ARAKAWA-SUZUKI FUNCTORS FOR WHITTAKER MODULES

by

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Abstract

In this thesis, we construct a family of exact functors from the category of Whittaker modules of the simple complex Lie algebra of type A_n to the category of finite-dimensional modules of the graded affine Hecke algebra of type A_ℓ. Using results of Backelin [3] and of Arakawa-Suzuki [1], we prove that these functors map standard modules to standard modules (or zero) and simple modules to simple modules (or zero). Moreover, we show that each simple module of the graded affine Hecke algebra appears as the image of a simple Whittaker module. Since the Whittaker category contains the BGG category $\mathcal{O}$ as a full subcategory, our results generalize results of Arakawa-Suzuki [1], which in turn generalize Schur-Weyl duality between finite-dimensional representations of $\mathrm{SL}_n(\mathbb{C})$ and representations of the symmetric group S_n.

For Aslyn Cora Zimmerman.

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Preface

The full historical (and mathematical) context of this thesis does not easily conform to a linear narrative structure. It is unlikely that any document of finite length will capture the multitude of individuals and events which gave representation theory its modern form. However, the utility of the narrative form for the purposes of communication is inescapable. We, therefore, find ourselves with the unfortunate task of imposing a simple structure onto the complex and often incomprehensible history of mathematics [1]. To the casualties of our oversimplification, we apologize[2].

The roots of this thesis, and representation theory more generally, can be traced back to Joseph Fourier, a paper rejected for publication, and the advent of harmonic analysis. While studying the diffusion of heat and the methodology of separation of variables, Joseph Fourier invented a technique for representing a function $f(x)$ on the interval $(-\pi, \pi)$ by a series of continuous multiplicative characters of the circle group (group homomorphisms $S^1 \to \mathbb{C}^\times$):

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \qquad c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

In a single paper, submitted in 1807, Fourier derived the heat equation and studied the representability of functions by trigonometric series[3] [28]. The representability of functions by multiplicative characters of groups became a central theme in harmonic analysis, leading to the decomposition of complex-valued functions on

[1]The historical content of this section is drawn heavily from [26–31].

[2]We specifically refer readers to [28, 29] for a much more thorough historical account of representation theory.

[3]The paper was blocked from publication due to objections concerning the incompatibility of Fourier series with the prevailing intuition of the time. Fourier submitted an expanded and revised version of the paper again in 1811, which was also rejected. Fourier's analytic theory of heat was eventually published as a book in 1822 [18].

finite abelian groups:

$$f(x) = \frac{1}{|G|} \sum_{\omega} \left(\sum_{y \in G} f(y)\overline{\omega(y)} \right) \omega(y)$$

where ω ranges over all multiplicative characters of G (group homomorphisms $G \to \mathbb{C}^{\times}$).

These fundamental results of harmonic analysis paved the way for a theory of multiplicative characters of finite non-abelian groups. The theory of characters and representations of finite groups was developed in the early 1900s, building on contributions from Frobenius, Burnside, Schur, and Noether [29]. Frobenius initiated the field in 1896 by developing a theory of characters for non-abelian finite groups. Brunside [12] and I. Schur [42] then shifted the focus of study to homomorphisms from a finite group into the group of invertible matrices. Noether then gave the modern definition of a representation of a group: A representation of a finite group G is a homomorphism π from G to the group of invertible linear transformations $\mathrm{GL}(V)$ of a finite-dimensional vector space V [2, page 528]. The corresponding character of G is defined to be the trace of the matrix corresponding to each group element:

$$\chi_{\pi}(g) = \mathrm{tr}(\pi(g)).$$

The celebrated representation theory of compact connected Lie groups was pioneered by Cartan and Weyl. In 1913, Cartan introduced highest weight representations of complex semisimple Lie algebras [13]. Suppose $\mathfrak{g}$ is a (finite-dimensional) complex semisimple Lie algebra. Suppose $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ (i.e. a nilpotent self-normalizing subalgebra). Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding set of roots and $W = W(\mathfrak{g}, \mathfrak{h})$ the corresponding Weyl group. Let $\Pi \subset \Delta^{+} \subset \Delta$ be a choice of simple roots and positive roots, respectively. Cartan showed that there is a one-to-one correspondence between dominant (with respect to Π) integral linear functionals $\lambda \in \mathfrak{h}^{*}$ and irreducible finite-dimensional representations of $\mathfrak{g}$. Weyl then developed an analytic theory for representations of compact connected Lie groups from 1924 to 1926, resulting in the Weyl character formula in 1925 and the Peter-Weyl theorem in 1926 [40, 48–51]. Let G be a compact connected Lie group with complexified Lie algebra $\mathfrak{g}$. For ease of exposition, we will additionally assume that G is simply connected. Let T be a maximal torus of G with complexified Lie algebra $\mathfrak{h}$. Let V_λ be an irreducible representation of $\mathfrak{g}$, corresponding to the dominant integral weight $\lambda \in \mathfrak{h}^*$ by Cartan's classification of irreducible representations of $\mathfrak{g}$ above. Let $\rho = \sum_{\Delta^+} \frac{1}{2}\alpha \in \mathfrak{h}^*$ denote the half sum of positive roots. The Weyl character formula

$$\chi(V_\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Delta^+} \left(e^{\alpha/2} - e^{-\alpha/2}\right)}$$

gives a continuous character of G corresponding to the representation V_λ in terms of characters[4]

$$e^\nu(H) = \nu(h) \text{ for } H = \exp(h),\, h \in \mathfrak{h}$$

of the the maximal torus T. The Maximal Torus Theorem of Weyl states that each element of G is conjugate to an element in T. This allows the character $\chi(V_\lambda)$ of T to be extended to a continuous character of G, resulting in a classification of continuous characters of irreducible representations of compact connected (simply connected) Lie groups.

The Peter-Weyl theorem completed the generalization of Fourier series to the setting of compact connected Lie groups by decomposing $L^2(G)$

$$L^2(G) \cong \widehat{\bigoplus_{\pi \in \widehat{G}}} V_\pi^* \otimes V_\pi,$$

where $\widehat{G}$ is the set of all finite-dimensional irreducible unitary representations of G. Additionally, the Peter-Weyl theorem shows that the linear span of all irreducible characters of G is dense in the space of square-integrable functions which are constant along G-conjugacy classes and the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of G is dense in $L^2(G)$.

In 1951, Harish-Chandra gave an algebraic classification of finite-dimensional representations of compact Lie group [21]. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ (i.e., a maximal solvable subalgebra) containing $\mathfrak{h}$. Working with the universal enveloping algebra $U(\mathfrak{g})$ of the complex semisimple Lie algebra $\mathfrak{g}$, we can define the Verma module $M(\lambda)$ by algebraic induction from the Borel subalgebra

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

[4] More work is required to show for which $\nu \in \mathfrak{h}^*$, e^ν extends uniquely to a well defined continuous function on T. See [43, Chapter 7].

where the nilradical of $\mathfrak{b}$ acts trivially on $\mathbb{C}_\lambda$ and $\mathfrak{h}$ acts by λ. Each Verma module has a unique irreducible quotient, denoted $L(\lambda)$, which is finite-dimensional precisely when λ is a dominant integral weight. The family of such modules was further studied by Verma in 1966 [46]. In 1971, Bernstein-Gelfand-Gelfand investigated these modules further, introducing a category of $U(\mathfrak{g})$-modules whose simple objects are given by the irreducible quotients $L(\lambda)$, for general $\lambda \in \mathfrak{h}^*$ [5]. The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ is defined to be the category of finitely generated $U(\mathfrak{g})$-modules which are locally $U(\mathfrak{b})$-finite and $U(\mathfrak{h})$-semisimple. A formula for the characters of irreducible modules in category $\mathcal{O}$ then becomes a natural extension of the Weyl character formula and was conjecturally described by Kazhdan and Lusztig in 1979.

The significance of the Kazhdan-Lusztig conjectures in representation theory is hard to overstate. Since the publication of the seminal paper *Representations of Coxeter groups and Hecke algebras* [24], multiple generations of mathematicians have been captivated by the sublime relationships uncovered between the combinatorial theory of Hecke algebras, the representation theory of complex semisimple Lie algebras, and the geometry of flag varieties. The eventual proof of the Kazhdan-Lusztig conjectures draws from techniques developed long before the conjectures were made, and ties together several seemingly unrelated branches of mathematics spanning decades of work by many profoundly insightful mathematicians.

In [24], Kazhdan and Lusztig begin by studying the properties of a deformation of the group algebra of a Coxeter group W, known as the Iwahori-Hecke algebra $\mathscr{H}_q$ associated with W. The Iwahori-Hecke algebra appears in several contexts in representation theory. One such realizations is as the convolution algebra of B-bi-invariant functions on G, where G is a split reductive linear algebraic group defined over the finite field $\mathbb{F}_q$, and B is a Borel subgroup of G. Kazhdan and Lusztig gave an algorithm for computing the change of basis matrix corresponding to two natural bases of $\mathscr{H}_q$ and conjectured that the algorithm would yield character formulas for the irreducible representations in category $\mathcal{O}$. These formulas provide the first appearance of the famous Kazhdan-Lusztig polynomials.

The proof of the Kazhdan-Lusztig conjectures, due independently to Brylinski-Kashiwara [11] and Beilinson-Bernstein [6], took only two years but required an amazingly vast range of mathematical techniques which had been under development for the past century. We will follow the independent progression of three key pieces, which came together in a wholly remarkable way.

The first key piece originated with a question concerning systems of differential equations, posed by Riemann in 1857 [26]. If we are given a system of m first order differential equations on U, a connected open subset of $\mathbb{CP}^1$, with regular singularities $u_0, \cdots, u_l \in U$, and a basis of the space of solutions, then we can construct a representation of the fundamental group of the punctured domain

$$\varphi : \pi_1(U - \{u_0, \cdots, u_l\}) \to GL(m, \mathbb{C})$$

called the monodromy of the system (here m is the size of the system of differential equations). Riemann asked for the natural converse of this construction. Given a connected open subset $U \subset \mathbb{CP}^1$, a set of points $\{u_0, \cdots, u_l\}$, and a representation

$$\varphi : \pi_1(U - \{u_0, \cdots, u_l\}) \to GL(m, \mathbb{C})$$

how many systems of first order differential equations with regular singularities have monodromy φ? Riemann showed that there is a unique system when $l = m = 2$, but the problem remained open in more general situations. In 1900, Hilbert included Riemann's question in his famous list of the twenty-three most important mathematical problems for the twentieth century. Hilbert asked whether we could find a system of differential equations with a prescribed set of (regular) singular points and fixed monodromy. This became known as Hilbert's twenty-first problem or the Riemann-Hilbert problem. The modern approach to this problem, known as the Riemann-Hilbert correspondence, was championed by Pierre Deligne and replaces notions of monodromy with locally constant sheaves, and systems of differential equations with holonomic $\mathcal{D}$-modules [26]. From the perspective of sheaf theory, the Riemann-Hilbert correspondence is an equivalence of categories between complexes of holonomic $\mathcal{D}$-modules with regular singularities on a smooth complex algebraic variety and a category of complexes of sheaves on said variety. This geometric theory became a crucial tool for the eventual proof of the Kazhdan-Lusztig conjectures.

Meanwhile, in the 1970s, Mark Goresky and Robert MacPherson were studying homology of singular spaces. They developed a new homology theory based on cocycles which intersected strata of the singular space according to a given rule which they called a perversity [19,20]. They proved that for topological pseudomanifolds, this new treatment of homology exhibits the holy grail of homology theories, the existence of Poincare duality. They called this new theory intersection homology. The next monumental step forward for intersection homology occurred at a Halloween party near Paris in 1976 when Pierre Deligne conjectured the existence of a complex of sheaves whose cohomology groups are equal to the intersection homology groups of Gorskey and MacPherson [27]. This serendipitous encounter opened the door for the modern machinery of derived categories and sheaf theory to be used in the study of singular spaces. For the next decade, Gorskey, MacPherson, and others worked on providing increasingly elegant treatments of intersection homology. Their work culminated in the realization of Deligne's complexes of sheaves as the irreducible objects in the category which was found to be equivalent to certain complexes of holonomic $\mathcal{D}$-modules by the Riemann-Hilbert correspondence. Thus, this new category which related the homology of singular spaces and the monodromies of systems of differential equations became known as the category of perverse sheaves.

The final piece in the puzzle is the localization theory of Beilinson-Bernstein, published in the 1981 paper *Localisation de $\mathfrak{g}$-modules* [6]. This fundamental paper ignited the field of geometric representation theory by building a bridge between modules of a Lie algebra and holonomic $\mathcal{D}$-modules on the corresponding flag variety. These developments advanced the proof of the Kazhdan-Lusztig conjectures by allowing the representation theoretic problem of computing irreducible characters to be phrased in terms of $\mathcal{D}$-modules. The Riemann-Hilbert correspondence then translates the character formulas to the language of perverse sheaves. Lastly, the Kazhdan-Lusztig conjectures can be solved by using Kazhdan and Lusztig's realization of the coefficients of Kazhdan-Lusztig polynomials as the dimensions of intersection homology groups on Schubert varieties.

In subsequent years, mathematicians in a variety of fields worked to emulate and generalize applications of intersection homology and perverse sheaves in representation theory. Two subsequent branches of development play a key role in this thesis.

The first branch is concerned with the generalization of the Kazhdan-Lusztig conjectures to a category of Whittaker modules. This category (studied in [3, 32, 36, 38, 39]) contains category $\mathcal{O}$ as a full subcategory and exhibits subtle obstacles when approached from the perspective of perverse sheaves. Namely, holonomic $\mathcal{D}$-modules obtained from localization can now have irregular singularities, preventing us from using the classical Riemann-Hilbert correspondence. This hurdle has been overcome through varying means. In [39] and [3], functors from category $\mathcal{O}$ to blocks in the Whittaker category are constructed as a means of translating Kazhdan-Lusztig theory for category $\mathcal{O}$ to the Whittaker category. This allows us to calculate the multiplicities of irreducible Whittaker modules in the composition series of standard Whittaker modules by reducing the calculation to the well-known case of highest weight modules. Recently, Romanov has developed a new proof of the Kazhdan-Lusztig conjectures for the Whittaker category by developing an algorithm for the computation of Whittaker Kazhdan-Lusztig polynomials directly in the category of holonomic $\mathcal{D}$-modules, rather than in the category of perverse sheaves [41].

The second branch of research that concerns us was developed by Bernstein-Zelevinsky, Zelevinsky, Kazhdan-Lusztig, and Lusztig [4, 25, 34, 35, 54]. Inspired by the work of Kazhdan and Lusztig, Zelevinsky developed a p-adic anolog of the Kazhdan-Lusztig conjectures for certain representations of $\mathrm{GL}(n, \mathbb{Q}_p)$. Kazhdan and Lusztig then refined Zelevinsky's conjecture to include split reductive p-adic Lie groups. However, instead of studying representations of p-adic groups directly, Kazhdan-Lusztig and Lusztig reduced the problem to the study of representations of the affine Hecke algebra by the Borel-Casselman correspondence, and later to the setting of finite-dimensional modules of the graded affine Hecke algebra by Lusztig [34]. In this setting, the graded affine Hecke algebra plays a role loosely analogous to that of the Lie algebra in the complex setting. For this reason (seeing as this thesis ultimately aims to draw connections to the Kazhdan-Lusztig theory of the Whittaker category for $\mathfrak{sl}_n$), we will formulate the p-adic Kazhdan-Lusztig conjectures in the setting of graded affine Hecke algebras (as was done in [1]). This formulation of the p-adic Kazhdan-Lusztig conjectures was proved in [35] by constructing an action of the graded affine Hecke algebra on certain geometric objects. Using Lusztig's geometric realization of standard and irreducible modules, the composition multiplicities of standard modules can then be directly related to the dimensions of intersection homology groups on certain affine algebraic varieties.

These two versions of the Kazhdan-Lusztig conjectures are related by a remarkable geometric observation due (independently) to Lusztig and Zelevinsky. In [54], Zelevinsky shows that for certain cases in type A_n, the p-adic Kazhdan-Lusztig polynomials (defined by intersection homology groups of affine algebraic varieties) match the original Kazhdan-Lusztig polynomials (defined by intersection homology groups of projective algebraic varieties). This observation leads one to ask whether there is a functorial relationship between category $\mathcal{O}$ and the category of graded affine Hecke algebra modules. Such a relationship was developed in [1] for the above setting, and in [16] for the setting of Harish-Chandra modules of GL$(n, \mathbb{R})$. Motivated by the results of Backelin [3] and Romanov [41], this thesis constructs a categorical relationship between Whittaker modules and graded affine Hecke algebra modules. In this way, the results of this thesis generalize the theory developed in [1].

Chapter 1

Introduction

Motivated by the study of Whittaker models of representations, Kostant defined a family of modules over the universal enveloping algebra, $U(\mathfrak{g})$, of a complex semisimple Lie algebra, $\mathfrak{g}$, and classified the irreducible modules contained in the family [32]. In [38], Miličić and Soergel give an axiomatic construction of a category of Whittaker modules, denoted by $\mathcal{N}$, which contains Kostant's family of $U(\mathfrak{g})$-modules as well as the classical Bernstein-Gelfand-Gelfand category $\mathcal{O}$. Suppose $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ with Cartan decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, and let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$. The category of Whittaker modules, denoted $\mathcal{N}$, is defined to be all $U(\mathfrak{g})$-modules which are finitely generated over $U(\mathfrak{g})$, locally finite over $U(\mathfrak{n})$, and locally finite over $Z(\mathfrak{g})$.

