Supplemental material for *A Dimension-reduced Pressure Solver for Liquid Simulations*

Ryoichi Ando[1], Nils Thürey[2], and Chris Wojtan[1]
[1]IST Austria, [2]Technische Universität München

December 24, 2014

Continuing with the notation from the paper, $p, \tilde{p}, \hat{\Phi}$ and U denote pressure, coarse pressure, modified distance function, and up-sampling matrix, respectively. To include surface tension forces, we extend our definition of the up-sampled pressure as follows:

$$p = \hat{\Phi} U \tilde{p} + (I - \hat{\Phi})\sigma H \tag{1}$$

where σ denotes the global surface tension coefficient and H the mean curvature given by the (spatially varying) Laplacian of a signed distance function. Note that at the free surface (when $\hat{\Phi}$ vanishes), $\hat{\Phi} U \tilde{p}$ is equal to zero and $p = \sigma H$, as required. Intuitively, Eq. (1) exactly meets the high-resolution Dirichlet boundary conditions at the free surface, and it transitions to the original dimension-reduced pressure within the liquid.

Before we start to incorporate this new term into the reduced system, it is important to keep in mind that this term will only modify the right-hand-side of the linear system. This will later on allow us to leave the pressure system unmodified, and separate the surface tension terms into a force applied to the velocities. We will first briefly review the ghost fluid method for the regular surface tension system.

Let us consider two pressure cells crossing a levelset surface. In this case, the pressure cell outside is treated as ghost cell p_G, and the other one (inside) as fluid cell p_F. Since the pressure at the surface due to surface tension is σH, the linear interpolation of these two cells right at the surface, with distance θ from p_F, should be σH:

$$(1-\theta) p_F + \theta p_G = \sigma H \tag{2}$$

Rearranging this equation yields:

$$p_G = \frac{1}{\theta}(\sigma H - p_F) + p_F \tag{3}$$

Thus, the gradient of these two cells is given by:

$$\frac{\Delta t}{\rho}\nabla p = \frac{\Delta t}{\rho}\frac{p_G - p_F}{\Delta x} = \frac{\Delta t}{\theta\rho}\Big(\frac{\sigma H - p_F}{\Delta x}\Big) \tag{4}$$

where Δx denotes the cell size at high resolution. Substituting Eq.(1) for p_F gives:

$$\frac{\Delta t}{\rho}\nabla p = \frac{\hat{\Phi}\Delta t}{\theta\rho}\Big(\frac{\sigma H - U\tilde{p}}{\Delta x}\Big). \tag{5}$$

The right hand side of this equation is separable into terms depending on $\tilde{p}$ (which need to be part of the pressure solve) and some that don't. Hence, those independent of $\tilde{p}$ will directly modify the velocity later on when it is made divergence free with the pressure gradient, and thus we can think of them as an external force that is directly applied to the fluid velocity. So, the external force for all cells at the surface is given by:

$$f_G = \frac{\hat{\Phi}\Delta t}{\theta\rho\Delta x}\sigma H. \tag{6}$$

However, in the context of re-sampling the velocity and pressure fields we cannot restrict our view to cells at the interface. Cells completely filled with fluid also introduce new right hand side terms. For two cells in the bulk of the liquid, the gradient of the pressure is computed with:

$$\frac{\Delta t}{\rho}\nabla p = \frac{\Delta t}{\rho}\frac{p_{i+1} - p_i}{\Delta x} \tag{7}$$

Substituting Eq.(1) for p_i and p_{i+1} yields:

$$\frac{\Delta t}{\rho}\nabla p = \frac{\Delta t}{\rho}\frac{\hat{\Phi}_{i+1}(U\tilde{p})_{i+1} - \hat{\Phi}_i(U\tilde{p})_i}{\Delta x} + \frac{\Delta t}{\rho}\frac{(1-\hat{\Phi}_{i+1})\sigma H_{i+1} - (1-\hat{\Phi}_i)\sigma H_i}{\Delta x} \tag{8}$$

Separating the terms independent of the pressure, just like before, gives the force term for fluid cells:

$$f_F = \frac{\Delta t}{\rho}\frac{(1-\hat{\Phi}_{i+1})\sigma H_{i+1} - (1-\hat{\Phi}_i)\sigma H_i}{\Delta x} \tag{9}$$

To summarize, we compute these forces for each cell face, forming a force vector $\boldsymbol{f}$. We apply $\boldsymbol{f}$ to the intermediate velocity $\hat{\boldsymbol{u}}$, to compute a new intermediate velocity field. The rest of the pressure solve can continue in an unmodified fashion. Note that including the surface tension terms into the right-hand-side would have also required us to include surface tension into the velocity update with the pressure gradient. As such, this force-based view is simpler to implement.