Exercise 1.1. Compute all matrix-matrix products $A_iB_j$ that are possible for the following matrices $A_1, \ldots, A_8$ and $B_1, \ldots, B_7$. Mark all combinations that cannot be multiplied by an $\times$. (28 points)

<table>
<thead>
<tr>
<th>$B_1$</th>
<th>$B_2$</th>
<th>$B_3$</th>
<th>$B_4$</th>
<th>$B_5$</th>
<th>$B_6$</th>
<th>$B_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>(1 0)</td>
<td>(1 2 3)</td>
<td>(0.5)</td>
<td>(−2)</td>
<td>(2 0 0)</td>
<td>(1 0 0 0)</td>
</tr>
<tr>
<td>(2)</td>
<td>(0 1)</td>
<td>(4 5 −7)</td>
<td>(0.25)</td>
<td>(−1)</td>
<td>(0 1 −1 0)</td>
<td>(0 2 0 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(−0.25)</td>
<td>(3)</td>
<td>(−1 0 0 1)</td>
<td>(0 0 3 0)</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(−0.5)</td>
<td>(−2)</td>
<td>(0 1 2 1)</td>
<td>(0 0 0 4)</td>
</tr>
</tbody>
</table>

$A_1$ (1 2)

$A_2$ (1 0)

$A_3$ (1 2)

$A_4$ (2 3 4)

$A_5$ (2 1 3)

$A_6$ (2 1 0)

$A_7$ (1 2 3)

$A_8$ (1 0 1)

Exercise 1.2. Show: any two diagonal matrices $\text{diag}(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n \times n}$ and $\text{diag}(\beta_1, \ldots, \beta_n) \in \mathbb{R}^{n \times n}$ commute. (4 points)

Hint: derive a simple rule for the matrix product of two diagonal matrices first.

Exercise 1.3. We know: to rotate a vector in $\mathbb{R}^2$ by an angle $\theta$ (ccw) we can multiply it by a matrix $R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$.

a) Show: any two such rotation matrices $R_\theta$ and $R_\eta$ commute. (2 points)

b) Use the geometry of rotations and the rules of matrix multiplication to show:

1) $\sin(\theta + \eta) = \sin \theta \cos \eta + \cos \theta \sin \eta$ (sine identity)

2) $\cos(\theta + \eta) = \cos \theta \cos \eta - \sin \theta \sin \eta$ (cosine identity)

Please turn page over...
Exercise 1.4. Which of the following matrices are rotation matrices? Justify your answer. (6 points)

\[
A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad A_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad A_5 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\sqrt{3}}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \quad A_6 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}
\]

Exercise 1.5. (6 points + 4 bonus points)
In the lecture we saw how to represent the Facebook friendship relation between n people as a matrix: \( A \in \mathbb{R}^{n \times n} \) with

\[
a_{ij} = \begin{cases} 1 & \text{if person } i \text{ and person } j \text{ are "friends"} \\ 0 & \text{otherwise}. \end{cases}
\]

Derive expression for the following functions in terms of the elements of \( A \):

- \( f_1(x) = \) "the number of friends that a person \( i \) has" \quad (for any person \( i = 1, \ldots, n \))
- \( f_2(x) = \) "the average number of friends a person has" \quad (for any \( i = 1, \ldots, n \))
- \( f_3(x) = \) "the number of mutual friends that person \( i \) has with person \( j \)" \quad (for any \( i,j = 1, \ldots, n \))

Bonus part:
Another interpretation of the number of mutual friends is how many paths of length 2 are in the friendship graph between two people.

- Derive expressions for the number of paths
  \( a) \) of length (exactly) 3,
  \( b) \) of length at most 3,
  \( c) \) of length exactly \( k \)
  Can you find a way to compute these efficiently?

- We call two people connected, if there is a path between them in the graph, regardless of its length. Otherwise, we call them disconnected. Find a way to make use of the matrix \( A \) for deciding whether two people are connected or not.

Exercise 1.6. (4 points + 4 bonus points)
In the lecture we had the following example for the dynamics of population counts:

\[
y = Ax \quad \text{with} \quad A = \begin{pmatrix} 0 & 1.3 \\ 0.5 & 0 \\ 0.7 & 0.5 \end{pmatrix} \in \mathbb{R}^{3 \times 2},
\]

where the vector \( x \in \mathbb{R}^3 \), \( x = \begin{pmatrix} x_{\text{young}} \\ x_{\text{adult}} \end{pmatrix} \) contains the number of individuals in the current generation, and \( y = \begin{pmatrix} y_{\text{young}} \\ y_{\text{adult}} \\ y_{\text{dead}} \end{pmatrix} \) contains the number of individuals in the next generation.

- How do we have to change the matrix if we don’t care about the number of dead individual and remove \( y_{\text{dead}} \) from the output vector?
- How do we have to change the matrix if we don’t change the dynamics, but include an additional entry \( x_{\text{dead}} \) as third entry to the input vector?

Bonus part:
In some cases, we might not be interested in the absolute number of individual, but in the relative proportions of the age groups. We can also write these proportions in vector form, and they will have the special properties that the sum of the entries is 1.

- How do we have to change \( A \) such that we do not change the dynamics, but we ensure that the elements of \( y \) are guaranteed to sum up to 1 if the elements of \( x \) do so?