The graded affine Hecke algebra naturally arises from the study of representations of reductive algebraic groups over p-adic fields. Inspired by the work of Kazhdan and Lusztig, Zelevinsky developed a p-adic anolog of the Kazhdan-Lusztig conjectures for smooth representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ containing Iwahori fixed vectors [53]. However, a p-adic analogue of the Kazhdan-Lusztig conjectures remained open for groups outside of type A. The full p-adic analogue of the Kazhdan-Lusztig conjectures was pioneered by Lusztig, and relied on contributions from Borel and Casselman. Borel and Casselman related smooth representations of reductive algebraic groups over p-adic fields containing Iwahori fixed vectors to representations of the affine Hecke algebra, completing what we refer to as the Borel-Casselman correspondence. Finally Lusztig constructed a reduction from the study of representations of the affine Hecke algebra to finite-dimensional modules of the graded affine Hecke algebra [34]. We will now define the graded affine Hecke algebra. Let W be the Weyl group of a semisimple complex Lie algebra $\mathfrak{g}$, $\Pi \subset \Delta^+$ be the set of simple and positive roots, respectively, corresponding to the choice of Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Let $S(\mathfrak{h})$ be the symmetric algebra of $\mathfrak{h}$. The graded affine Hecke algebra $\mathbb{H}$ is the associative algebra generated by $\mathbb{C}[W]$ and $S(\mathfrak{h})$ subject to the relations

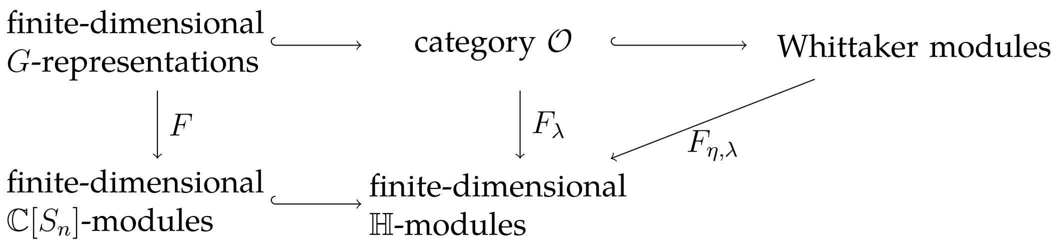
$$s_\alpha \cdot h - s_\alpha(h) \cdot s_\alpha = \langle \alpha, h \rangle \text{ for all } \alpha \in \Pi \text{ and } h \in \mathfrak{h}.$$

In this setting, the graded affine Hecke algebra plays a role loosely analogous to that of the Lie algebra in the complex setting. To complete the p-adic analog of the Kazhdan-Lusztig conjectures, Lusztig constructed algebra isomorphisms between graded affine Hecke algebras and higher endomorphism algebras of certain perverse sheaves [35]. The multiplicity of irreducible representations in the composition series of standard representations is then directly related to the geometry of orbits of a Levi subgroup L_σ on $\mathfrak{g}_1(\sigma) = \{x \in \mathfrak{g} : \text{ad}(\sigma)x = x\}$, for $\sigma \in \mathfrak{h}$. These discoveries illustrate the subtle relationships between the combinatorial representation theory of S_n, the geometry of L_σ orbits on $\mathfrak{g}_1(\sigma)$, and the representation theory of $\text{GL}_n(\mathbb{Q}_p)$.

Schur-Weyl duality relates finite-dimensional representations of $G = \text{GL}(V)$ to finite-dimensional representations of the symmetric group S_n. These connections can be interpreted as a functor, $F(X) = \text{Hom}_G(\mathbb{1}, X \otimes V^{\otimes n})$, from the category of finite-dimensional representations of G to the category of finite-dimensional representations of S_n. The Schur-Weyl duality of G and S_n implies that this functor maps irreducible G-representations to irreducible S_n-representations (or zero). Arakawa and Suzuki generalized the classical Schur-Weyl duality by constructing an action of $\mathbb{H}$ on the tensor product representation $X \otimes V^{\otimes n}$, and then by defining a functor for each Verma module $M(\lambda)$ of highest weight $\lambda \in \mathfrak{h}^*$

$$F_\lambda(X) = \text{Hom}_{U(\mathfrak{g})}(M(\lambda), X \otimes V^{\otimes n})$$

which map irreducible objects in category $\mathcal{O}$ to irreducible $\mathbb{H}$-modules [1] (under certain assumptions on λ). The combinatorial and geometric classification of irreducible Whittaker modules and irreducible graded affine Hecke algebra modules provide a foundation for generalizing the functorial relationships described in [1]. This thesis uses techniques developed by Kostant [32], Backelin [3], and Miličić-Soergel [38, 39], to construct a family of Arakawa-Suzuki type fuctors $F_{\eta,\lambda}$ for the category of Whittaker modules, completing the following diagram.



Moreover, each irreducible graded affine Hecke algebra module is obtained in this way. We can thus view our result as an algebraic realization of the geometric observation of Lusztig and Zelevinsky and the corresponding implications on the Kazhdan Lusztig theories for each category.

1.1 Main results

Let $\mathfrak{g}$ be a reductive complex Lie algebra and $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra and Borel subalgebra of $\mathfrak{g}$, respectively. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be the nilradical of $\mathfrak{b}$, and $\Pi \subset \Delta^+$ be the set of simple and positive roots corresponding to $\mathfrak{b}$, respectively. For a root $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \mathrm{ad}(h)x = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}$, and let $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$ denote the half sum of positive roots. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$. Let $\mathcal{N}_\mathfrak{g}$ denote the category of Whittaker modules defined above. For $\mathfrak{g} = \mathfrak{sl}_\ell(\mathbb{C})$, we define a family of exact functors from the category $\mathcal{N}_\mathfrak{g}$ to the category of finite-dimensional modules for the graded affine Hecke algebra of the Coxeter system associated with $(\mathfrak{g}, \mathfrak{b})$. We prove (under natural assumptions) that standard objects are mapped to standard objects (or zero) and irreducible objects are mapped to irreducible objects (or zero). The category $\mathcal{N}_\mathfrak{g}$ contains the BGG category $\mathcal{O}$ as a full subcategory, and when we restrict to $\mathcal{O}$ we recover Arakawa-Suzuki functors [1]. When we restrict to finite-dimensional $\mathfrak{sl}_n(\mathbb{C})$-modules, we recover the classical Schur-Weyl duality between finite-dimensional representations of $SL_n(\mathbb{C})$ and finite-dimensional representations of the symmetric group S_n.

We will now briefly review the notation needed to define the functor. Let $\eta \in \mathrm{ch}\mathfrak{n} := (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ be a character of $\mathfrak{n}$, $\Pi_\eta = \{\alpha \in \Pi : \eta|_{\mathfrak{g}_\alpha} \neq 0\}$, and W_η be the Weyl group generated by the reflections s_α for $\alpha \in \Pi_\eta$. Let $\mathfrak{p}_\eta \subset \mathfrak{g}$ be the corresponding parabolic subalgebra containing $\mathfrak{b}$ with $\mathrm{ad}\mathfrak{h}$-stable Levi decomposition $\mathfrak{p}_\eta = \mathfrak{l}_\eta \oplus \mathfrak{n}^\eta$. Let $\mathfrak{z}$ be the center of the $\mathfrak{l}_\eta$ and $\mathfrak{s} := [\mathfrak{l}_\eta, \mathfrak{l}_\eta]$. Set $\mathfrak{b}_\eta = \mathfrak{b} \cap \mathfrak{l}_\eta$ and let $\mathfrak{n}_\eta$ be its nilradical. Let $Z(\mathfrak{l}_\eta)$ be the center of $U(\mathfrak{l}_\eta)$, and $\xi_\eta : \mathfrak{h}^* \to \mathrm{Max}Z(\mathfrak{l}_\eta)$ be induced by the relative

Harish-Chandra homomorphism for $U(\mathfrak{l}_\eta)$. Let $V = \mathbb{C}^n$ be the standard representation of $\mathfrak{g}$. For a locally $Z(\mathfrak{g})$-finite $U(\mathfrak{g})$-module X, let $X^{[\lambda]}$ denote the subspace consisting of vectors with generalized $Z(\mathfrak{g})$-infinitesimal character corresponding to $\lambda \in \mathfrak{h}^*$ via the Harish-Chandra homomorphism. Let

$$\mathcal{N}_{\mathfrak{g}}(\eta) = \{M \in \mathcal{N}_{\mathfrak{g}} : \forall\, m \in M \text{ and } u \in \mathfrak{n}, \exists\, k \in \mathbb{Z}_{\geq 0} \text{ so that } (u - \eta(u))^k m = 0\}.$$

For a $U(\mathfrak{g})$-module $X \in \mathcal{N}_{\mathfrak{g}}(\eta)$, let

$$X_{\lambda_{\mathfrak{z}}} = \{x \in X : \forall z \in \mathfrak{z}, \quad (z - \lambda(z))^k x = 0\}$$

denote the generalized $\mathfrak{z}$-weight space corresponding to $\lambda \in \mathfrak{h}^*$ restricted to $\mathfrak{z} \subset \mathfrak{h}$. Given a $U(\mathfrak{g})$-module X, define

$$H^0_\eta(\mathfrak{n}_\eta, X) := \{x \in X : ux - \eta(u)x = 0 \quad \forall u \in \mathfrak{n}_\eta\}.$$

A vector $v \in H^0_\eta(\mathfrak{n}_\eta, X)$ is called a Whittaker vector. For a $U(\mathfrak{g})$-module $X \in \mathcal{N}$ we define the following functor

$$F_{\ell,\eta,\lambda}(X) := H^0_\eta\left(\mathfrak{n}_\eta, \left(X \otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right),$$

for $\ell \in \mathbb{N}$, $\eta \in \operatorname{ch}\mathfrak{n}$, and $\lambda \in \mathfrak{h}^*$. Following [1], in Section 3.4 we define an action of the graded affine Hecke algebra $\mathbb{H}$ corresponding to the root datum for SL_ℓ on the $U(\mathfrak{g})$-module $X \otimes V^{\otimes \ell}$. The action of $\mathbb{H}$ commutes with the action of $U(\mathfrak{g})$, and induces an $\mathbb{H}$-module structure on $H^0_\eta\left(\mathfrak{n}_\eta, \left(X \otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right)$. We can therefore view $F_{\ell,\eta,\lambda}$ as a functor from $\mathcal{N}$ to the category of $\mathbb{H}$-modules. This family of functors posses a number of nice properties that we will explore in this paper and elsewhere. Our main result is the following (Theorem 5.2.4).

Theorem 1.1.1. *For $\lambda \in \mathfrak{h}^*$ dominant, $F_{\ell,\eta,\lambda}$ is an exact functor from $\mathcal{N}_{\mathfrak{g}}(\eta)$ to the category of finite-dimensional modules of the graded affine Hecke algebra corresponding to $\mathfrak{g}$. Moreover, if λ is a dominant integral weight such that $W_\eta = \{w \in W : w(\lambda + \rho) = \lambda + \rho\}$, and $X \in \mathcal{N}_{\mathfrak{g}}(\eta)$ is irreducible with infinitesimal character corresponding to λ, then $F_{n,\eta,\lambda}(X)$ is irreducible or zero.*

Chapter 6 describes ongoing research in collaboration with Anna Romanov. There we give an algebraic classification of contravariant forms on standard Whittaker modules. We show that (in general) Whittaker modules admit multiple linearly independent contravariant forms (Corollary 6.0.14).

Theorem 1.1.2. *Let $\eta \in ch\mathfrak{n}$, $\lambda \in \mathfrak{h}^*$, and Γ be the space of contravariant forms on the standard Whittaker module $std_{\mathcal{N}}(\lambda, \eta)$. Then*

$$dim\Gamma = |W_\eta|.$$

Chapter 2

Whittaker modules

In this chapter we will introduce the theory of Whittaker modules, as well as prove several results which will be needed in Chapter 5. We will begin by following the Miličić-Soergel [38] construction of the category of Whittaker modules. For $w \in W$ and $\lambda \in \mathfrak{h}^*$ define the dot action of W on $\mathfrak{h}^*$ by

$$w \bullet \lambda = w(\lambda + \rho) - \rho \quad \text{where} \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

The Poincaré-Birkhoff-Witt basis theorem for $U(\mathfrak{g})$ relative to the triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{h} \oplus \mathfrak{n}$ gives us the vector space decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n})$. The Harish-Chandra homomorphism

$$\xi^\sharp : Z(\mathfrak{g}) \to U(\mathfrak{h})$$

is the projection map from $Z(\mathfrak{g})$ to $U(\mathfrak{h})$ by the above decomposition. This induces a map

$$\xi : \mathfrak{h}^* \to \text{Max}Z(\mathfrak{g}),$$

where $\lambda \in \mathfrak{h}^*$ maps to $\ker(\lambda \circ \xi^\sharp)$. It is well known that $\xi(\lambda) = \xi(\mu)$ if and only if $W \bullet \lambda = W \bullet \mu$. As in the introduction, suppose $\eta \in \text{ch}\mathfrak{n} := (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ is a character of $\mathfrak{n}$ with corresponding set of simple roots $\Pi_\eta = \{\alpha \in \Pi : \eta|_{\mathfrak{g}_\alpha} \neq 0\}$ and Weyl group W_η generated by the reflections s_α for $\alpha \in \Pi_\eta$. Let $\mathfrak{p}_\eta \subset \mathfrak{g}$ be the corresponding parabolic subalgebra containing $\mathfrak{b}$ with ad$\mathfrak{h}$-stable Levi decomposition $\mathfrak{p}_\eta = \mathfrak{l}_\eta \oplus \mathfrak{n}^\eta$. Let $\bar{\mathfrak{n}}^\eta$ be the orthocomplement (with respect to the Killing form) of $\mathfrak{p}_\eta$ in $\mathfrak{g}$, so that we have the decomposition $\mathfrak{g} = \bar{\mathfrak{n}}^\eta \oplus \mathfrak{p}_\eta$. We will use $\mathfrak{z}$ to denote the center of $\mathfrak{l}_\eta$. Set $\mathfrak{b}_\eta = \mathfrak{b} \cap \mathfrak{l}_\eta$ and let $\mathfrak{n}_\eta$ be its nilradical. Let $\xi_\eta^\sharp : Z(\mathfrak{l}_\eta) \to S(\mathfrak{h})$ be the Harish-Chandra homomorphism of $\mathfrak{l}_\eta$. That is, the projection $Z(\mathfrak{l}_\eta)$ to $U(\mathfrak{h})$ induced by the decomposition $U(\mathfrak{l}_\eta) = U(\mathfrak{h}) \oplus (\bar{\mathfrak{n}}_\eta U(\mathfrak{l}_\eta) + U(\mathfrak{l}_\eta)\mathfrak{n}_\eta)$ given by the Poincaré-Birkhoff-Witt basis theorem for $U(\mathfrak{l}_\eta)$. Let $\xi_\eta : \mathfrak{h}^* \to \mathrm{Max}Z(\mathfrak{l}_\eta)$ be the map induced from $\xi_\eta^\sharp$. We say that an A-module is locally finite if the A-span of any element of the module is finite-dimensional.

We can now define the category of *Whittaker modules*, denoted by $\mathcal{N}_\mathfrak{g}$, to be the full subcategory of all $U(\mathfrak{g})$-modules which are finitely generated over $U(\mathfrak{g})$, locally finite over $U(\mathfrak{n})$, and locally finite over $Z(\mathfrak{g})$.

We have two decompositions for $\mathcal{N}$. The action of $Z(\mathfrak{g})$ decomposes $\mathcal{N}$ into a direct sum

$$\mathcal{N} = \bigoplus_{\chi \in \mathrm{Max}Z(\mathfrak{g})} \mathcal{N}(\chi),$$

where $\mathcal{N}(\chi)$ denotes modules in $\mathcal{N}$ that are annihilated by a power of χ. In other words, each object in $\mathcal{N}$ decomposes into a direct sum of modules with generalized infinitesimal character (see Proposition 2.0.3). Similarly, the action of $\mathfrak{n}$ gives us a decomposition

$$\mathcal{N} = \bigoplus_{\eta \in \mathrm{ch}\mathfrak{n}} \mathcal{N}(\eta),$$

where $\mathcal{N}(\eta) = \{M \in \mathcal{N} : \forall\, m \in M \text{ and } u \in \mathfrak{n}, \exists\, k \in \mathbb{Z}_{\geq 0} \text{ so that } (u - \eta(u))^k m = 0\}$. Combining these decompositions, we have

$$\mathcal{N} = \bigoplus_{\chi,\eta} \mathcal{N}(\chi, \eta),$$

where $\mathcal{N}(\chi,\eta) = \mathcal{N}(\chi) \cap \mathcal{N}(\eta)$. For each choice $\chi \in \mathrm{Max}Z(\mathfrak{g})$ and $\eta \in \mathrm{ch}\mathfrak{n}$, we see that the category $\mathcal{N}(\chi,\eta)$ contains the module $U(\mathfrak{g})/\chi U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\eta$. Therefore each category $\mathcal{N}(\chi,\eta)$ is non-empty. Moreover, each simple object L in $\mathcal{N}$ is contained in $\mathcal{N}(\chi,\eta)$ for some choice of χ and η depending on L.

Proposition 2.0.3. [38, Section 1] The categories $\mathcal{N}(\chi,\eta)$ are closed under subquotients and extensions in the category of $U(\mathfrak{g})$-modules.

The following propositions will be useful when proving that the functors defined in the introduction are exact.

Proposition 2.0.4. [38, Section 1] Every object $X \in \mathcal{N}$ admits a decomposition

$$X = \bigoplus_{[\lambda]} X^{[\lambda]},$$

where $X^{[\lambda]} \in \mathcal{N}(\xi(\lambda))$ and λ ranges over coset representatives of $W\backslash\mathfrak{h}^*$. Moreover, the functor from $\mathcal{N}$ to $\mathcal{N}(\xi(\lambda))$ which maps X to $X^{[\lambda]}$ is exact.

Theorem 2.0.5. *[36], [39, Theorem 2.5] Every object $X \in \mathcal{N}$ has finite length.*

Sketch of Proof. While McDowell proved this theorem algebraically in [36], this fact follows easily from the geometric perspective of [39]. Miličić and Soergel show that the Beilinson and Bernstein localization of an object $X \in \mathcal{N}$ with infinitesimal character $\xi(\lambda)$ is a holonomic $\mathcal{D}_\lambda$-module. Since we are assuming that each object in $\mathcal{N}$ is finitely generated and locally $Z(\mathfrak{g})$-finite, it follows that each object in $\mathcal{N}$ has finite length. □

Remark. We will use the notation $\mathcal{N}_\mathfrak{g}$ when we want to emphasize the semisimple complex Lie algebra $\mathfrak{g}$ for which we are considering the category of Whittaker modules.

