Structure of complexes and filtrations.

1. Filtrations and collapses.
2. Circumspheres and radius functions.
3. Structure of a simplex.
Filtrations and collapses.

Setting: \( \alpha \) increased from 0 to \( \infty \).

Various filtrations:

For \( \alpha \leq \alpha' \) we have

\[
\text{Čech}_\alpha P \subseteq \text{Čech}_{\alpha'} P
\]

\[
\text{Del}_\alpha \subseteq \text{Del}_{\alpha'} P
\]

\[
\text{Harp}_\alpha P \subseteq \text{Harp}_{\alpha'} P
\]

Our ultimate goal is to find a smaller complex than \( \text{Del}_\alpha \), which encodes the same topological information.

A sequence of complexes:

\[
K_0 \subseteq K_1 \subseteq \ldots \subseteq K_n
\]

is called a filtration.
Delanay filtrations

Consider Del$_3(P), P \subseteq \mathbb{R}^2$ of size 3:

\[ d_0 = 0 \quad d_1 > d_0 \quad d_2 > d_0 \quad d_3 > d_2 \quad d_4 > d_3 \]

After $d_0 = 0$, the connectivity changes 4 times. For another pointset (with an obtuse triangle):

\[ d_0 = 0 \quad d_1 > d_0 \quad d_2 > d_0 \quad d_3 > d_2 \]

Only 2 connectivity changes.

Now between $d_2$ and $d_3$, two simplices enter:

\[ \text{A} \quad \text{B} \]

But there is no change in \( \sigma \) connectivity!

In fact we can collapse \( \text{B} \) onto \( \text{A} \) \((\text{BVA})\):

\[ \text{B} \quad \Rightarrow \quad \text{A} \]

\[ \text{Simpcicial collapse: \, combinatorially, we just removed the pair } \sigma \leq I, \text{\ which we call a collapse.} \]
Circumspheres and radius functions.

We are interested in the first (smallest) radius $\lambda$ for which simplex $\Delta$ appears in a filtration. This is the value of the radius function, $R: \text{Del} \ P \to \mathbb{R}$.

Circumsphere of $Q \subseteq \mathbb{R}^d$ is a $d$-dimensional sphere that passes through each point in $Q$. If $\text{card} \ Q = d+1$, it is unique. The unique circumsphere of $a,b,c \subseteq \mathbb{R}^2$ for some of the infinitely many circumpheres is again unique.
Delaunay radius function.

Recall that Del. simplices appear when the restricted Voronoi regions meet.

Claim: The radius function of a simplex $\mathcal{S}$ is the radius of its smallest empty circumsphere.

Analysis:

Out of all the point equidistant from $a$ and $b$, $c$ is the closest point that is not in the interior of $\text{Vor}(X)$. Point $c$ is also the center of the smallest empty (avoiding $x$) circumsphere of $\{a, b, x\}$. In this case it's also the circumsphere of $\{a, b, x\}$. 
Radius function in $\mathbb{R}^3$.

Observe that several simplices share the big sphere as their smallest empty circumsphere. The following simplices have the same value of a radius function: $\{a,b,c,d\}$, $\{a,b\}$, $\{b, c\}$, $\{c, d\}$.

After collapsing them away, we get:
III. Structure of a simplex.

We visualize the face relations of a simplex (or any complex) using Hasse diagrams:

\[ \phi \xrightarrow{\text{shortcut for } \{a,b\}} \phi \xrightarrow{\text{face relation}} \text{consider the empty set} \]

The cone of a simplex \( \mathcal{S} = \{ V_0, \ldots, V_k \} \) is \( \mathcal{S}' = \mathcal{S} \cup \{ V_{k+1} \} \).

Repeating coning gives higher dim. simplices and their hyper-cubical face lattices.
Intervals.

So far we saw that two consecutive Delaunay filtration levels can differ by 1, 2 or 4 simplices. The last one had the structure:

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abcd
\_\_\_\_\_
abc \_\_\_\_
\_\_\_\_
\_\_
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which is isomorphic to the Hasse diagram of an edge.

We define an interval in the face lattice as:

\[
[\pi \# q] = \{ s : \pi \leq s \leq q \}
\]

Later we show that collapses in Del. filtrations may only occur as intervals, and use this fact to define the Hasp complex onto which the Del. complex collapses.