To prove that a function is not linear, it suffices to provide a counterexample, i.e. a case in which property \( i) \) or \( ii) \) is violated.

Exercise 1.2. Which of these function \( f_i : \mathbb{R} \to \mathbb{R} \) are linear? Justify your answers. (8 points)

- \( f_1(x) = 0 \)
- \( f_2(x) = 1 \)
- \( f_3(x) = 1x \)
- \( f_4(x) = 0x \)
- \( f_5(x) = \begin{cases} -|x| & \text{for } x \leq 0 \\ |x| & \text{otherwise.} \end{cases} \)
- \( f_6(x) = (x + 1)^2 - (x - 1)^2 \)
- \( f_7(x) = \sqrt{x^2} \)
- \( f_8(x) = \sqrt[3]{x^3} \)

Exercise 1.3. For which values of \( a, b, c \in \mathbb{R} \) is the function \( f(x) = ax^2 + bx + c \) linear? (3 points)

Exercise 1.4. Which of these function \( f_i : \mathbb{R}^n \to \mathbb{R} \) are linear? (6 points)

- \( f_9(x) = \sum_{j=2,4,6,\ldots,n} x_j \) (for even \( n \in \mathbb{N} \))
- \( f_{10}(x) = \sum_{j=1,2,4,\ldots,n} (\frac{1}{2})^j x_j \)
- \( f_{11}(x) = \max(x_1, \ldots, x_n) - \min(x_1, \ldots, x_n), \) where max and min pick the largest and smallest value from a list

Exercise 1.5. Let \( f_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \) be the function that rotates a two-dimensional vector counterclockwise by the angle \( \theta \). Prove geometrically, i.e. without using the example about rotation matrices from the lecture, that \( f_\theta \) is linear. (4 points)

Exercise 1.6. Prove or disprove the following two statements: (4 points)

- For any linear function \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \), the function \( h : \mathbb{R}^2 \to \mathbb{R} \) given by \( h(x_1, x_2) = f(x_1) + g(x_2) \) is linear.
- For any linear function \( f : \mathbb{R} \to \mathbb{R} \) and \( g : \mathbb{R} \to \mathbb{R} \), the function \( h : \mathbb{R}^2 \to \mathbb{R} \) given by \( h(x_1, x_2) = f(x_1) \cdot g(x_2) \) is linear.
Exercise 1.7. Identify the matrices corresponding to the following linear functions \( f : \mathbb{R}^n \to \mathbb{R}^m \):

\[
\begin{align*}
& f_{12}(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} & & f_{13}(x) = (x_1 + x_2 + \cdots + x_n) & & f_{14}(x) = x & & f_{15}(x) = \pi x & & f_{16}(x) = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{pmatrix}
\end{align*}
\]

Exercise 1.8. Compute the following matrix-vector multiplications, if possible.

\[
\begin{align*}
& \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & & \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix} & & \begin{pmatrix} \frac{1}{2} & \frac{1}{3} \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \begin{pmatrix} \frac{1}{2} \end{pmatrix} & & \begin{pmatrix} 0 & \sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} & & \begin{pmatrix} \sqrt{2} \\ -\sqrt{2} \end{pmatrix} \\
& \begin{pmatrix} -\frac{\pi}{2} & -\frac{\pi}{2} \end{pmatrix} & & \begin{pmatrix} 2\pi \\ 0 \end{pmatrix} & & \begin{pmatrix} 2\pi \\ 0 \end{pmatrix} & & \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix} & & \begin{pmatrix} -1 \\ 0 \end{pmatrix} & & \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \end{pmatrix} & & \begin{pmatrix} -1 \\ 1 \end{pmatrix} & & \begin{pmatrix} 1 \end{pmatrix}
\end{align*}
\]

Exercise 1.9. In the lecture, we studied least-squares parameter estimation. Show that for fixed data \( x^{(1)}, \ldots, x^{(n)} \in \mathbb{R} \), the optimal coefficient \( c^* \) is a linear function of the vector of observations \( y = \begin{pmatrix} y^{(1)} \\ \vdots \\ y^{(n)} \end{pmatrix} \in \mathbb{R}^n \).

Exercise 1.10. Use least-squares regression to estimate the equation of a linear function from the following samples:

\[
\begin{array}{c|cccccccc}
  x & 4 & -3 & 3 & 2 & 5 & -2 & -2 & 1 \\
  y & -5.5 & 4.9 & -5.0 & -3.9 & -8.1 & 0.9 & 1.9 & -0.1 \\
\end{array}
\]

Exercise 1.11. (8 points bonus)

For any polynomial \( p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \) of degree at most \( n \), we call \( (a_n, a_{n-1}, \ldots, a_1, a_0) \in \mathbb{R}^{n+1} \) its coefficient vector.

- Show: the function \( D : \mathbb{R}^{n+1} \to \mathbb{R}^n \) that maps the coefficient vector of a polynomial \( p(t) \) to the coefficient vector of its derivative \( \frac{d}{dt} p(t) \) is a linear function.
- What is the matrix of \( D \)?

Exercise 1.12. (8 point bonus)

Let \( x^{(1)}, x^{(n)} \) be \( n \) increasing positions on the \( x \)-axis, and let \( y^{(1)}, \ldots, y^{(n)} \) be some function values at these positions. We can form the piece-wise linear (actually: piece-wise affine) function by ”connecting the dots” (see Figure). For \( i = 1, \ldots, n-1 \), we denote the slope of the piecewise linear approximation between \( x_i \) and \( x_{i+1} \) by \( s_i \).

- Show: function \( D \) that maps the values \( y_1, \ldots, y_n \) to the slopes \( s_1, \ldots, s_{n-1} \) is linear from \( \mathbb{R}^n \) to \( \mathbb{R}^{n-1} \).
- What’s the matrix of \( D \)?