2.1 Classification of irreducible Whittaker modules

In this section we will review the classification of standard and irreducible Whittaker modules [36]. This algebraic classification generalizes the theory of Verma modules, and uses induction to define standard objects which have unique irreducible quotients. We will begin by defining the modules studied in [32] for the Lie algebra $\mathfrak{l}_\eta$. For any ideal $I \subset Z(\mathfrak{l}_\eta)$ define the $U(\mathfrak{l}_\eta)$-module

$$Y(I,\eta) = U(\mathfrak{l}_\eta)/IU(\mathfrak{l}_\eta) \otimes_{U(\mathfrak{n}_\eta)} \mathbb{C}_\eta.$$

Building on work of Kostant, McDowell showed that $Y(I,\eta)$ is irreducible for each $I \in \mathrm{Max}Z(\mathfrak{l}_\eta)$ [32,36]. When $\eta = 0$, $\mathfrak{l}_\eta = \mathfrak{h}$ and $Y(I,\eta)$ is a one dimensional $\mathfrak{h}$-module with weight $\lambda \in \mathfrak{h}^*$ corresponding to the maximal ideal $I \in \mathrm{Max}S(\mathfrak{h})$. Following the classical theory of Verma modules, we define a $U(\mathfrak{g})$-module $\mathrm{std}_\mathcal{N}(\lambda,\eta)$ by algebraic induction from $\mathfrak{l}_\eta$ to $\mathfrak{g}$. Recall that $\xi_\eta(\lambda) \in \mathrm{Max}Z(\mathfrak{l}_\eta)$ is induced from the relative Harish-Chandra homomorphism on $U(\mathfrak{l}_\eta)$. Define the standard Whittaker module corresponding to the pair (λ,η) by

$$\mathrm{std}_\mathcal{N}(\lambda,\eta) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_\eta)} Y(\xi_\eta(\lambda),\eta),$$

where we extend the $\mathfrak{l}_\eta$ action on Y to an action of $\mathfrak{p}_\eta$ by letting $\mathfrak{n}^\eta$ act trivially.

Proposition 2.1.1. [38, Proposition 2.1]

(a) $\text{std}_{\mathcal{N}}(\lambda, \eta) \cong \text{std}_{\mathcal{N}}(\mu, \eta)$ if and only if $W_\eta \bullet \lambda = W_\eta \bullet \mu$,

(b) $\text{std}_{\mathcal{N}}(\lambda, \eta)$ has a unique simple quotient, denoted $\text{irr}_{\mathcal{N}}(\lambda, \eta)$, and

(c) $\text{Ann}_{U(\mathfrak{g})}\text{std}_{\mathcal{N}}(\lambda, \eta) = \xi(\lambda)U(\mathfrak{g})$.

Proposition 2.1.2. [38, Corollary 2.5] For all $\lambda \in \mathfrak{h}^*$ the module

$$U(\mathfrak{g})/\xi(\lambda)U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\eta \in \mathcal{N}(\xi(\lambda), \eta)$$

admits a filtration with subquotients isomorphic to $\text{std}_{\mathcal{N}}(w \bullet \lambda, \eta)$, where w ranges over any choice of coset representatives of $W_\eta \backslash W$.

We can conclude from Proposition 2.1.2 and Proposition 2.0.3 that $\text{std}_{\mathcal{N}}(\lambda, \eta)$ is an object in $\mathcal{N}(\xi(\lambda), \eta)$. If $\eta = 0$, then $\text{std}_{\mathcal{N}}(\lambda, 0)$ is the usual Verma module with highest weight λ, and $\mathcal{N}(\xi(\lambda), \eta = 0)$ contains the BGG category $\mathcal{O}_\lambda$ with generalized infinitesimal character $\xi(\lambda)$.

Theorem 2.1.3. *[36, Proposition 2.4] As $U(\mathfrak{l}_\eta)$-modules,*

$$std_{\mathcal{N}}(\lambda, \eta) \cong U(\bar{\mathfrak{n}}^\eta) \otimes_{\mathbb{C}} Y(\xi_\eta(\lambda), \eta).$$

Moreover, the center of $\mathfrak{l}_\eta$ (denoted $\mathfrak{z}$) acts semisimply on $std_{\mathcal{N}}(\lambda, \eta)$, and the $\mathfrak{z}$-weight spaces $U(\bar{\mathfrak{n}}^\eta)_{\gamma_\mathfrak{z}}$ are finite-dimensional $U(\mathfrak{l}_\eta)$-modules.

To avoid the unfortunate notational confusion between $\mathfrak{z}$-weight spaces and $\mathfrak{h}$-weight spaces, we will reluctantly use double subscrits and denote the generalized $\mathfrak{z}$-weight space of a module X by $X_{\lambda_\mathfrak{z}}$, for $\lambda \in \mathfrak{z}^*$. The generalized $\mathfrak{h}$-weight space corresponding to $\gamma \in \mathfrak{h}^*$ will be denoted by the usual notation X_γ (where γ has no subscript). The following theorem appears as Theorem 4.6 in [32], and will be crucial to our understanding of the structure of Whittaker modules.

Theorem 2.1.4. *[32, Theorem 4.6] Let F be a finite-dimensional $U(\mathfrak{g})$-module. Assume η is nondegenerate (i.e. $\mathfrak{l}_\eta = \mathfrak{g}$), and let Y be an irreducible Whittaker module in $\mathcal{N}(\xi(\lambda), \eta)$. Let $T = F \otimes Y$ be the tensor product $U(\mathfrak{g})$-module. Then T is an object in $\mathcal{N}(\eta)$ and composition series length equal to $dim F$. In particular,*

$$dim H^0_\eta(\mathfrak{n}, T) = dim F.$$

Suppose dim $F = k$. Then there is a composition series $0 = T_0 \subset T_1 \subset \cdots \subset T_k = T$ such that $T_i/T_{i-1} \cong Y(\xi(\lambda+\nu_i), \eta)$, where ν_i are the weights of F (counting multiplicities) ordered corresponding to a $\bar{\mathfrak{b}}$-invariant flag of F.

Observe that the above filtration is induced by a $\bar{\mathfrak{b}}$-invariant flag of F, rather than a $\mathfrak{b}$-invariant flag, as would be the case for category $\mathcal{O}$.

Corollary 2.1.5. [38, Lemma 5.12] Let F be a finite-dimensional $U(\mathfrak{g})$-module. Let η be any character of $\mathfrak{n}$. Let $T = F \otimes \mathrm{std}_{\mathcal{N}}(\lambda, \eta)$ for some $\lambda \in \mathfrak{h}^*$. Then T has a filtration with subquotients $\mathrm{std}_{\mathcal{N}}(\lambda + \nu, \eta)$ for weights ν of F (counting multiplicity).

Proof. We begin with the definition of standard Whittaker modules

$$T = F \otimes \mathrm{std}_{\mathcal{N}}(\lambda, \eta) = F \otimes \big(U(\mathfrak{g}) \otimes_{\mathfrak{p}_\eta} Y_{\mathfrak{l}_\eta}(\xi_\eta(\lambda), \eta)\big).$$

The Mackey isomorphism gives us the isomorphism of $U(\mathfrak{g})$-modules

$$T \cong U(\mathfrak{g}) \otimes_{\mathfrak{p}_\eta} \big(F \otimes Y_{\mathfrak{l}_\eta}(\xi_\eta(\lambda), \eta)\big).$$

Now we can apply Theorem 2.1.4 to $F \otimes Y_{\mathfrak{l}_\eta}(\xi_\eta(\lambda), \eta)$. Since parabolic induction is an exact functor, it descends to the level of Grothendieck groups. Therefore the composition factors of T are of the form $U(\mathfrak{g}) \otimes_{\mathfrak{p}_\eta} Y_{\mathfrak{l}_\eta}(\xi_\eta(\lambda + \nu), \eta) = \mathrm{std}_{\mathcal{N}}(\lambda + \nu, \eta)$ for weights ν of F (counting multiplicity). □

The following theorem concludes the classification of simple objects in $\mathcal{N}$.

Theorem 2.1.6. *[36], [38, Theorem 2.6] Each simple $U(\mathfrak{g})$-module contained in $\mathcal{N}$ is isomorphic to irr$_{\mathcal{N}}(\lambda, \eta)$ for some choice of $\eta \in$ ch$\mathfrak{n}$ and $\lambda \in \mathfrak{h}^*$.*

Proof. Each simple object L in $\mathcal{N}$ has an infinitesimal character and is contained in $\mathcal{N}(\xi(\lambda), \eta)$ for some $\lambda \in \mathfrak{h}^*$ and $\eta \in$ ch$\mathfrak{n}$. Therefore L is a subquotient of

$$U(\mathfrak{g})/\xi(\lambda)U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}_\eta.$$

Proposition 2.1.2 implies that L is a subquotient of $\mathrm{std}_{\mathcal{N}}(w \bullet \lambda, \eta)$ for some $w \in W$. Any simple subquotient of $\mathrm{std}_{\mathcal{N}}(w \bullet \lambda, \eta)$ must have a $\mathfrak{z}$-weight $\mu \in \mathfrak{z}^*$ so that $L_{\mu_\mathfrak{z}} \neq 0$ and $\mathfrak{n}^\eta L_{\mu_\mathfrak{z}} = 0$. There exists $I \in \mathrm{Max}\ Z(\mathfrak{l}_\eta)$ so that $\mathrm{Hom}_{\mathfrak{l}_\eta}(Y(I, \eta), L_{\mu_\mathfrak{z}}) \neq 0$. Therefore $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{std}_{\mathcal{N}}(\nu, \eta), L) \neq 0$ for $\nu \in \mathfrak{h}^*$ such that $\xi_\eta(\nu) = I$. This proves that $L \cong \mathrm{irr}_{\mathcal{N}}(\nu, \eta)$ since L is simple. □

Remark. It follows from Proposition 2.1.1 and Theorem 2.1.6 that the irreducible objects of $\mathcal{N}(\xi(\lambda), \eta)$ are parametrized by double cosets $W_\eta \backslash W / W_\lambda$, where $W_\lambda = \{w \in W : w \bullet \lambda = \lambda\}$.

2.2 Whittaker vectors and $\mathfrak{z}$-weight vectors of Whittaker modules

For a $U(\mathfrak{g})$-module X, let

$$X^{\mathfrak{n}} = \{x \in X | ux - \eta(u)x = 0 \text{ for all } u \in \mathfrak{n}\}$$

be the subspace of $\mathfrak{n}$-invariant vectors for the η-twisted action of $\mathfrak{n}$ on X. Let

$$H_\eta^\bullet(\mathfrak{n}, X)$$

denote η-twisted $\mathfrak{n}$-Lie algebra cohomology of X, i.e. the right derived functor of the $\mathfrak{n}$-invariants functor $X \mapsto X^{\mathfrak{n}}$ (for the η-twisted action of $\mathfrak{n}$ on X). An *η-Whittaker vector* (or Whittaker vector if the context is clear) of $X \in \mathcal{N}$ is a vector $v \in X$ so that $nv = \eta(n)v$ for all $n \in \mathfrak{n}$. Equivalently, an η-Whittaker vector of $X \in \mathcal{N}$ is a vector contained in $H_\eta^0(\mathfrak{n}, X)$.

Lemma 2.2.1. [39, Lemma 5.8] For $\mathfrak{g}$ semisimple and η nondegenerate ($\mathfrak{l}_\eta = \mathfrak{g}$), the functor $H_\eta^0(\mathfrak{n}, \cdot)$ from the category $\mathcal{N}(\eta)$ to the category of $Z(\mathfrak{g})$-modules is exact.

Sketch. First we prove that $H_\eta^i(\mathfrak{n}, V) = 0$ when $i \geq 1$ and V is an irreducible Whittaker module. Let N be a connected algebraic group with Lie algebra $\mathfrak{n}$, equipped with a morphism of N into the group of inner automorphisms of $\mathfrak{g}$ such that its differential induces an injection of $\mathfrak{n}$ into $\mathfrak{g}$. In [39], Miličić and Soergel show that an irreducible Whittaker module viewed as a $U(\mathfrak{n})$-module with η-twisted action of $\mathfrak{n}$, for nondegenerate η, is isomorphic to the differential of the natural action of N on the algebra of regular functions $R(C(w_0))$ on the open cell $C(w_0)$ of the flag variety of $\mathfrak{g}$. Under this isomorphism, the Whittaker vector generating the irreducible Whittaker module is mapped to the spherical vector $\mathbf{1} \in R(C(w_0))$. Notice that $C(w_0)$ is a single N orbit on the flag variety of $\mathfrak{g}$. Since, N and $C(w_0)$ are connected, N acts transitively on $C(w_0)$, and dim$N$ = dim$C(w_0)$, we can conclude that N and $C(w_0)$ are isomorphic as varieties. We can therefore consider the corresponding action of $\mathfrak{n}$ on the algebra of regular functions on N. We will proceed by induction on the dimension of $\mathfrak{n}$, following the proof of Lemma 1.9 in [37]. If dim$\mathfrak{n} = 1$, then

$\mathfrak{n}$ is abelian. Since N is an affine space, it follows that $R(N)$ is a polynomial algebra, and $H^i(\mathfrak{n}, R(N))$ is the cohomology of the Koszul complex with coefficients in $R(N)$. By the Poincaré lemma, we have that $H^i(\mathfrak{n}, R(N)) = 0$ for $i \geq 1$. If dim$\mathfrak{n} > 1$, we consider the commutator subgroup $N' = (N, N)$, with Lie algebra $\mathfrak{n}' \subsetneq \mathfrak{n}$. By the induction hypothesis, the Hochschild-Serre spectral sequence of Lie algebra cohomology collapses, giving us the equality

$$H^p(\mathfrak{n}/\mathfrak{n}', R(N/N')) = H^p(\mathfrak{n}, R(N)).$$

for $p \geq 0$. Since $\mathfrak{n}/\mathfrak{n}'$ is abelian, we can conclude (by the first part of the proof) that $H^p(\mathfrak{n}/\mathfrak{n}', R(N/N'))$ is zero for $p > 0$. Therefore $H^i_\eta(\mathfrak{n}, V) = 0$ for all $i \geq 1$ and irreducible Whittaker modules V. To complete the proof we will now consider an arbitrary Whittaker module V. Since V has finite length, we will proceed by induction on the length of V. Suppose V has length n, and consider the short exact sequence

$$0 \to S \to V \to I \to 0,$$

where S is a submodule with length $n-1$ and I is an irreducible quotient. By the strong induction assumption, $H^i_\eta(\mathfrak{n}, S) = 0$ for $i \geq 1$. The long exact sequence of Lie algebra cohomology shows that $H^i(\mathfrak{n}, V) = H^i(\mathfrak{n}, I)$ for $i \geq 1$. By induction on the length of V we see that $H^i(\mathfrak{n}, V) = 0$ for $i \geq 1$. □

Recall the Cartan decomposition $\mathfrak{l}_\eta = \bar{\mathfrak{n}}_\eta \oplus \mathfrak{h} \oplus \mathfrak{n}_\eta$ for the Levi subalgebra $\mathfrak{l}_\eta$, and the decomposition of $\mathfrak{g}$ given by considering the subalgebra $\mathfrak{p}_\eta$ and its orthocomplement $\bar{\mathfrak{n}}^\eta$ (with respect to the Killing form):

$$\mathfrak{g} = \bar{\mathfrak{n}}^\eta \oplus \mathfrak{p}_\eta = \bar{\mathfrak{n}}^\eta \oplus \mathfrak{l}_\eta \oplus \mathfrak{n}^\eta.$$

Let $\mathfrak{s} := [\mathfrak{l}_\eta, \mathfrak{l}_\eta]$ be the semisimple part of the Levi subalgebra $\mathfrak{l}_\eta$. For $\gamma \in \mathfrak{z}^*$, let

$$X_{\gamma_\mathfrak{z}} := \{x \in X : \forall z \in \mathfrak{z} \quad (z - \gamma_\mathfrak{z}(z))^k x = 0\}.$$

Proposition 2.2.2. Objects in $\mathcal{N}_\mathfrak{g}$ are locally $U(\mathfrak{z})$-finite. Moreover, for any $\gamma \in \mathfrak{z}^*$, the functor

$$\begin{array}{ccc} \mathcal{N}_\mathfrak{g}(\eta) & \to & \mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s}) \\ X & \mapsto & X_{\gamma_\mathfrak{z}} \end{array}$$

is exact. Here we use the notation $\eta_{\mathfrak{s}}$ to denote the restriction of η (viewed as a function on $\mathfrak{n}$) to $\mathfrak{n}_\eta = \mathfrak{n} \cap \mathfrak{s}$.

Proof. We will begin by showing that the functor described above is well defined. Our first step is to show that objects in $\mathcal{N}_{\mathfrak{g}}(\eta)$ are locally $U(\mathfrak{z})$-finite. Since every object $X \in \mathcal{N}(\eta)$ has finite length, we will argue by induction on the length of X. Consider a Jordan-Holder filtration of X:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = X$$

where V_i/V_{i-1} is irreducible. Assume X is irreducible. Then there exists λ so that the unique irreducible quotient of $\text{std}_{\mathcal{N}}(\lambda, \eta)$ is isomorphic to X. Since $\mathfrak{z}$ acts semisimply on $\text{std}_{\mathcal{N}}(\lambda, \eta)$, it will act semisimply on the quotient $\text{irr}_{\mathcal{N}}(\lambda, \eta)$, which is isomorphic to X. It follows that $U(\mathfrak{z})$ acts locally finitely on X. Now we will proceed with the inductive step. Assume that V_{n-1} is locally $U(\mathfrak{z})$-finite. Then

$$0 \to V_{n-1} \to V_n \to V_n/V_{n-1} \to 0$$

is an exact sequence of $U(\mathfrak{z})$-modules. We aim to show that $\dim U(\mathfrak{z})v < \infty$ for any $v \in V_n$. The above exact sequence restricts to

$$0 \to K \to U(\mathfrak{z})v \to I \to 0$$

where K =Ker: $U(\mathfrak{z})v \to V_n/V_{n-1}$ and I =Im: $U(\mathfrak{z})v \to V_n/V_{n-1}$. Since $U(\mathfrak{z})v$ is clearly finitely generated as a $U(\mathfrak{z})$-module, and the category of finitely generated $U(\mathfrak{z})$-modules is closed under extensions, we can conclude that K and I are finitely generated. Since K and I are both finitely $U(\mathfrak{z})$-generated $U(\mathfrak{z})$ submodules of locally finite $U(\mathfrak{z})$-modules, they are finite-dimensional. Therefore $\dim U(\mathfrak{z})v < \infty$. Therefore any object $X \in \mathcal{N}(\eta)$ is locally $U(\mathfrak{z})$-finite.

