6. Menger Theory

Triangulations
Euler characteristic
Triangulation of a disk
Manifolds
Closed surfaces
Classification
Height function
More. Function
More equally

Euler characteristic:

\[ \chi(\mathcal{X}) = \sum_{p \geq 0} (-1)^p s_p(\mathcal{X}). \]

Example:

\[ \mathcal{X} = \mathbb{S}^2 \]
\[ k = \text{disk} \]

\[ \chi = 4 - 6 + 2 = 2 \]
\[ \text{torus} \]

\[ \chi = 9 - 27 + 18 = 0 \]

Triangulation of a disk:

\[ k \text{ is triangulation of } \mathcal{X} = \mathbb{R}^2. \]

Lemma:

\[ \chi(k) = 1. \]

Proof by induction:

\[ \chi(\Delta) = 3 - 3 + 1 = 1. \]

Case 1. Remove edge and triangle.

Case 2. Remove vertex, two edges, one triangle.

There is always such a triangle:

(i) If \( \lambda \) interior vertex, then one must be connected to both or in Case 1.

(ii) If \( \lambda \) interior vertex, then there exists an ear or in Case 2.
An orientable is top. space X in which every pt. has nbhd. homeo. to $\mathbb{R}^n$.

Examples for $n=2$:

- Sphere, $X=S^2$
- Torus, $X=\mathbb{C}P^2$
- Klein bottle, $X=\mathbb{C}P^2$ with twisted boundary

Non-examples:

- Disk, $X=D^2$
- Two spheres, $X=S^2 \times S^2$

Question:
Given $n$ closed 1, 2, ..., $n$ edges is it triangulable? \\
How bad can you test whether a triangulation is 2-dimenstional?

Closed surfaces:

A closed surface is a compact 2-manifold (without boundary).

$k$-fold torus is connected sum of $k$ tori: $T_k = T \# T \# \ldots \# T$

$\chi(T_k) = k \cdot \chi(T) - 3(k-1)$

$= 2 - 2k$.

2-fold is orientable if its triangulation have consistent orientations:

Projection space is set of edges

$RP^2$ meaning through origin

$X = \frac{1}{2} \chi(sphere) = 1$
Height function.

\( M \) is compact \( 2 \)-manifold.
\( f: M \rightarrow \mathbb{R} \) is smooth function.

Example:
\[
\begin{array}{c|c}
\text{Max} & \text{Min} \\
\hline
\text{Val} & \text{Val} \\
\end{array}
\]

\( f(M) \) is a circle.
\( x \) is a critical point of \( f \).

The Hessian of \( x \) is:
\[
H(x) = \begin{bmatrix}
\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\
\frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2}
\end{bmatrix}
\]

\( x \) is non-degenerate critical point if \( \det H(x) \neq 0 \).

Morse function.

\( f: M \rightarrow \mathbb{R} \) is Morse function if all critical points are non-degenerate.

Non-examples:
- circle of maximum
- circle of minimum
- non-degenerate critical points are isolated
- compactness is finite

Examples:
- monkey saddle
- saddle
- max

Theorem: The Morse functions are dense in the set of smooth functions on a manifold.

Morse inequality.

\( M \) is compact \( n \)-manifold.
\( f: M \rightarrow \mathbb{R} \) is Morse function.

The index of a critical point \( x \) of \( f \) is the number of critical points of \( f \) in each direction below.

\[ c_p(f) = \# \text{critical points of } f \text{ in index } p. \]

Theorem: \( \sum_{p \geq 0} (-1)^p c_p(f) = \chi(M) \).

Proof (for tame, closed, and bounded):
- regular with cond.
- add vertices in order of height.
- \( \chi(M) = \chi(\partial M) \).

Min:
\[ x_{\text{min}} = x_{\text{max}} + 1 \]

Saddles:
\[ x_{\text{max}} = x_{\text{min}} - 1 \]

Max:
\[ x_{\text{max}} = x_{\text{min}} + 1 \]