Exercise 4.1. Solve the following systems of linear equations:

(a) \[
\begin{align*}
x_2 &+ 2x_3 + 3x_4 = 0 \\
x_1 + 2x_2 + 3x_3 + 4x_4 &= 0 \\
2x_1 + 3x_2 + 4x_3 &= 5x_4 = 0 \\
3x_1 + 4x_2 + 5x_3 &= 6x_4 = 0 \\
\end{align*}
\]

(b) \[
\begin{align*}
-6x_1 + 6x_2 + 2x_3 - 2x_4 &= 2 \\
-9x_1 + 8x_2 + 3x_3 - 2x_4 &= 3 \\
-3x_1 + 2x_2 + x_3 &= 1 \\
-15x_1 + 14x_2 + 5x_3 - 4x_4 &= 5 \\
\end{align*}
\]

Exercise 4.2 (Affine Subspaces). A subset \(F \subseteq \mathbb{R}^n\) is called an affine subspace if either \(F = \emptyset\), or there exists a linear subspace \(U \subseteq \mathbb{R}^n\) and a vector \(a \in \mathbb{R}^n\) such that

\[F = a + L := \{a + u : u \in U\}.\]

If \(F\) is nonempty, the dimension of \(F\) is defined as the dimension of \(U\) (this makes sense by Part (b) below). (If \(F\) is empty, its dimension is defined as \(-1\).)

(a) Show that every affine subspace of \(\mathbb{R}^2\) is of one of the following types: (i) \(F = \emptyset\); (ii) \(F = \{a\}\) has a single element; (iii) \(F\) is a line, not necessarily passing through the origin; (iv) \(F = \mathbb{R}^2\). (2 points)

(b) Show that if \(F\) is a nonempty affine subspace of \(\mathbb{R}^n\) and if \(F = a + U = a' + U'\) for vectors \(a, a' \in \mathbb{R}^n\) and linear subspaces \(U\) and \(U'\), then \(U = U'\). Thus the linear subspace \(U\) in the definition of a nonempty affine subspace is uniquely determined. Does it also necessarily hold that \(a = a'\)? (4 points)

(c) Consider a system of linear equations \(Ax = b\), where \(A \in \mathbb{R}^{m \times n}\) and \(b \in \mathbb{R}^m\). Show that the set of solutions \(\text{Sol}(A, b)\) is an affine subspace of \(\mathbb{R}^n\). (4 points)

(d) Suppose that \(A\) is in row echelon form. How can you tell the dimension of \(\text{Sol}(A, b)\) just by looking at \(A\) and \(b\)? (4 points)

Exercise 4.3. Prove the following statements from the lecture. For any invertible \(A, B \in \mathbb{R}^{n \times n}\):

a) the matrix \(A^{-1}\) is invertible and its inverse is \(A\), i.e. \((A^{-1})^{-1} = A\), (4 points)

b) \(AB\) is invertible, and its inverse is \(B^{-1}A^{-1}\), i.e. \((AB)^{-1} = B^{-1}A^{-1}\),

Exercise 4.4. Which of the following matrices are invertible? (6 points)

\[
\begin{align*}
a) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & b) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & c) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & d) \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix} & e) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & f) \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix}
\end{align*}
\]

Exercise 4.5. The rank of a matrix \(A \in \mathbb{R}^{m \times n}\) is defined as the maximum number of linearly independent columns of \(A\) (we view each column as a vector in \(\mathbb{R}^m\)).

What’s the rank of the matrices from the previous exercise? (6 points)

\[
\begin{align*}
a) & \quad & b) & \quad & c) & \quad & d) & \quad & e) & \quad & f) & \quad
\end{align*}
\]

Bonus. Show that if \(A\) is in row echelon form then the definintion of rank we gave in class agrees with the definition of rank given here. (4 points)

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Exercise 4.6. Identify all pairs of vectors from the following set that are orthogonal to each other for every $a, b, c \in \mathbb{R}$:

- in $\mathbb{R}^2$: $x_1 = \begin{pmatrix} a \\ b \end{pmatrix}$, $x_2 = \begin{pmatrix} b \\ -a \end{pmatrix}$, $x_3 = \begin{pmatrix} -b \\ a \end{pmatrix}$, $x_4 = \begin{pmatrix} a-b \\ b-a \end{pmatrix}$.

- in $\mathbb{R}^3$: $x_5 = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}$, $x_6 = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix}$, $x_7 = \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix}$, $x_8 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

- in $\mathbb{R}^n$: $x_9 = \begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}$, $x_{10} = \begin{pmatrix} b \\ b \\ \vdots \\ b \end{pmatrix}$, $x_{11} = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$, $x_{12} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

Bonus: for which of the other cases exist non-zero values of $a, b, c \in \mathbb{R}$ that do make the vectors orthogonal? (4 points)

Exercise 4.7. Prove: for any $x \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}$: $\| \lambda x \| = |\lambda| \| x \|$. (4 points)

Exercise 4.8. Prove: $\| \cdot \| : \mathbb{R}^n \to \mathbb{R}$, given by $x \mapsto \| x \|$, is not a linear function. (4 points)

Exercise 4.9. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with $x \perp y$.

a) Use the definition of $\| \cdot \|$ and the properties of the inner product to show:

$$\| x + y \|^2 = \| x \|^2 + \| y \|^2 \quad (\text{Pythagorean Identity}).$$

b) Give an example that shows that the identity can be violated if $x \not\perp y$.

Exercise 4.10. Let $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ be arbitrary.

Use the definition of $\| \cdot \|$ and the properties of the inner product to show:

$$\| x + y \|^2 + \| x - y \|^2 = 2(\| x \|^2 + \| y \|^2) \quad (\text{Parallelogram Identity}).$$