Since $\mathfrak{z}$ is a nilpotent Lie algebra that acts locally finitely on modules in $\mathcal{N}(\eta)$, given an exact sequence of Whittaker modules $0 \to A \to B \to C \to 0$, we get an exact sequence of $U(\mathfrak{z})$-modules by taking generalized weight spaces

$$0 \to A_{\gamma_{\mathfrak{z}}} \to B_{\gamma_{\mathfrak{z}}} \to C_{\gamma_{\mathfrak{z}}} \to 0$$

Here the morphisms are obtained by restricting the maps in the above exact sequence to the subspaces of generalized weight vectors.

We now turn our attention to showing that these generalized weight spaces are in fact Whittaker modules for the semisimple part of the Levi subalgebra. Again, for $X \in \mathcal{N}(\eta)$, consider a Jordan-Holder filtration of X:

$$0 = V_0 \subset V_1 \subset \cdots \subset V_N = X$$

where V_i/V_{i-1} is irreducible. We will show $X_{\gamma_\mathfrak{z}} \in \mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$ by induction on the length of X. Assume $N = 1$. Then X is irreducible. We aim to show $X_{\gamma_\mathfrak{z}} \in \mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. There exists λ so that the unique irreducible quotient of $\text{std}_\mathcal{N}(\lambda, \eta)$ is isomorphic to X. Theorem 2.1.3 implies that $\text{std}_\mathcal{N}(\lambda, \eta)_{\gamma_\mathfrak{z}} \cong F \otimes Y_\mathfrak{s}(\xi_\mathfrak{s}(\lambda), \eta_\mathfrak{s})$ as $U(\mathfrak{s})$-modules, where F is a finite-dimensional $U(\mathfrak{s})$-module. Since F is finite-dimensional, we can apply Theorem 4.6 of [32] and conclude that $\text{std}_\mathcal{N}(\lambda, \eta)_{\gamma_\mathfrak{z}}$ has finite length as a $U(\mathfrak{s})$-module, and has compositions factors contained in $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. Since $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$ is closed under extensions and $\text{std}_\mathcal{N}(\lambda, \eta)_{\gamma_\mathfrak{z}}$ has finite length, we can further conclude that $\text{std}_\mathcal{N}(\lambda, \eta)_{\gamma_\mathfrak{z}}$ is an object in $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. Since $\mathfrak{z}$ acts semisimply on $\text{std}_\mathcal{N}(\lambda, \eta)$, we get the following exact sequence of $U(\mathfrak{s})$-modules:

$$\text{std}_\mathcal{N}(\lambda, \eta)_{\gamma_\mathfrak{z}} \to X_{\gamma_\mathfrak{z}} \to 0.$$

Since $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$ is closed under quotients, we can conclude that $X_{\gamma_\mathfrak{z}} \in \mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. Now we will proceed with the inductive step. Assume that $(V_{n-1})_{\gamma_\mathfrak{z}} \in \mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. Then

$$0 \to (V_{n-1})_{\gamma_\mathfrak{z}} \to (V_n)_{\gamma_\mathfrak{z}} \to (V_n/V_{n-1})_{\gamma_\mathfrak{z}} \to 0$$

is an exact sequence of $U(\mathfrak{s})$-modules. Since $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$ is closed under extensions, and V_n/V_{n-1} is irreducible, we can conclude that $(V_n)_{\gamma_\mathfrak{z}} \in \mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. In summary, we have the following exact sequence of $U(\mathfrak{z})$-modules

$$0 \to A_{\gamma_\mathfrak{z}} \to B_{\gamma_\mathfrak{z}} \to C_{\gamma_\mathfrak{z}} \to 0$$

where the morphisms are restrictions of $U(\mathfrak{g})$-module morphisms, and the objects are naturally $U(\mathfrak{s})$-modules contained in $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. Therefore this is an exact sequence in $\mathcal{N}_\mathfrak{s}(\eta_\mathfrak{s})$. $\square$

2.3 The category of highest weight modules

If we specialize to the case when $\eta = 0$, we recover the category of highest weight modules. The standard objects $\text{std}_\mathcal{N}(\lambda, \eta = 0)$ in this category are the usual Verma modules, with corresponding unique irreducible quotients. Instead of retaining the cumbersome notation $\mathrm{std}_{\mathcal{N}}(\lambda, 0)$, we will use the standard notation $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ to denote a Verma module with highest weight $\lambda \in \mathfrak{h}^*$, and $L(\lambda)$ to denote the unique irreducible quotient of $M(\lambda)$. We will also use the abbreviated notation $\mathcal{O}'$ to denote the highest weight category ($\mathcal{O}' := \mathcal{N}(\eta = 0)$). Notice that the BGG category $\mathcal{O}$ is a full subcategory of $\mathcal{O}'$. In this section we will review the Kazhdan-Lusztig theory for highest weight modules.

Suppose $\Pi \subset \Delta^+$ is a set of simple and positive roots (respectively) of a root system $\Delta \subset \mathfrak{h}^*$. Let $S_\Pi \subset W$ be the set of reflections in $\mathfrak{h}^*$ through hyperplanes orthogonal to simple roots Π. We will use the notation $[X]$ for a $U(\mathfrak{g})$-module X in $\mathcal{O}'$ to denote the representative of X in the integral Grothendieck group of $\mathcal{O}'$.

We define the Iwahori-Hecke algebra $\mathscr{H}$ of a Coxeter system (W, S) to be the algebra over the ring of Laurent polynomials with integral coefficients and indeterminate $q^{1/2}$, with basis T_w for $\mathscr{H}$, indexed by $w \in W$, subject to the relations:

$$\begin{aligned} T_w T_y &= T_{wy} && \text{if } l(wy) = l(w) + l(y) \\ (T_s + 1)(T_s - q) &= 0 && \text{if } s \in S \end{aligned}$$

As $\mathbb{Z}[q^{-1/2}, q^{1/2}]$-modules, we have $\mathscr{H} \cong \bigoplus_{w \in W} \mathbb{Z}[q^{-1/2}, q^{1/2}] T_w$. We can further define an involution $x \mapsto \bar{x}$ of $\mathscr{H}$ by

$$\overline{\sum_{w \in W} p_w T_w} = \sum_{w \in W} \bar{p}_w T_{w^{-1}}^{-1}$$

where $p_w \in \mathbb{Z}[q^{-1/2}, q^{1/2}]$, $\overline{q^{1/2}} = q^{-1/2}$, and T_w^{-1} is the inverse of T_w in $\mathscr{H}$.

Theorem 2.3.1. *[24, Theorem 1.1] For any $w \in W$, there is a unique element $C_w \in \mathscr{H}$ such that*

$$\begin{aligned} \overline{C_w} &= C_w \\ C_w &= \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{(l(w)-l(y))/2} \overline{P_{y,w}} T_y \end{aligned}$$

where $y \leq w$ is the Bruhat order on W, $P_{y,w}$ is a polynomial in q of degree less than or equal to $\frac{1}{2}(l(w) - l(y) - 1)$ for $y < w$ and $P_{w,w} = 1$.

We can give an alternative geometric description of the Kazhdan-Lusztig polynomials $P_{y,w}$ using the intersection cohomology complex on the flag variety $\mathcal{B}$ (see [26] for an introduction to intersection cohomology complexes).

Theorem 2.3.2. *[24] For $y, w \in W$, such that $C(y) \subset \overline{C(w)}$, let $IC_x(C(w))$ denote the stalk at a point $x \in C(y) \subset \overline{C(w)}$ of the intersection cohomology complex supported on the closure of the Bruhat cell $C(w) \subset \mathcal{B}$. Then*

$$P_{y,w} = \sum_i q^i \dim H^{2i}(IC_x(C(w)))$$

for a chosen point $x \in C(y)$. The polynomial $P_{y,w}$ is independent of the choice of point $x \in C(y)$, since various choices will yield stalks with equal dimension.

The following is the famous Kazhdan-Lusztig conjecture for category $\mathcal{O}$, proved independently by Beilinson-Bernstein in [6] and Brylinski-Kashiwara in [11].

Theorem 2.3.3. *[6] [11] Suppose $\lambda \in \mathfrak{h}^*$ is regular, integral, and dominant. Let*

$$M(w \bullet \lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{w \bullet \lambda}$$

be the Verma module of highest weight $w \bullet \lambda$. Let w_0 be the longest element in W. In the Grothendieck group of $\mathcal{O}_\lambda$, we have the following equalities

$$\begin{aligned} [M(w \bullet \lambda)] &= \sum_{w \leq y} P_{w,y}(1)[L(y \bullet \lambda)] \\ [L(w \bullet \lambda)] &= \sum_{w \leq y} (-1)^{l(ww_0) - l(yw_0)} P_{yw_0, ww_0}(1)[M(y \bullet \lambda)] \end{aligned}$$

where $l(w)$ is the length of $w \in W$ and $P_{w,y}$ are the Kazhdan-Lusztig polynomials associated with the Coxeter system (W, S_Π).

Corollary 2.3.4. Suppose $\mu \in \mathfrak{h}^*$ is integral and dominant. In the Grothendieck group of $\mathcal{O}_\mu$ we have the following equality

$$[M(w \bullet \mu)] = \sum_{y \in I} P_{w,y}(1)[L(y \bullet \mu)]$$

where $I = \{y \geq w : y$ is the longest element in the coset $yW_\mu\}$.

Proof. We will begin by restating Theorem 2.3.3 for $\lambda = 0$:

$$[M(w \bullet 0)] = \sum_{y \geq w} P_{w,y}(1)[L(y \bullet 0)].$$

Now we will apply the translation functor T_0^μ which maps $X \in \mathcal{O}_0$ to $(X \otimes F_\mu)^{[\mu]}$, where F_μ is the finite-dimensional $U(\mathfrak{g})$-module with highest weight μ. It is well known ([22, Chapter 7]) that $T_0^\mu(M(w \bullet 0)) = M(w \bullet \mu)$ for all $w \in W$. Notice that if w and y are in the same coset $yW_\mu = wW_\mu$, then $T_0^\mu(M(w \bullet 0)) = T_0^\mu(M(y \bullet 0))$.

If there exists $y \in wW_\mu$ such that $l(y) \geq l(w)$, then there exists a simple reflection $s_\alpha \in W_\mu$ so that $L(w \bullet 0)$ is a subquotient of $M(w \bullet 0)/M(ws_\alpha \bullet 0)$. However, T_0^μ maps both $M(w \bullet 0)$ and $M(ws_\alpha \bullet 0)$ to $M(w \bullet \mu)$. Therefore $T_0^\mu(L(w \bullet 0)) = 0$.

Since T_0^μ is an exact functor which maps $M(w \bullet 0)$ to $M(w \bullet \mu)$, exactly one of the composition factors of $M(w \bullet 0)$ must be mapped to $L(w \bullet \mu)$. Suppose $L(y \bullet 0)$ is mapped to $L(w \bullet \mu)$ for some $y \notin wW_\mu$. Since $M(y \bullet 0)$ is mapped to $M(y \bullet \mu)$, and $L(y \bullet 0)$ is a composition factor of $M(y \bullet 0)$, we can conclude that $L(w \bullet \mu)$ is a composition factor of $M(y \bullet \mu)$, which is a contradiction. Therefore we can assume that $L(wx \bullet 0)$ is mapped to $L(w \bullet \mu)$ for some $x \in W_\mu$. But we have already shown that $T_0^\mu(L(wx \bullet \mu)) = 0$ if wx is not the longest element in wW_μ. Therefore the only option left is that $T_0^\mu(L(w \bullet 0)) = L(w \bullet \mu)$ for w the longest element in wW_μ. Finally, we have the equality

$$\begin{aligned}
[T_0^\mu(M(w \bullet 0))] &= \sum_{y \geq w} P_{w,y}(1)[T_0^\mu(L(y \bullet 0))] \\
[M(w \bullet \mu)] &= \sum_{y \in I} P_{w,y}(1)[L(y \bullet \mu)]
\end{aligned}$$

where $I = \{y \geq w : y \text{ is the longest element in } yW_\mu\}$. □

2.4 Backelin functors

The following functor developed by Backelin in [3] will be crucial to our proof that standard objects in the Whittaker category are mapped (by the functor described in the introduction) to standard objects (or zero) in the category of graded affine Hecke algebra modules.

The following construction is inspired by ideas in [32], and is developed in [3]. For $\gamma \in \mathfrak{h}^*$, let X_γ denote the generalized γ-weight space of X. Let $P(V)$ denote the set of nonzero generalized weights of a $U(\mathfrak{g})$-module V.

Definition. Let X be a $U(\mathfrak{g})$-module contained in $\mathcal{O}'$. The *completion module* of X, denoted $\overline{X}$, is the direct product

$$\overline{X} = \prod_{\gamma \in P(X)} X_\gamma$$

An element of $\overline{X}$ is a formal infinite sum

$$v = \sum_{\gamma \in P(X)} v_\gamma \in \overline{X}$$

where $v_\gamma \in X_\gamma$. We can extend the $U(\mathfrak{g})$-module structure of X to the completion by defining

$$u_\nu v := \sum_{\gamma \in P(X)} u_\nu v_{\gamma-\nu}$$

where $u_\nu \in U(\mathfrak{g})_\nu$ (the ν-weight space of $U(\mathfrak{g})$ viewed as a $\mathfrak{h}$ module with that ad action of $\mathfrak{h}$), and $u_\nu v_{\gamma-\nu} = 0$ if $\gamma - \nu \notin P(X)$.

Remark. If we consider duality in $\mathcal{O}'$, we can give an alternate description of $\overline{X}$. Let $Y' = \mathrm{Hom}_{\mathbb{C}}(Y, \mathbb{C})$ be the linear dual of a vector space, and $X^* = \bigoplus_\lambda X'_\lambda$ where X_λ is a generalized weight space of X. The $U(\mathfrak{g})$-module structure on X defines a $U(\mathfrak{g})$-module structure on X' and X^*. We have that $\overline{X} \cong (X^*)'$ as $U(\mathfrak{g})$-modules [3].

Lemma 2.4.1. Let F be a finite-dimensional $U(\mathfrak{g})$-module, and X an object in $\mathcal{O}'$. Then $F \otimes_{\mathbb{C}} \overline{X} = \overline{F \otimes_{\mathbb{C}} X}$.

Proof. Any element of $\overline{F \otimes X}$ is a formal infinite sum $\sum t_\tau$, where t_τ is a weight vector of $F \otimes X$ with weight τ. Each weight vector t_τ can be written as a sum of simple tensors of weight vectors of F and X:

$$t_\tau = \sum_{\alpha+\beta=\tau} f_\alpha \otimes x_\beta.$$

Therefore, we can write elements of $\overline{F \otimes X}$ as $\sum_J f_\alpha \otimes x_\beta$, where $J = P(F) \times P(X)$, $f_\alpha \in F$ is a weight vector of F, and $x_\beta \in X$ is a weight vector of X. Let $\{v_i : i \in I\}$ be a basis of F. We can write each element f_α as $f_\alpha = \sum_I a_i(\alpha) v_i$, where $a_i(\alpha) \in \mathbb{C}$. Therefore,

$$\sum_J f_\alpha \otimes x_\beta = \sum_J \sum_I a_i(\alpha) v_i \otimes x_\beta = \sum_I \sum_J a_i(\alpha) v_i \otimes x_\beta = \sum_I v_i \otimes \Big(\sum_J a_i(\alpha) x_\beta \Big).$$

We can rewrite the sum $\sum_J a_i(\alpha) x_\beta$ as an formal infinite sum of generalized weight vectors of X (for each i), since each x_β is already a generalized weight vector, and since there are only finitely many $(\alpha, \beta) \in J$ for a fixed β. This shows that

$$\sum_I v_i \otimes \big(\sum_J a_i(\alpha) x_\beta \big) \in F \otimes \overline{X}$$

since I is a finite set. Therefore $\overline{F \otimes X} \subseteq F \otimes \overline{X}$. The reverse inclusion is straightforward. □

Definition. We will define the subspace of η-finite vectors

$$X_\eta = \{x \in X : U_\eta(\mathfrak{n})^k x = 0 \quad \text{for some } k \in \mathbb{Z}_{\geq 0}\}$$

where $U_\eta(\mathfrak{n})$ is the kernel of $\eta : U(\mathfrak{n}) \to \mathbb{C}$.

Definition. Let $\overline{\Gamma}_\eta$ be the functor from the highest weight category $\mathcal{O}'$ to the category of Whittaker modules $\mathcal{N}(\eta)$ defined as follows:

$$\begin{aligned} \overline{\Gamma}_\eta : \mathcal{O}' &\to \mathcal{N}(\eta) \\ X &\mapsto \left(\overline{X}\right)_\eta \end{aligned}$$

Theorem 2.4.2. *[3, Proposition 6.9] $\overline{\Gamma}_\eta$ is an exact functor with the following properties*

(a) $\overline{\Gamma}_\eta(M(\lambda)) = \mathit{std}_{\mathcal{N}}(\lambda, \eta)$ *for all* $\lambda \in \mathfrak{h}^*$,

(b) $\overline{\Gamma}_\eta(L(\lambda)) = \mathit{irr}_{\mathcal{N}}(\lambda, \eta)$ *if* λ *is* $\mathfrak{n}_\eta$*-antidominant, and*

(c) $\overline{\Gamma}_\eta(L(\lambda)) = 0$ *if* λ *is not* $\mathfrak{n}_\eta$*-antidominant.*

The following lemma will be crucial to the proof of Theorem 2.4.2.

Lemma 2.4.3. [3, Lemma 6.5] For each Verma module $M(\lambda)$, we have the following equality of $U(\mathfrak{g})$-modules:

$$\overline{\Gamma}_\eta(M(\lambda)) = U(\mathfrak{g}) \cdot H^0_\eta(\mathfrak{n}, \overline{\Gamma}_\eta(M(\lambda))).$$

In other words, $\overline{\Gamma}_\eta(M(\lambda))$ is generated (as a $U(\mathfrak{g})$-module) by Whittaker vectors.

Proof. The proof follows directly from Theorem 3.8, Lemma 3.9, and Theorem 4.4 of [32]. □

Proposition 2.4.4. Let F be a finite-dimensional $U(\mathfrak{g})$-module and $\lambda \in \mathfrak{h}^*$. Then

$$\overline{\Gamma}_\eta(M(\lambda) \otimes F) = \overline{\Gamma}_\eta(M(\lambda)) \otimes F.$$

Proof. By Lemma 2.4.1, we have that $\overline{M(\lambda) \otimes F} = \overline{M(\lambda)} \otimes F$. If $v \in \overline{M(\lambda)}_\eta$, then $U_\eta(\mathfrak{n})^k(v \otimes f) = 0$ for any $f \in F$ and some $k >> 0$. Therefore, we have the inclusion $\overline{M(\lambda)}_\eta \otimes F \subseteq \overline{M(\lambda) \otimes F}_\eta$. From Corollary 2.1.5 and Theorem 2.4.2, we know that $\overline{M(\lambda)}_\eta \otimes F$ has a standard filtration with quotients $\mathrm{std}_{\mathcal{N}}(\lambda + \tau, \eta)$ for each $\tau \in P(F)$ (including multiplicity). Similarly, we have that $M(\lambda) \otimes F$ has a standard filtration with quotients $M(\lambda + \tau)$ for $\tau \in P(F)$. Since $\overline{\Gamma}_\eta$ is exact, and maps $M(\lambda + \tau)$ to $\mathrm{std}_{\mathcal{N}}(\lambda + \tau, \eta)$, we can conclude that $\overline{M(\lambda)}_\eta \otimes F$ and $\overline{M(\lambda) \otimes F}_\eta$ have isomorphic standard filtrations. □

2.5 Composition series of Whittaker modules

We can now use Theorem 2.4.2 to calculate the multiplicity of irreducible Whittaker modules in the composition series of standard Whittaker modules. Although Backelin's results apply to nonintegral weights, we will focus on the setting of integral weights.

Theorem 2.5.1. *[38], [3, Theorem 6.2] Assume $\lambda \in \mathfrak{h}^*$ is dominant and integral. In the integral Grothendieck group of $\mathcal{N}(\xi(\lambda), \eta)$, we have*

$$[std_{\mathcal{N}}(w \bullet \lambda, \eta)] = \sum_{y \in I} P_{w,y}(1)[irr_{\mathcal{N}}(y \bullet \lambda, \eta)]$$

where the sum is taken over $I = \{y \geq w : y \text{ is the longest element of } W_\eta y W_\lambda\}$ and $P_{w,y}$ are the Kazhdan-Lusztig polynomials of the Coxeter system (W, S_Π).

Proof. We can apply Corollary 2.3.4 to get an equality in the Grothendieck group of $\mathcal{O}'$:

$$[M(w \bullet \lambda)] = \sum_{y \in J} P_{w,y}(1)[L(y \bullet \lambda)],$$

where $J = \{y \geq w : y$ is the longest element in $yW_\lambda\}$.

Lemma 2.5.2. Suppose $\lambda \in \mathfrak{h}^*$ is integral and dominant, let y_λ be the longest element in the coset yW_λ, and ${}_\eta y_\lambda$ be the longest element in the double coset $W_\eta y W_\lambda$. Then $y_\lambda = {}_\eta y_\lambda$ if and only if $y \bullet \lambda$ is $\mathfrak{n}_\eta$-antidominant.

By Lemma 2.5.2 and Theorem 2.4.2, we get the following equalities in the integral Grothendieck group of $\mathcal{N}(\xi(\lambda), \eta)$:

$$[\mathrm{std}_{\mathcal{N}}(w \bullet \lambda, \eta)] = \sum_{y \in I} P_{w,y}(1)[\mathrm{irr}_{\mathcal{N}}(y \bullet \lambda, \eta)],$$

where $I = \{y \geq w : y = y_\lambda = {}_\eta y_\lambda\} = \{y \geq w : y$ is the longest element in $W_\eta y W_\lambda\}$.

□

Corollary 2.5.3. We can give an alternative description of the multiplicity formulas in terms of the geometry of the flag variety as follows.

$$[\mathrm{std}_{\mathcal{N}}(w \bullet \lambda, \eta) : \mathrm{irr}_{\mathcal{N}}(y \bullet \lambda, \eta)] = \sum_i \dim H^i(\mathrm{IC}_x(C({}_\eta y_\lambda)))$$

for $x \in C(w)$.

Chapter 3

Graded affine Hecke algebra modules

Hecke algebras appear naturally in the representation theory of algebraic groups over p-adic fields. Notably, affine Hecke algebras were used in the proof of the Deligne-Langlands conjecture for irreducible representations of p-adic groups in [25]. As motivation for the study of graded affine Hecke algebra, we will briefly review (following [47]) the local Langlands conjecture for split reductive groups over a non-archimedean local field F of characteristic 0, as well as the corresponding Deligne-Langlands conjecture.

Suppose $(X, R, Y, R^\vee)$ is the root datum of a split connected reductive group $G(F)$ over F. Let ${}^\vee G$ denote the complex reductive dual group, $\Gamma = \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group, $k = \mathrm{res}F$ the residue field of F with $|k| = q_F$, and $\mathrm{Fr} \in \mathrm{Gal}(\overline{k}/k)$ the Frobenius endomorphism. Consider the short exact sequence

$$1 \to I_F \to \Gamma \to \mathrm{Gal}(\overline{k}/k) \to 1$$

where I_F is the inertia subgroup of Γ. Define the Weil group W_F to be the inverse image of the group generated by the Frobenius endomorphism, so that

$$1 \to I_F \to W_F \xrightarrow{p} \mathbb{Z} \to 1.$$

Define the Weil-Deligne group by $W_F' = \mathbb{C} \rtimes W_F$ where $wzw^{-1} = q_F^{p(w)} z$ for $w \in W_F$ and $z \in \mathbb{C}$. A *Langlands parameter* is a continuous homomorphism $\phi : W_F' \to {}^\vee G$ such that $\phi(w)$ is semisimple for $w \in W_F$ and $\phi(z)$ is unipotent for $z \in \mathbb{C}$. Two Langlands parameters are equivalent if they are conjugate by the natural action of ${}^\vee G$. We say that ϕ is *unramified* if $\phi|_{I_F}$ is trivial. Notice that the unramified Langlands parameters are determined by pairs (s, N) with $s = \phi(\mathrm{Fr}) \in {}^\vee G$ semisimple and $N \in {}^\vee\mathfrak{g}$ nilpotent such that $\mathrm{Ad}(s)(N) = q_F N$. Alternatively, if ${}^\vee G(s)$ denotes the centralizer of s in ${}^\vee G$, the equivalence classes of unramified Langlands parameters are determined by ${}^\vee G(s)$ orbits on ${}^\vee\mathfrak{g}_{q_F}(s) = \{x \in {}^\vee\mathfrak{g} : \mathrm{Ad}(s)(x) = q_F x\}$.

In this setting the local Langlands conjecture asserts that for each equivalence class of Langlands parameters, denoted ϕ, there is an associated finite subset of irreducible admissible representations of $G(F)$, called the L-packet of ϕ. Moreover, these L-packets partition the set of equivalence classes of irreducible admissible representations of $G(F)$. The Deligne-Langlands conjecture, reformulated in this setting by Lusztig, further refines this association by identifying the irreducible unramified representations in each L-packet Π_ϕ. Let $Z_{{}^\vee G}(\phi)$ be the centralizer of a Langlands parameter ϕ in ${}^\vee G$. Let $A_{{}^\vee G}(\phi)$ be the component group of $Z_{{}^\vee G}(\phi)$. Then the Deligne-Langlands conjecture shows that the unramified representations contained in Π_ϕ are parametrized by irreducible representations of $A_{{}^\vee G}(\phi)$ which appear in the equivariant K-theory of $\mathcal{B}_\phi$, the variety of Borel subgroups of ${}^\vee G$ containing s and e^N.

Borel and Casselman developed a categorical equivalence between unramified admissible representations of $G(\mathbb{Q}_p)$ and finite-dimensional modules of the convolution algebra of compactly supported smooth functions on $G(\mathbb{Q}_p)$ which are I-bi-invariant for a fixed Iwahori subgroup I of $G(\mathbb{Q}_p)$. In [25], Kazhdan and Lusztig used the Borel-Casselman equivalence to prove the Deligne-Langlands conjecture in the setting of affine Hecke algebras.

A further reduction we will use is due to Lusztig [34]. By introducing a filtration on the affine Hecke algebra, Lusztig constructs a corresponding graded algebra, whose representation theory is closely related that of the affine Hecke algebra. The representation theory of the graded affine Hecke aglebra is in many ways easier to study and can be thought of as analogous to the Lie algebra of a Lie group. Specifically, the graded affine Hecke algebra can be studied using methods of equivariant homology. With these tools available, Lusztig was able to construct standard and irreducible modules for the graded affine Hecke algebra, as well as compute the composition series of standard modules in terms of intersection homology [35].

In this section, we will review an algebraic construction of standard and irreducible graded affine Hecke algebra modules due to Evens [17], as well as the corresponding composition series in terms of the geometric parametrization of standard modules due to Lusztig. Finally, we will discuss a useful parametrization of standard and irreducible modules in terms of combinatorial data due to Zelevinsky in the case where G is of type A_n [54].

3.1 Graded affine Hecke algebras

We will now define the graded affine Hecke algbera introduced by Lusztig [34]. Let $(X, R, Y, R^\vee, \Pi)$ be a based root datum, with $V^* = \mathbb{C} \otimes_{\mathbb{Z}} X$ and $V = \mathbb{C} \otimes_{\mathbb{Z}} Y$. Let W be the reflection group generated by simple reflections s_α for $\alpha \in \Pi$.

Definition. The graded (degenerate) affine Hecke algebra $\mathbb{H}$ of the based root datum $(X, R, Y, R^\vee, \Pi)$ is the unital associative algebra over $\mathbb{C}$ generated by $\{t_w : w \in W\}$ and $\{t_h : h \in V\}$ (with unit t_e, e the identity element of W), subject to the relations:

(a) The map $w \mapsto t_w$ from $\mathbb{C}[W]$ to $\mathbb{H}$ is an algebra injection,

(b) the map $h \mapsto t_h$ from $S(V)$ to $\mathbb{H}$ is an algebra injection, and

(c) the generators satisfy the following commutation relation

$$t_{s_\alpha} t_h - t_{s_\alpha(h)} t_{s_\alpha} = \langle \alpha, h \rangle \text{ for all } \alpha \in \Pi \text{ and } h \in V.$$

For notational convenience we will write w instead of t_w and h instead of t_h. Rewriting the commutation relations in condition (c), we get

$$s_\alpha h - s_\alpha(h) s_\alpha = \alpha(h) \text{ for all } \alpha \in \Pi \text{ and } h \in V.$$

Remark. As vector spaces, $\mathbb{H} \cong \mathbb{C}[W] \otimes S(V)$.

Let

$$\mathfrak{a}^* = \{x \in V^* : x(\alpha^\vee) = 0 \quad \forall \alpha \in \Pi\} \text{ and } \mathfrak{a} = \{x \in V : \alpha(x) = 0 \quad \forall \alpha \in \Pi\}.$$

We will define the based root datum $(X_{ss}, R, Y_{ss}, R^\vee, \Pi)$ by considering the subsets of X and Y which are perpendicular to $\mathfrak{a}^*$ and $\mathfrak{a}$ respectively. Let

$$X_{ss} = \{x \in X : x(a) = 0 \quad \forall a \in \mathfrak{a}\} \text{ and } Y_{ss} = \{y \in Y : a'(y) = 0 \quad \forall a' \in \mathfrak{a}^*\}.$$

Then $S(\mathfrak{a})$ is in the center of $\mathbb{H}$ and we have the decomposition

$$\mathbb{H} \cong \mathbb{H}_{ss} \otimes S(\mathfrak{a})$$

where $\mathbb{H}_{ss}$ is the graded affine Hecke algebra associated to the root datum

$$(X_{ss}, R, Y_{ss}, R^\vee, \Pi).$$

Lemma 3.1.1. [34, Lemma 4.5] The center $Z(\mathbb{H})$ of $\mathbb{H}$ is

$$Z(\mathbb{H}) = S(V)^W.$$

Moreover, central characters of $\mathbb{H}$ (and maximal ideals of $S(V)^W$) are parametrized by W orbits of $\lambda \in V^*$ (with the usual action $w\lambda$ of W on V^*). Let χ_λ denote the maximal ideal in $S(V)^W$ corresponding to $\lambda \in V^*$.

Let $\mathcal{H}$ denote the category of finite-dimensional $\mathbb{H}$-modules and $\mathcal{H}_\lambda$ denote the subcategory of finite-dimensional $\mathbb{H}$-modules with central character corresponding to the maximal ideal χ_λ.

3.2 Classification of irreducible modules

In this section, we will review three types of classifications of (finite-dimensional) simple $\mathbb{H}$-modules. First, we review the geometric construction of standard $\mathbb{H}$-modules, following [35]. These $\mathbb{H}$-modules have unique irreducible quotients and parametrize isomorphism classes of simple $\mathbb{H}$-modules. We will then review the algebraic approach of [17], parametrizing simple modules by pairs $(\mathbb{H}_\mathfrak{p}, U)$, where $\mathbb{H}_\mathfrak{p}$ is a parabolic subalgebra of $\mathbb{H}$ and U is a tempered representation of $\mathbb{H}_\mathfrak{p}$. Finally, we will consider the combinatorial parametrization developed in [4] for the case when $\mathbb{H}$ corresponds to root datum of type A_n.

3.2.1 Geometric classification

Let G be the connected complex reductive group with root datum $(X, R, Y, R^\vee, \Pi)$, with Lie algebra $\mathfrak{g}$, and flag variety $\mathcal{B}$. For $\sigma \in V$, let $L_\sigma = \{g \in G : \mathrm{Ad}(g)(\sigma) = \sigma\}$ and $\mathfrak{g}_1(\sigma) = \{x \in \mathfrak{g} : \mathrm{ad}(x)(\sigma) = \sigma\}$. Let $\mathcal{N} := \{x \in \mathfrak{g} : \mathrm{ad}(x) \text{ is nilpotent}\}$ denote the nilpotent cone in $\mathfrak{g}$. Consider the Springer resolution of the nilpotent cone

$$\begin{array}{rcl} \widetilde{\mathcal{N}} := \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} : x \in \mathfrak{b}\} & \xrightarrow{\mu} & \mathcal{N} \\ (x, \mathfrak{b}) & \mapsto & x \end{array}$$

Let $\mathcal{N}^\sigma$ denote the subvariety of $\mathcal{N}$ fixed by $\mathrm{ad}(\sigma)$, $\widetilde{\mathcal{N}}^\sigma = \{(x, \mathfrak{b}) \in \mathcal{N}^\sigma \times \mathcal{B} : x, \sigma \in \mathfrak{b}\}$, and μ^σ denote the restriction of μ to $\widetilde{\mathcal{N}}^\sigma$. Let $\mathcal{C}_{\widetilde{\mathcal{N}}^\sigma}$ denote the constant perverse sheaf on $\widetilde{\mathcal{N}}^\sigma$. Let $\mathcal{P}^g_{\mathbb{H}}(\sigma)$ denote the set of pairs $(\mathcal{O}, \mathcal{E})$ such that $\mathcal{O}$ is an L_σ orbit in $\mathfrak{g}_1(\sigma)$ and $\mathcal{E}$ is an L_σ-equivariant local system on $\mathcal{O}$ such that $\mathrm{IC}(\mathcal{O}, \mathcal{E})$ appears in the decomposition of $\mu^\sigma_* \mathcal{C}_{\widetilde{\mathcal{N}}^\sigma}$. In [35] (cf. [14]), Lusztig constructs an action of $\mathbb{H}$ on the vector space $H^\bullet(i^!_x(\mu^\sigma_*(\mathcal{C}_{\widetilde{\mathcal{N}}^\sigma})))^\chi$, where $i_x : \{x\} \hookrightarrow \mathfrak{g}$ with $x \in \mathfrak{g}_1(\sigma)$, and χ is a representation of the component group of $Z_G(\sigma, x)$, the simultaneous centralizer of σ and x in G, corresponding to the local system $\mathcal{E}$. Since μ^σ is a proper map, we can identify the above vector space with the homology of the fiber of μ^σ at x. Let $\widetilde{\mathcal{N}}^\sigma_x$ denote the fiber of μ^σ at $x \in \mathfrak{g}_1(\sigma) \subset \mathcal{N}^\sigma$. As (non-graded) vector spaces, we have

$$H_\bullet(\widetilde{\mathcal{N}}^\sigma_x)^\chi \cong H^\bullet(i^!_x(\mu^\sigma_*(\mathcal{C}_{\widetilde{\mathcal{N}}^\sigma})))^\chi.$$

The $\mathbb{H}$-module $H_\bullet(\widetilde{\mathcal{N}}^\sigma_x)^\chi$ only depends on the L_σ orbit of x and the local system $\mathcal{E}$. Therefore, we will denote this module by $\mathrm{std}_{\mathbb{H}}(\mathcal{O}, \mathcal{E})$, and refer to it as a standard $\mathbb{H}$-module corresponding to the geometric parameters $(\mathcal{O}, \mathcal{E}) \in \mathcal{P}^g_{\mathbb{H}}(\sigma)$.

Theorem 3.2.1. *[33] Each simple $\mathbb{H}$-module is isomorphic to the quotient of a standard $\mathbb{H}$-module.*

Theorem 3.2.2. *[35, Corollary 8.18] Let $\lambda \in \mathfrak{h}^*$ be dual to $\sigma \in \mathfrak{h}$ by the trace form on $\mathfrak{g}$. The set of isomorphism classes of simple $\mathbb{H}$-modules with central character χ_λ is naturally in 1 to 1 correspondence with $\mathcal{P}^g_{\mathbb{H}}(\sigma)$.*

We will therefore use the notation $\mathrm{irr}_{\mathbb{H}}(\mathcal{O}, \mathcal{E})$ to denote the irreducible module corresponding to parameter $(\mathcal{O}, \mathcal{E}) \in \mathcal{P}^g_{\mathbb{H}}(\sigma)$.

3.2.2 Algebraic classification

Consider a subset $\Pi_{\mathfrak{p}}$ of Π, and the corresponding roots (coroots) $R_{\mathfrak{p}}$ (resp. $R^\vee_{\mathfrak{p}}$) generated by α (resp. $\alpha^\vee$) for $\alpha \in \Pi_{\mathfrak{p}}$. Then $(X, R_{\mathfrak{p}}, Y, R^\vee_{\mathfrak{p}}, \Pi_{\mathfrak{p}})$ is a root datum. Let $\mathbb{H}_{\mathfrak{p}}$ be the graded affine Hecke algebra associated to the root datum $(X, R_{\mathfrak{p}}, Y, R^\vee_{\mathfrak{p}}, \Pi_{\mathfrak{p}})$. Let $\mathfrak{a}$ be as in 3.1, and $\mathbb{H}_{\mathfrak{s}}$ denote corresponding subalgebra in the decomposition

$$\mathbb{H}_{\mathfrak{p}} = \mathbb{H}_{\mathfrak{s}} \otimes S(\mathfrak{a}).$$

Theorem 3.2.3. *[17, Theorem 2.1]*

(a) *Let V be an irreducible $\mathbb{H}$-module. Then V is a quotient of $\mathbb{H}\otimes_{\mathbb{H}_{\mathfrak{p}}} U$, where $U = \tilde{U} \boxtimes \mathbb{C}_\nu$ is such that $\tilde{U}$ is a tempered $\mathbb{H}_{\mathfrak{s}}$-module and $\mathbb{C}_\nu$ is a character of $S(\mathfrak{a})$ with $\nu \in \mathfrak{a}^*$ and $Re\langle \nu, \alpha\rangle > 0$ for all $\alpha \in \Pi - \Pi_{\mathfrak{p}}$. We will refer to $\mathbb{H} \otimes_{\mathbb{H}_{\mathfrak{p}}} U$ a standard module, and denote it by $std_{\mathbb{H}}(\mathbb{H}_{\mathfrak{p}}, U)$.*

(b) *If U is as in (a), then $\mathbb{H} \otimes_{\mathbb{H}_{\mathfrak{p}}} U$ has a unique irreducible quotient, which we will denote by $irr_{\mathbb{H}}(\mathbb{H}_{\mathfrak{p}}, U)$.*

(c) *If $irr_{\mathbb{H}}(\mathbb{H}_{\mathfrak{p}}, U) \cong irr_{\mathbb{H}}(\mathbb{H}_{\mathfrak{p}'}, U')$, then $\Pi_{\mathfrak{p}} = \Pi_{\mathfrak{p}'}$ and $U \cong U'$.*

3.2.3 Combinatorial classification

Let $\mathfrak{g} = \mathfrak{sl}_\ell(\mathbb{C})$. For convenience, let $\mathfrak{h}$ be the Cartan subalgebra consisting of diagonal matrices in $\mathfrak{g}$, and $\mathfrak{b}$ be the Borel subalgebra consisting of upper triangular matrices in $\mathfrak{g}$. It is occasionally useful to represent elements in $\mathfrak{h}$ using the standard basis ϵ_i of $\mathfrak{t}$, where $\mathfrak{t}$ is the space of diagonal matrices in $\mathfrak{gl}_\ell(\mathbb{C})$, and ϵ_i is the matrix with 1 in the ith diagonal entry, and 0 elsewhere. Let $\mathbb{H}$ be the graded affine Hecke algebra of a root datum associated with $(\mathfrak{g}, \mathfrak{b})$.

Remark. When we want to emphasize that we are considering the graded affine Hecke algebra corresponding to the root datum of $\mathfrak{sl}_k(\mathbb{C})$ for some k, we will use the notation $\mathbb{H}_k$.

Finite-dimensional irreducible $\mathbb{H}$-modules are parametrized by combinatorial objects which we will refer to as multisegments. Define a segment to be a finite sequence of complex numbers $\{a_n\}$, such that any two consecutive terms differ by 1, i.e. $a_n - a_{n-1} = 1$ for all n. A multisegment is a finite ordered collection of segments. Define the support $\underline{\tau}$ of a multisegment τ to be the multiset of all elements (so as to keep track of multiplicity) of all segements of the multisegment τ. Since we are considering $\mathfrak{sl}_\ell$, we will only require multisegments with zero trace. More precisely, let MS be the set of multisegments τ such that $\sum_{x\in\underline{\tau}} x = 0$. For $\lambda \in \mathfrak{h}^*$, let $\lambda' \in \mathfrak{h}$ be the image of λ when we identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using the trace form. For $\lambda \in \mathfrak{h}^*$, set

$$\text{MS}(\lambda) = \{\tau \in \text{MS} | \underline{\tau} = \underline{\lambda}\}$$

where we view $\underline{\lambda}$ as a multiset consisting of the diagonal entries of $\lambda' \in \mathfrak{h}$. If $\tau, \sigma \in MS(\lambda)$, then define an equivalence relation by $\tau \sim \sigma$ if τ and σ have the same segments (with a possibly different ordering). Let $MS_\circ(\lambda)$ denote the set of equivalence classes of $MS(\lambda)$. We will proceed by building a standard object in $\mathbb{H}$ from a class of multisegments $\tilde{\tau} \in MS_\circ(\lambda)$. Let $\tilde{\tau}$ be represented by $\tau = \{\{a_1, a_1+1, a_1+2, \cdots, a_1+(l_1-1)\}, \cdots, \{a_r, a_r+1, a_r+2, \cdots, a_r+(l_r-1)\}\} \in MS(\lambda)$ where $\mathrm{Re}(a_i + \frac{1}{2}(l_i-1)) \geq \mathrm{Re}(a_{i+1} + \frac{1}{2}(l_{i+1}-1))$ for all i. Consider $\mathfrak{sl}(l_1) \oplus \cdots \oplus \mathfrak{sl}(l_r)$ as a block diagonal subalgebra of $\mathfrak{g}$. Let $R_\mathfrak{l}$, $R_\mathfrak{l}^\vee$, and $\Pi_\mathfrak{l}$ be the set of roots, co-roots, and simple roots for $\mathfrak{sl}(l_1) \oplus \cdots \oplus \mathfrak{sl}(l_r)$, respectively, chosen so that $\Pi_\mathfrak{l} \subset \Pi$. Then the graded affine Hecke algebra (denoted $\mathbb{H}_\mathfrak{p}$) associated with the root datum $(X, R_\mathfrak{l}, Y, R_\mathfrak{l}^\vee, \Pi_\mathfrak{l})$ decomposes as

$$\mathbb{H}_\mathfrak{p} = \mathbb{H}_{ss} \otimes S(\mathfrak{a})$$

where $\mathbb{H}_{ss} \cong \mathbb{H}_{l_1} \otimes \cdots \otimes \mathbb{H}_{l_r}$ is isomorphic to the graded affine Hecke algebra of the root datum associated with $\mathfrak{sl}(l_1) \oplus \cdots \oplus \mathfrak{sl}(l_r)$ and $S(\mathfrak{a})$ is as in Section 3.1.

Let $\gamma_i = \sum_{k=1}^{l_i}(a_i + k - 1)\epsilon_k \in \mathfrak{t}^*$ be viewed as an element in $\mathfrak{h}_{l_i}^*$, the dual of the Cartan subalgebra of diagonal matrices intersected with the $\mathfrak{sl}(l_i)$ block of $\mathfrak{g}$. Define the discrete series representation $\delta_{\tilde{\tau}}$ of $\mathbb{H}_{ss}$ to be

$$\delta_{\tilde{\tau}} = \delta_1 \boxtimes \cdots \boxtimes \delta_r$$

where $\delta_i = \mathbb{C}_{\gamma_i}$ is the one dimensional representation of $\mathbb{H}_{l_i}$ (the graded affine Hecke algebra of the root datum associated with the algebra $\mathfrak{sl}(l_i)$) where $\mathfrak{h}_{l_i}$ acts by weight γ_i and W_{l_i} acts by the sign representation. We will denote the standard $\mathbb{H}_\ell$-module corresponding to $\tilde{\tau}$ by

$$\mathrm{std}_\mathbb{H}(\tilde{\tau}) = \mathbb{H}_\ell \otimes_{\mathbb{H}_\mathfrak{p}} (\delta_{\tilde{\tau}} \boxtimes \mathbb{C}_\gamma)$$

where $\mathbb{C}_\gamma$ is the character of $S(\mathfrak{a})$ given by $\gamma = \sum \gamma_i \in \mathfrak{h}^*$ restricted to $\mathfrak{a}$. Notice that the module $\mathrm{std}_\mathbb{H}(\tilde{\tau})$ is generated as a $\mathbb{C}[W]$-module by the vector

$$\mathbb{1} = e \otimes 1_1 \otimes \cdots \otimes 1_r \otimes 1_\nu$$

where $e \in \mathbb{H}$ is the identity element, $1_i \in \delta_i$ is the identity in $\mathbb{C}_{\gamma_i}$, and 1_ν is the identity in $\mathbb{C}_\nu$. Additionally $\mathbb{1}$ is an $\mathfrak{h}$-weight vector with weight $\zeta_{\tilde{\tau}}$ given by:

$$\zeta_{\tilde{\tau}}(e_j^\vee) = a_i + j - \sum_{k=1}^{i-1} l_k - 1 \quad \text{for } \sum_{k=1}^{i-1} l_k < j \leq \sum_{k=1}^{i} l_k.$$

Here it is notationally easiest to define $\zeta_{\tilde{\tau}}$ as an element of $\mathfrak{t}^*$, but we will only consider the restriction of $\zeta_{\tilde{\tau}}$ to $\mathfrak{h}$.

Proposition 3.2.4. The standard $\mathbb{H}_\ell$-module $\mathrm{std}_{\mathbb{H}}(\tilde{\tau})$ has a unique simple quotient denoted $\mathrm{irr}_{\mathbb{H}}(\tilde{\tau})$.

Proof. This follows directly from Theorem 3.2.3, which states that $\mathrm{std}_{\mathbb{H}}(\tilde{\tau})$ has a unique irreducible quotient if $\gamma = \sum \gamma_i$ satisfies

$$\mathrm{Re}(\gamma(\alpha^\vee)) \geq 0 \quad \forall \alpha \in \Pi - \Pi_{\mathfrak{l}}.$$

This is guaranteed by our choice of multisegment representative of $\tilde{\tau}$ chose to satisfy the condition

$$\mathrm{Re}\big(a_i + \frac{1}{2}(l_i - 1)\big) \geq \mathrm{Re}\big(a_{i+1} + \frac{1}{2}(l_{i+1} - 1)\big)$$

for all i. □

Theorem 3.2.5. *[4] Suppose that $\lambda \in \mathfrak{h}^*$ corresponds to $\sigma \in \mathfrak{h}$ by the trace form on $\mathfrak{g}$. There is a one-to-one correspondence between multisgements $MS_\circ(\lambda)$ and geometric parameters $\mathcal{P}^g_{\mathbb{H}}(\sigma)$.*

Now we will construct an $\mathbb{H}_\ell$-module from a pair $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu \in P(V^{\otimes \ell}) = \{\gamma \in \mathfrak{h}^* : (V^{\otimes \ell})_\gamma \neq 0\}$. There exists $(\ell_1, \cdots, \ell_n) \in \mathbb{Z}^n_{\geq 0}$ so that $\ell = \sum_i \ell_i$ and

$$\lambda - \mu \equiv \sum_{i=1}^{n} \ell_i \epsilon_i \bmod \mathbb{C}\left(\sum_{i=1}^{n} \epsilon_i\right) \tag{3.1}$$

We will now define the following multisegment corresponding to the pair $\lambda, \mu \in \mathfrak{h}^*$, with λ dominant.

$$\delta_{\lambda,\mu} = \{\{(\mu+\rho)(\epsilon_1^\vee), \cdots, (\mu+\rho)(\epsilon_1^\vee)+\ell_1-1\}, \cdots, \{(\mu+\rho)(\epsilon_n^\vee), \cdots, (\mu+\rho)(\epsilon_n^\vee)+\ell_n-1\}\}$$

and set $\mathrm{std}_{\mathbb{H}}(\lambda, \mu) := \mathrm{std}_{\mathbb{H}}(\tilde{\delta}_{\lambda,\mu})$. The standard module $\mathrm{std}_{\mathbb{H}}(\lambda, \mu)$ is a cyclic module with a cyclic weight vector $\mathbb{1}$, whose weight $\zeta_{\lambda,\mu}$ is given by

$$\zeta_{\lambda,\mu}(\epsilon_j^\vee) = (\mu + \rho)(\epsilon_i^\vee) + j - \sum_{r=1}^{i-1} \ell_r - 1 \quad \text{for } \sum_{r=1}^{i-1} \ell_r < j \leq \sum_{r=1}^{i} \ell_r.$$

Since λ is dominant, $\mathrm{std}_{\mathbb{H}}(\lambda, \mu)$ has a unique simple quotient denoted $\mathrm{irr}_{\mathbb{H}}(\lambda, \mu)$.

Lemma 3.2.6. [1, Lemma 3.3.2] Let $P(V^{\otimes \ell})$ be the set of non-trivial weights of $V^{\otimes \ell}$. If λ and μ are integral weights such that $\lambda - \mu \in P(V^{\otimes \ell})$, then

$$\dim \mathbf{std}_{\mathbb{H}}(\lambda, \mu) = \dim \left(V^{\otimes \ell}\right)_{\lambda-\mu}.$$

Remark. Notice that the multisegment $\delta_{\lambda, w \circ \lambda}$ is an element of $MS(\lambda + \rho)$, and the $\mathbb{H}_\ell$-module $\mathrm{std}_{\mathbb{H}}(\lambda, w \bullet \lambda)$ has central character $\chi_{\lambda+\rho}$.

3.3 The composition series of standard modules

We will now review the p-adic analogue of the Kazhdan-Lusztig conjectures. Using Lusztig's geometric realization of $\mathbb{H}$-modules [35](cf. Section 3.2.1), we can relate the multiplicity of simple $\mathbb{H}$-modules in the composition series of standard $\mathbb{H}$-modules to stalks of intersection cohomology complexes.

Theorem 3.3.1. *[35, Corollary 10.7] Suppose we have two geometric parameters*

$$(\mathcal{O}, \mathcal{E}), (\mathcal{O}', \mathcal{E}') \in \mathcal{P}^g_{\mathbb{H}}(\sigma)$$

such that $\mathcal{O} \subset \overline{\mathcal{O}'}$. Let $IC_y(\mathcal{O}', \mathcal{E}')$ denote the stalk at $y \in \mathcal{O} \subset \overline{\mathcal{O}'}$ of the intersection cohomology complex on $\mathcal{O}'$ corresponding to the local system $\mathcal{E}'$. Let $H^i(IC_y(\mathcal{O}', \mathcal{E}'))_{\mathcal{E}}$ denote the $\mathcal{E}$ isotypic component of $H^i(IC_y(\mathcal{O}', \mathcal{E}'))$, where we view $\mathcal{E}$ as a representation of the component group of $Z_G(\sigma, x)$ (where σ and x are as in Section 3.2.1). Then

$$[\mathit{std}_{\mathbb{H}}(\mathcal{O}, \mathcal{E}) : \mathit{irr}_{\mathbb{H}}(\mathcal{O}', \mathcal{E}')] = \sum_i \dim H^i(IC_y(\mathcal{O}', \mathcal{E}'))_{\mathcal{E}}.$$

When $\mathbb{H}$ is a graded affine Hecke algebra corresponding to a root datum of type A_n, then we can reformulate the multiplicity formula in terms of the combinatorial parametrization of [4], thus proving a conjecture of Zelevinsky [54]. We will thus proceed with the notation of Section 3.2.3. Suppose $\tau \in \mathrm{MS}_\circ(\lambda)$ is a multisegment consisting of segments of size l_1 through l_r. Let $\sigma \in \mathfrak{h}$ correspond to $\lambda \in \mathfrak{h}^*$ by the trace form. Let $x_\tau \in \mathfrak{g}_1(\sigma)$ denote the nilpotent matrix in $\mathfrak{g}$ with a nonzero entry in the i^{th} row and $(i+1)^{\text{th}}$ column for each $\sum_{j=1}^{k} l_j \leq i < \sum_{j=1}^{k+1} l_j$, and zero entries elsewhere. Let X_τ be the L_σ orbit of x_τ. This gives us a bijection between $\mathrm{MS}_\circ(\lambda)$ and L_σ orbits on $\mathfrak{g}_1(\sigma)$ (cf. [53], [16, Equation 4.2])

$$\begin{array}{ccc} \mathrm{MS}_\circ(\lambda) & \longleftrightarrow & L_\sigma \backslash \mathfrak{g}_1(\sigma) \\ \tau & \mapsto & X_\tau. \end{array}$$

For $G = SL_n$, we have that the component group of $Z_G(\sigma, x)$ is non trivial. However, the only irreducible L_σ-equivariant local system which appears in the decomposition of $\mu_!^\sigma \mathcal{C}_{\widetilde{N}^\sigma}$ is the trivial local system. So the multisegment $\tau \in \mathrm{MS}_\circ(\lambda)$ corresponds to geometric parameter $(X_\tau, \mathbb{C}_{X_\tau}) \in \mathcal{P}_{\mathbb{H}}^g(\sigma)$.

Corollary 3.3.2. For $\tau, \gamma \in \mathrm{MS}_\circ(\lambda)$, we have that

$$[\mathrm{std}_{\mathbb{H}}(\tau) : \mathrm{irr}_{\mathbb{H}}(\gamma)] = \sum_{i \geq 0} \dim H^i(\mathrm{IC}_y(X_\gamma, \mathbb{C}_{X_\gamma}))$$

for $y \in X_\tau$.

3.4 Action of the graded affine Hecke algebra on $X \otimes V^{\otimes \ell}$

Throughout this section, let X be a $U(\mathfrak{g})$-module. Let $B = \{E_i\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the trace form. For notational convenience, let

$$\begin{aligned}
\tau_r(x) &= 1^{\otimes r-1} \otimes x \otimes 1^{\otimes \ell - r+1} \in U(\mathfrak{g})^{\otimes \ell+1}, \\
\tau_{r,s}(y,z) &= 1^{\otimes r-1} \otimes y \otimes 1^{\otimes s-r-1} \otimes z \otimes 1^{\otimes \ell - s+1} \in U(\mathfrak{g})^{\otimes \ell+1}, \text{ and} \\
\tau_{r,s,t}(x,y,z) &= 1^{\otimes r-1} \otimes x \otimes 1^{\otimes s-r-1} \otimes y \otimes 1^{\otimes t-s-1} \otimes z \otimes 1^{\otimes \ell - t+1} \in U(\mathfrak{g})^{\otimes \ell+1}.
\end{aligned}$$

We will define an operator $\Omega_{i,j} \in \mathrm{End}(X \otimes V^{\otimes \ell})$ by

$$\Omega_{i,j} := \sum_{E \in B} \tau_{i,j}(E, E).$$

Consider the map Θ from $\mathbb{H}_\ell$ to $\mathrm{End}(X \otimes V^{\otimes \ell})$ defined by

$$\begin{aligned}
\Theta(s_i) &= -\Omega_{i,i+1} \quad \text{for } 1 \leq i < \ell, \\
\Theta(\epsilon_k) &= \frac{n-1}{2} + \sum_{0 \leq j < k} \Omega_{j,k} \quad \text{for } 1 \leq k \leq \ell,
\end{aligned}$$

where ϵ_k is as in Section 3.2.3 and s_i is the simple reflection defined by $s_i(\epsilon_i) = \epsilon_{i+1}$.

Lemma 3.4.1. [1] As operators on $X \otimes V^{\otimes \ell}$, we have the following equalities

$$\begin{aligned}
[\Theta(\epsilon_i), \Theta(\epsilon_j)] &= 0, \text{ and} \\
\Theta(s_i)\Theta(\epsilon_j) - \Theta(s_i(\epsilon_j))\Theta(s_i) &= \alpha_i(\epsilon_j^\vee) \quad \text{for } 1 \leq i < \ell \text{ and } 1 \leq j \leq \ell.
\end{aligned}$$

Proof. Let $\kappa(\cdot,\cdot)$ denote the Killing form on $\mathfrak{g}$. We will begin with the following relations:

$$\begin{aligned}
[\Omega_{ij}, \Omega_{jk}] &= \sum_{E\in B} \tau_{i,j}(E,E^*) \sum_{E\in B} \tau_{j,k}(E,E^*) - \sum_{E\in B} \tau_{j,k}(E,E^*) \sum_{E\in B} \tau_{i,j}(E,E^*) \\
&= \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}(E_1, E_1^*E_2, E_2^*) - \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}(E_1, E_2E_1^*, E_2^*) \\
&= \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}(E_1, [E_1^*, E_2], E_2^*) \\
&= \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}\big(E_1, \sum_{E_l} \kappa([E_1^*, E_2], E_l)E_l^*, E_2^*\big) \\
&= \sum_{E_1,E_2\in B\times B} \sum_{E_l} B([E_1^*, E_2], E_l)\tau_{i,j,k}(E_1, E_l^*, E_2^*) \\
&= \sum_{E_1,E_2,E_3\in B^3} \kappa([E_1^*, E_3], E_2)\tau_{i,j,k}(E_1, E_2^*, E_3^*) \\
&= -\sum_{E_1,E_2,E_3\in B^3} \kappa([E_2, E_3], E_1^*)\tau_{i,j,k}(E_1, E_2^*, E_3^*).
\end{aligned}$$

$$\begin{aligned}
[\Omega_{ij}, \Omega_{ik}] &= \sum_{E\in B} \tau_{i,j}(E,E^*) \sum_{E\in B} \tau_{i,k}(E,E^*) - \sum_{E\in B} \tau_{i,k}(E,E^*) \sum_{E\in B} \tau_{i,j}(E,E^*) \\
&= \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}(E_1E_2, E_1^*, E_2^*) - \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}(E_2E_1, E_1^*, E_2^*) \\
&= \sum_{E_1,E_2\in B\times B} \tau_{i,j,k}\left(\sum_{E_l} \kappa([E_1, E_2], E_l^*)E_l, E_1^*, E_2^*\right) \\
&= \sum_{E_1,E_2\in B\times B} \sum_{E_l} \kappa([E_1, E_2], E_l^*)\tau_{i,j,k}(E_l, E_1^*, E_2^*) \\
&= \sum_{E_1,E_2,E_3\in B^3} \kappa([E_2, E_3], E_1^*)\tau_{i,j,k}(E_1, E_2^*, E_3^*).
\end{aligned}$$

Therefore, we have that

$$[\Omega_{ij}, \Omega_{ik}] + [\Omega_{ij}, \Omega_{jk}] = 0 \tag{3.2}$$

for i, j, k distinct. If i, j, k, l are distinct, then clearly

$$[\Omega_{ij}, \Omega_{kl}] = 0.$$

This implies

$$[\Theta(\epsilon_i), \Theta(\epsilon_j)] = 0$$

and

$$\begin{aligned}
\Theta(s_i)\Theta(\epsilon_j) - \Theta(\epsilon_{s_i(j)})\Theta(s_i) &= -\Omega_{i(i+1)}\left(\frac{n-1}{2} + \sum_{0\le k<j} \Omega_{kj}\right) \\
&\quad + \left(\frac{n-1}{2} + \sum_{0\le k<s_i(j)} \Omega_{ks_i(j)}\right)\Omega_{i(i+1)} \\
&= -\sum_{0\le k<j} \Omega_{i(i+1)}\Omega_{kj} + \sum_{0\le k<s_i(j)} \Omega_{ks_i(j)}\Omega_{i(i+1)} \\
&= 0 \quad \text{if } j \neq i, i+1 \text{ by Equation 3.2.}
\end{aligned}$$

Suppose $j = i$. Then

$$\begin{aligned}
\Theta(s_i)\Theta(\epsilon_j) - \Theta(\epsilon_{s_i(j)})\Theta(s_i) &= -\sum_{0\le k<i} \Omega_{i(i+1)}\Omega_{ki} \\
&\quad + \sum_{0\le k<i+1} \Omega_{k(i+1)}\Omega_{i(i+1)} \\
&= \Omega^2_{i(i+1)} - \sum_{0\le k<i} \Omega_{i(i+1)}\Omega_{ki} + \Omega_{k(i+1)}\Omega_{i(i+1)} \\
&= \Omega^2_{i(i+1)} \\
&= 1.
\end{aligned}$$

Now suppose $j = i+1$. Then

$$\begin{aligned}
\Theta(s_i)\Theta(\epsilon_j) - \Theta(\epsilon_{s_i(j)})\Theta(s_i) &= -\sum_{0\le k<i+1} \Omega_{i(i+1)}\Omega_{k(i+1)} + \sum_{0\le k<i} \Omega_{ki}\Omega_{i(i+1)} \\
&= -\Omega^2_{i(i+1)} - \sum_{0\le k<i} \Omega_{i(i+1)}\Omega_{k(i+1)} - \Omega_{ki}\Omega_{i(i+1)} \\
&= -1.
\end{aligned}$$

Therefore, we have

$$s_i\Theta(\epsilon_j) - \Theta(s_i(\epsilon_j))s_i = \alpha_i(\epsilon_j^\vee) \quad \text{for } 1 \le i < \ell \text{ and } 1 \le j \le \ell.$$

□

Proposition 3.4.2. [1], [15] For any $U(\mathfrak{g})$-module X, $X \otimes V^{\otimes \ell}$ is an $\mathbb{H}_\ell$-module (with the action of $\mathbb{H}_\ell$ defined by Θ). Moreover, the action of $\mathbb{H}_\ell$ commutes with the action of $\mathfrak{g}$.

Proof. We will prove that the action of $\mathfrak{g}$ on $X \otimes V^{\otimes \ell}$ commutes with the action of $\Omega_{r,s}$ for all r, s. The action of $\mathfrak{g}$ on $X \otimes V^{\otimes \ell}$ is given by

$$z.x \otimes v_1 \otimes \cdots \otimes v_\ell = \left(\sum_{r=0}^{\ell} \tau_r(z) \right).x \otimes v_1 \otimes \cdots \otimes v_\ell.$$

For notational convenience, let

$$\tau(z) = \sum_{r=0}^{\ell} \tau_r(z).$$

We will calculate $[\tau(z), \Omega_{r,s}]$ as elements of $T(\mathfrak{g})$, the tensor algebra of $\mathfrak{g}$. Let $\kappa(\cdot, \cdot)$ denote the Killing form on $\mathfrak{g}$.

$$\begin{aligned}
[\tau(z), \Omega_{r,s}] &= [\tau_r(z) + \tau_s(z), \sum_i \tau_{r,s}(E_i, E_i^*)] \\
&= \sum_i \tau_{r,s}(zE_i, E_i^*) + \tau_{r,s}(E_i, zE_i^*) - \tau_{r,s}(E_i z, E_i^*) - \tau_{r,s}(E_i, E_i^* z) \\
&= \sum_i \tau_{r,s}([z, E_i], E_i^*) + \tau_{r,s}(E_i, [z, E_i^*]) \\
&= \sum_i \tau_{r,s}\big(\sum_j \kappa([z, E_i], E_j^*) E_j, E_i^*\big) + \tau_{r,s}\big(E_i, \sum_j \kappa([z, E_i^*], E_j) E_j^*\big) \\
&= \sum_i \left(\sum_j \kappa([z, E_i], E_j^*) \right) \tau_{r,s}(E_j, E_i^*) \\
&\quad + \left(\sum_j \kappa([z, E_i^*], E_j) \right) \tau_{r,s}(E_i, E_j^*) \\
&= \left(\sum_{i,j} \kappa([z, E_i], E_j^*) \right) \tau_{r,s}(E_j, E_i^*) + \left(\sum_{i,j} \kappa([z, E_j^*], E_i) \right) \tau_{r,s}(E_j, E_i^*) \\
&= \left(\sum_{i,j} \kappa(z, [E_i, E_j^*]) \right) \tau_{r,s}(E_j, E_i^*) + \left(\sum_{i,j} -\kappa(z, [E_i, E_j^*]) \right) \tau_{r,s}(E_j, E_i^*) \\
&= 0.
\end{aligned}$$

Since the action of $\mathfrak{g}$ commutes with $\Omega_{r,s}$ for all r and s, the action of $\mathbb{H}_\ell$ on $X \otimes V^{\otimes \ell}$ commutes with $\mathfrak{g}$. $\square$

Chapter 4

Equivariant maps between geometric parameters

In this chapter, we will illuminate the relationship between the multiplicities of irreducible modules in the composition series of standard objects in $\mathcal{N}$ and $\mathcal{H}$ for type A_n. Theorem 4.0.3 shows that there is a natural correspondence between our parametrizations of irreducible objects in each category. This correspondence allows us to compare the multiplicity formulas given by the Kazhdan-Lusztig polynomials in each category. Theorem 4.0.6, which is the geometric core of our main result, shows that the multiplicity formulas agree under the correspondence in Theorem 4.0.3.

Let $V = \mathbb{C}^n$, $G = \mathrm{SL}(V)$, and $\mathfrak{g} = \mathrm{Lie}(G)$. Let σ be a semisimple element of $\mathfrak{g}$, and $\mathfrak{g}_i$ be the i-eigenspace of $\mathrm{ad}(\sigma)$. For convenience, we will assume σ is diagonal. Let

$$\mathfrak{l} = \mathfrak{g}_0, \qquad \mathfrak{u} = \bigoplus_{i>0} \mathfrak{g}_i, \qquad \mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}.$$

There is a decomposition $V = V_1 \oplus \cdots \oplus V_k$ such that

$$\begin{aligned} \mathfrak{l} &= \{x \in \mathfrak{g} | x(V_i) \subset V_i\} \\ \mathfrak{u} &= \{x \in \mathfrak{g} | x(V_i) \subset \bigoplus_{j<i} V_j\}. \end{aligned}$$

Finally, let $L \subset G$ and $P \subset G$ be such that $\mathrm{Lie}(L) = \mathfrak{l}$ and $\mathrm{Lie}(P) = \mathfrak{p}$. Consider the action of L on $\mathfrak{g}_1$ defined by

$$g.x = (g^t)^{-1} x g^t,$$

where g^t denotes the transpose of $g \in L$.

Definition. An orbit of L on $\mathfrak{g}_1$ is called a *graded nilpotent class* [54].

Theorem 4.0.3. *[54] Let $F(V)$ be the variety of conjugates of $\mathfrak{p}$ (which we will later identify with the partial flag variety G/P). Let N^t denote the transpose of $N \in \mathfrak{g}_1$ and consider the following map*

$$\begin{aligned} \varphi : \mathfrak{g}_1 &\to F(V) \\ N &\mapsto (1+N^t) \bullet \mathfrak{p} := (1+N^t)\mathfrak{p}(1+N^t)^{-1}. \end{aligned}$$

We have that

(a) φ is equivariant for the action of L, and

(b) $\varphi(L \bullet N) = L(1+N^t) \bullet \mathfrak{p}$ is dense in $P(1+N^t) \bullet \mathfrak{p}$.

Proof. We have that φ is equivariant because conjugation by L fixes $\mathfrak{p}$, and

$$\begin{aligned} (1+(g.N)^t) \bullet \mathfrak{p} &= (1+gN^tg^{-1})\mathfrak{p}(1+gN^tg^{-1})^{-1} = g(1+N^t)g^{-1}\mathfrak{p}g(1+N^t)^{-1}g^{-1} \\ &= g(1+N^t) \bullet \mathfrak{p} \end{aligned}$$

Fix $N \in \mathfrak{g}_1$. Let $\mathfrak{l}' = (1+N^t)\bullet\mathfrak{l}$, $\mathfrak{p}' = (1+N^t)\bullet\mathfrak{p}$, $V_i' = (1+N^t)V_i$, and $W = (1+N^t)\mathfrak{u} \subset \mathrm{End}(V)$. We will reduce the proof of the theorem to the following lemma.

Lemma 4.0.4. The following map is a well defined isomorphism of vector spaces.

$$\begin{aligned} \mathfrak{l} \cap \mathfrak{p}' \oplus W &\to \mathfrak{p} \cap \mathfrak{p}' \\ (A, \phi) &\mapsto A + \phi. \end{aligned}$$

The following corollary of the above lemma implies part (b) of the theorem.

Corollary 4.0.5. (a) $\dim(\mathfrak{p} \cap \mathfrak{p}') = \dim(\mathfrak{l} \cap \mathfrak{p}') + \dim(\mathfrak{u})$,

(b) $\dim \mathrm{Stab}_P(\mathfrak{p}') = \dim \mathrm{Stab}_L(\mathfrak{p}') + \dim(\mathfrak{u})$, and

(c) $L(1+N^t) \bullet \mathfrak{p}$ is dense in $P(1+N^t) \bullet \mathfrak{p}$.

Therefore, we will proceed by proving the above lemma. First we will show that the map is well defined. Since $A \in \mathfrak{l} \cap \mathfrak{p}'$ we have

$$A(V_i) \subset V_i \quad \text{and} \quad A(V_i') \subset \bigoplus_{j \leq i} V_j'.$$

Notice that $(1+N^t)V_i \subset V_i \oplus V_{i+1}$. So

$$\phi(V_i) \subset \bigoplus_{j<i} V_j' \subset \bigoplus_{j \leq i} V_j.$$

Therefore $(A+\phi)(V_i) \subset \bigoplus_{j\leq i} V_j$ and $A+\phi \in \mathfrak{p}$. Now we will show that $A+\phi \in \mathfrak{p}'$. This follows from

$$\begin{aligned} A(V_i') &\subset \bigoplus_{j\leq i} V_j', \text{ and} \\ \phi(V_i') &\subset \phi(V_i)+\phi(V_{i+1}) \subset \bigoplus_{j<i+1} V_j'. \end{aligned}$$

Therefore the map is well defined. Now we will show that the map is an isomorphism. Since $\mathfrak{l}\cap\mathfrak{p}'\cap W = \emptyset$, the map is injective. So to complete the proof we must show that the map is surjective. Let $\tilde{A} \in \mathfrak{p}\cap\mathfrak{p}'$ and fix $v_i \in V_i$. We want to write

$$\tilde{A}(v_i) = A(v_i) + \phi(v_i)$$

for some $A \in \mathfrak{l}\cap\mathfrak{p}'$ and some $\phi \in W$. We will proceed by reverse induction on i. Suppose $i = k$.

$$\tilde{A}(v_k) = u_1' + \cdots + u_k'$$

with $u_i' \in V_i'$. Since $u_k' \in V_k' = V_k$, set

$$\begin{aligned} A(v_k) &:= u_k' \in V_k, \text{ and} \\ \phi(v_k) &:= u_1' + \cdots + u_{k-1}' \in \bigoplus_{j<k} V_j'. \end{aligned}$$

Now assume A and ϕ are defined for all $v_j \in V_j$ with $j > i$. Fix $v_i \in V_i$. Let $v_i' = (1+N^t)v_i \in V_i'$ and $v_{i+1} = -N^t v_i \in V_{i+1}$. Then

$$\tilde{A}(v_i) = \tilde{A}(v_i') + \tilde{A}(v_{i+1}).$$

By induction $\tilde{A}(v_{i+1}) = A(v_{i+1}) + \phi(v_{i+1})$. Since $\tilde{A} \in \mathfrak{p}'$ there are $w_j' \in V_j'$ so that

$$\tilde{A}(v_i') = w_1' + \cdots + w_i'.$$

So

$$\tilde{A}(v_i) = (w_1' + \cdots + w_i') + (u_1' + \cdots + u_i') + A(v_{i+1}) \in \bigoplus_{j\leq i} V_j.$$

Notice that $w_i' + u_i' + A(v_{i+1}) \in V_i$. So

$$\tilde{A}(v_i) \quad = \quad A(v_i) + \phi(v_i),$$

where

$$\begin{aligned} A(v_i) &:= w_i' + u_i' + A(v_{i+1}), \text{ and} \\ \phi(v_i) &:= w_1' + \cdots + w_{i-1}' + u_1' + \cdots + u_{i-1}'. \end{aligned}$$

□

Since φ is L-equivariant, we can define an induced map on L orbits of $\mathfrak{g}_1$ as follows (now identifying $F(V)$ with G/P):

$$\begin{aligned} \Phi : L\backslash\mathfrak{g}_1 &\to P\backslash G/P \\ Q &\mapsto P \cdot \varphi(Q). \end{aligned}$$

Now we will relate the above map to the classifications of irreducible objects in Chapter 2 and Chapter 3. Recall the notation $\mathcal{N}(\xi(\lambda), \eta)$, a subcategory of Whittaker modules (defined in Chapter 2), and $\mathcal{P}^g_{\mathbb{H}}(\sigma)$, the geometric parametrization of irreducible modules for the graded affine Hecke algebra (defined in Chapter 3). Suppose $\sigma \in \mathfrak{h}$ is the dual of $\lambda + \rho$ by the trace form. The double cosets $W_\eta\backslash W/W_\lambda$ parametrize the set of irreducible Whittaker modules in $\mathcal{N}(\xi(\lambda), \eta)$ (cf. Theorem 2.1.6). Such double cosets are in one-to-one correspondence with P_η orbits on G/P_λ (where P_η is a Lie subgroup of G with Lie algebra $\mathfrak{p}_\eta$). Therefore, we denote the set of P_η orbits on G/P_λ by $\mathcal{P}^g_{\mathcal{N}}(\eta, \lambda)$, and use this set to parametrize the irreducible objects in $\mathcal{N}(\xi(\lambda), \eta)$. Similarly, the geometric parameters $\mathcal{P}^g_{\mathbb{H}}(\sigma)$ can be identified with the set of L orbits on $\mathfrak{g}_1$ (see the discussion following Theorem 3.3.1). If we assume that $W_\lambda = W_\eta$, then $P_\eta = P_\lambda$, and we get the induced map on geometric parameters:

$$\begin{aligned} \Psi : \mathcal{P}^g_{\mathcal{N}}(\eta, \lambda) &\to \mathcal{P}^g_{\mathbb{H}}(\sigma) \cup \{\emptyset\} \\ \mathcal{Q} &\mapsto \begin{cases} \Phi^{-1}(\mathcal{Q}) \text{ if } \mathcal{Q} \in \text{im}\Phi \\ \emptyset \text{ otherwise} \end{cases} . \end{aligned}$$

Applying Theorem 2.5.1, we get the following equality of multiplicities.

Theorem 4.0.6. *Assume λ is integral, dominant, and $W_\eta = W_\lambda$. Then for $\mathcal{Q}, \mathcal{O} \in \mathcal{P}^g_{\mathcal{N}}(\eta, \lambda)$, we have*

$$[std_{\mathcal{N}}(\mathcal{Q}), irr_{\mathcal{N}}(\mathcal{O})] = [std_{\mathbb{H}}(\Psi(\mathcal{Q})), irr_{\mathbb{H}}(\Psi(\mathcal{O}))]$$

if $\Psi(\mathcal{Q}), \Psi(\mathcal{O}) \neq \emptyset$.

Proof. We will begin by applying Corollary 2.5.3 to the left hand side to get

$$[\mathrm{std}_{\mathcal{N}}(\mathcal{Q}), \mathrm{irr}_{\mathcal{N}}(\mathcal{O})] = \sum_i \dim H^i(\mathrm{IC}_x(C({}_\eta y_\lambda)))$$

where $x \in C(w)$ and $C(w)$ is the open B orbit in $\mathcal{Q}$, and $C({}_\eta y_\lambda)$ is the open B orbit in $\mathcal{O}$. Applying Corollary 3.3.2 to the right hand side, we get

$$[\mathrm{std}_{\mathbb{H}}(\Psi(\mathcal{Q})), \mathrm{irr}_{\mathbb{H}}(\Psi(\mathcal{O}))] = \sum_i \dim H^i(\mathrm{IC}_y(\Psi(\mathcal{O})))$$

for $y \in \Psi(\mathcal{Q})$. Since the map φ is stratum preserving, continuous, and has dense image in each stratum S of G/P, we have that

$$\dim H^i(\mathrm{IC}_x(C({}_\eta y_\lambda))) = \dim H^i(\mathrm{IC}_y(\Psi(\mathcal{O})))$$

for each i ([54], [26, Section 4.8]). □

Therefore, under the assumptions of Theorem 4.0.6, if there is an exact functor $F : \mathcal{N}(\xi(\lambda), \eta) \to \mathcal{H}_{\lambda+\rho}$ such that

$$F(\mathrm{std}_{\mathcal{N}}(\mathcal{O})) = \begin{cases} \mathrm{std}_{\mathbb{H}}(\Psi(\mathcal{O})) \text{ if } \Psi(\mathcal{O}) \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

then we can conclude that

$$F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O})) = \begin{cases} \mathrm{irr}_{\mathbb{H}}(\Psi(\mathcal{O})) \text{ if } \Psi(\mathcal{O}) \neq \emptyset \\ 0 \text{ otherwise.} \end{cases}$$

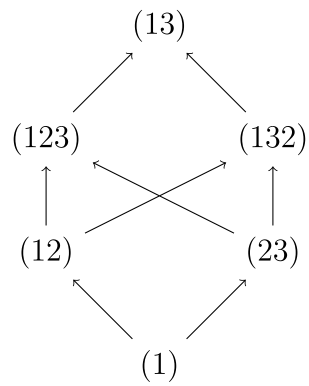
The content of the remainder of the paper will be focused on constructing such a functor for each choice of $\xi(\lambda)$ and η.

4.1 Example

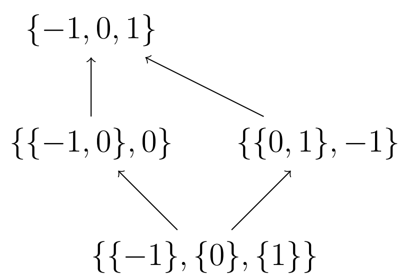
Let $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$, with Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices in $\mathfrak{g}$, and Borel subalgebra $\mathfrak{b}$ consisting of upper triangular matrices. Let $\sigma = (1, 0, -1) \in \mathfrak{h}$, $\alpha = (1, -1, 0)$ and $\beta = (0, 1, -1) \in \mathfrak{h}^*$. We have that $\mathfrak{g}_1 = \mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta$, $\mathfrak{l} = \mathfrak{h}$, $L = \{\text{diagonal matrices in } SL(3, \mathbb{C})\}$, $P = B$, and $F(V) = G/B$.

$$\begin{array}{ccc} \varphi : \mathfrak{g}_1 & \to & G/B \\ \begin{bmatrix} 0 & * & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix} & \mapsto & \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & * & 1 \end{bmatrix} B \end{array}$$

The Bruhat order on G/B is given by the following diagram



where B orbits on G/B are labelled by elements of $W = S_3$. Here $(1) \in S_3$ corresponds to the closed B-orbit on G/B. Similarly, we can consider the closure order on L orbits on $\mathfrak{g}_1$



where L orbits on $\mathfrak{g}_1$ are indexed by multisegments as in Section 3.2.3. Since $W_\lambda = (1)$, in order to relate the multiplicity formulas for $\mathcal{N}(\xi(0), \eta)$ and $\mathcal{H}_\rho$, we must choose $\eta = 0$. Suppose F is an exact functor from $\mathcal{N}(\xi(0), \eta = 0)$ to $\mathcal{H}_\rho$ such that

$$\begin{aligned}
F(\mathrm{std}_{\mathcal{N}}(\mathcal{O}_{(1)})) &= \mathrm{std}_{\mathbb{H}}(\mathcal{Q}_{\{\{-1\},\{0\},\{1\}\}}) \\
F(\mathrm{std}_{\mathcal{N}}(\mathcal{O}_{(12)})) &= \mathrm{std}_{\mathbb{H}}(\mathcal{Q}_{\{\{-1,0\},0\}}) \\
F(\mathrm{std}_{\mathcal{N}}(\mathcal{O}_{(23)})) &= \mathrm{std}_{\mathbb{H}}(\mathcal{Q}_{\{\{0,1\},-1\}}) \\
F(\mathrm{std}_{\mathcal{N}}(\mathcal{O}_{(123)})) &= \mathrm{std}_{\mathbb{H}}(\mathcal{Q}_{\{-1,0,1\}}) \\
F(\mathrm{std}_{\mathcal{N}}(\mathcal{O}_{(132)})) &= 0 \\
F(\mathrm{std}_{\mathcal{N}}(\mathcal{O}_{(13)})) &= 0
\end{aligned}$$

Then we can conclude that

$$\begin{aligned}
F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O}_{(1)})) &= \mathrm{irr}_{\mathbb{H}}(\mathcal{Q}_{\{\{-1\},\{0\},\{1\}\}}) \\
F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O}_{(12)})) &= \mathrm{irr}_{\mathbb{H}}(\mathcal{Q}_{\{\{-1,0\},0\}}) \\
F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O}_{(23)})) &= \mathrm{irr}_{\mathbb{H}}(\mathcal{Q}_{\{\{0,1\},-1\}}) \\
F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O}_{(123)})) &= \mathrm{irr}_{\mathbb{H}}(\mathcal{Q}_{\{-1,0,1\}}) \\
F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O}_{(132)})) &= 0 \\
F(\mathrm{irr}_{\mathcal{N}}(\mathcal{O}_{(13)})) &= 0
\end{aligned}$$

Chapter 5

Arakawa-Suzuki functors for Whittaker modules

In order to proceed in constructing exact functors from the category of Whittaker modules to the category of graded affine Hecke algebra modules, we will need to review some of the results of Arakawa-Suzuki.

5.1 Highest weight modules

Following [1], we can define a functor from $\mathcal{O}' = \mathcal{N}(\eta = 0)$ to the category of vector spaces as follows:

$$F_{\ell,\lambda}(X) := (X \otimes V^{\otimes \ell})_{\lambda}^{[\lambda]},$$

where V is the canonical representation of $\mathfrak{g} = \mathfrak{sl}(V)$. Since the action of $\mathbb{H}_\ell$ on $X \otimes V^{\otimes \ell}$ commutes with the action of $\mathfrak{g}$ (Proposition 3.4.2), $F_{\ell,\lambda}(X)$ has the structure of an $\mathbb{H}_\ell$-module. In this section we will show that the above functor maps standard modules in $\mathcal{O}'$ to standard modules in $\mathcal{H}$.

Proposition 5.1.1. [1, Proposition 1.4.2, Remark 1.4.3] For λ dominant, we have the following bijections

$$\begin{aligned} \mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M(\mu) \otimes V^{\otimes \ell}) &\cong H^0(\mathfrak{n}, M(\mu) \otimes V^{\otimes \ell})_\lambda \\ &\cong H_0(\bar{\mathfrak{n}}, M(\mu) \otimes V^{\otimes \ell})_\lambda \\ &\cong (M(\mu) \otimes V^{\otimes \ell})_\lambda^{[\lambda]}. \end{aligned}$$

Proposition 5.1.2. [1, Proposition 2.1.1] For λ dominant, the functor $F_{\ell,\lambda}$ is exact.

5.1.1 Images of Verma modules

We will now review the main results of [1]. For a $U(\mathfrak{g})$-module X where $\mathfrak{h}$ acts semisimply, let $P(X) := \{\lambda \in \mathfrak{h}^* : X_\lambda \neq 0\}$ be the set of all non trivial weights of X.

Theorem 5.1.3. *[1] For λ, μ integral weights with λ dominant, there is an isomorphism of $\mathbb{H}_\ell$-modules*

$$(M(\mu) \otimes V^{\otimes \ell})_\lambda^{[\lambda]} \cong \begin{cases} std_{\mathbb{H}}(\lambda, \mu) & \text{if } \lambda - \mu \in P(V^{\otimes \ell}) \\ 0 & \text{otherwise.} \end{cases}$$

Sketch of Proof. In [1], Arakawa and Suzuki prove Theorem 5.1.3 with two lemmata. First they show that for integral weight μ and integral dominant weight λ, the natural inclusion $(V^{\otimes \ell})_{\lambda-\mu} \hookrightarrow (M(\mu) \otimes V^{\otimes \ell})_\lambda$ given by $u \mapsto v_\mu \otimes u$ (where v_μ is a highest weight vector of $M(\mu)$) induces the following isomorphism of $W_\ell (= S_\ell)$ representations

$$(V^{\otimes \ell})_{\lambda-\mu} \to H^0(\mathfrak{n}, M(\mu) \otimes V^{\otimes \ell})_\lambda.$$

Next they construct a projection $(M(\mu) \otimes V^{\otimes \ell})_\lambda \to H^0(\mathfrak{n}, M(\mu) \otimes V^{\otimes \ell})_\lambda$ by identifying $H^0(\mathfrak{n}, M(\mu) \otimes V^{\otimes \ell})_\lambda = \big((M(\mu) \otimes V^{\otimes \ell})/\bar{\mathfrak{n}}(M(\mu) \otimes V^{\otimes \ell})\big)_\lambda$. Let $u_{\lambda,\mu}$ be the image of $\tilde{u}_{\lambda,\mu} = v_\mu \otimes u_1^{\otimes \ell_1} \otimes \cdots \otimes u_n^{\ell_n} \in (M(\mu) \otimes V^{\otimes \ell})_\lambda$ under this projection, where the ℓ_i are chosen in Equation 3.1. The second lemma is a calculation of the action of $\mathbb{H}_\ell$ on $u_{\lambda,\mu}$. They show that

$$\Theta(\epsilon_k) u_{\lambda,\mu} = \zeta_{\lambda,\mu}(\epsilon_i^\vee) u_{\lambda,\mu},$$

where $\zeta_{\lambda,\mu}$ is a character of $\mathfrak{t}$ depending on λ and μ (defined in [1]). We can then observe that we have a surjective $\mathbb{H}_\ell$-module morphism from $\mathrm{std}_{\mathbb{H}}(\lambda, \mu)$ to

$$H^0(\mathfrak{n}, M(\mu) \otimes V^{\otimes \ell})_\lambda$$

which maps $\mathbb{1}$ to $u_{\lambda,\mu}$. Injectivity follows from Lemma 3.2.6, and the theorem is proved by observing that $H_0(\bar{\mathfrak{n}}, M(\mu) \otimes V^{\otimes \ell})_\lambda = (M(\mu) \otimes V^{\otimes \ell})_\lambda^{[\lambda]}$ when λ is dominant. □

We can restate this theorem using the geometric parametrizations of standard and irreducible objects in the respective categories as follows.

Corollary 5.1.4. Suppose λ is regular, integral, and dominant, and $\eta = 0$. The geometric parameters for $\mathcal{N}(\xi(\lambda), \eta)$ are B orbits on the flag variety G/B. Let $C(w)$ denote the B orbit corresponding to $w \in W$.

$$\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M(w \bullet \lambda) \otimes V^{\otimes n}) = \begin{cases} \mathrm{std}_{\mathbb{H}}(\Psi(C(w))) \text{ if } \Psi(C(w)) \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

Notice that $\Psi(C(w)) \neq \emptyset$ precisely when $\lambda - w \bullet \lambda \in P(V^{\otimes n})$.

5.2 Whittaker modules

For $\lambda \in \mathfrak{h}^*$, we define the following functor from $\mathcal{N}(\eta)$ to the category of finite-dimensional vector spaces:

$$F_{\ell,\eta,\lambda}(X) := H^0_\eta\left(\mathfrak{n}_\eta, (X \otimes V^{\otimes \ell})^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right).$$

Since the action of $\mathbb{H}_\ell$ on $X \otimes V^{\otimes \ell}$ commutes with the action of $\mathfrak{g}$ (Proposition 3.4.2), $F_{\ell,\eta,\lambda}(X)$ has the structure of an $\mathbb{H}_\ell$-module. This allows us to view $F_{\ell,\eta,\lambda}$ as a functor from $\mathcal{N}(\eta)$ to the category of finite-dimensional $\mathbb{H}_\ell$-modules.

Proposition 5.2.1. For λ dominant, $F_{\ell,\eta,\lambda}$ is an exact functor from $\mathcal{N}(\eta)$ to $\mathcal{H}$.

Proof. Following Proposition 2.0.4, Lemma 2.2.1, and Proposition 2.2.2, we can see that $F_{\ell,\eta,\lambda}$ is exact when viewed as a functor from $\mathcal{N}(\eta)$ to the category of vector spaces. In order to see that $F_{\ell,\eta,\lambda}$ takes exact sequences of $U(\mathfrak{g})$-modules to exact sequences of $\mathbb{H}_\ell$-modules (not just vector spaces), we will show that given a morphism of $U(\mathfrak{g})$-modules ϕ, the linear map $F_{\ell,\eta,\lambda}(\phi)$ is actually a morphism of $\mathbb{H}_\ell$-modules. Suppose $\phi : A \to B$ is a morphism of $U(\mathfrak{g})$-modules. Recall the operators $\Omega_{i,j} \in U(\mathfrak{g})^{\otimes \ell+1}$ from Section 3.4. Let $\phi' : A \otimes V^{\otimes \ell} \to B \otimes V^{\otimes \ell}$ be defined by $\phi'(a \otimes v_1 \otimes \cdots \otimes v_\ell) = \phi(a) \otimes v_1 \otimes \cdots \otimes v_\ell$. If $a \in A$, then $\Omega_{i,j}(\phi'(a \otimes v_1 \otimes \cdots \otimes v_\ell)) = \Omega_{i,j}(\phi(a) \otimes v_1 \otimes \cdots \otimes v_\ell) = \phi'(\Omega_{i,j}(a \otimes v_1 \otimes \cdots \otimes v_\ell))$ then $F_{\ell,\eta,\lambda}(\phi)$. Since the action of $\mathbb{H}_\ell$ is defined in terms of the operators $\Omega_{i,j}$, we have that $h\phi'(a \otimes v_1 \otimes \cdots \otimes v_\ell) = \phi'(h(a \otimes v_1 \otimes \cdots \otimes v_\ell))$ for $h \in \mathbb{H}_\ell$. It follows that $F_{\ell,\eta,\lambda}(\phi)$ is a morphism of $\mathbb{H}_\ell$-modules. Hence $F_{\ell,\eta,\lambda}$ is an exact functor from $\mathcal{N}(\eta)$ to $\mathcal{H}$. □

5.2.1 Images of standard Whittaker modules

In this section, we will calculate the image of standard Whittaker modules.

Theorem 5.2.2. *The functor $\overline{\Gamma}_\eta$ induces a morphism of $\mathbb{H}_\ell$-modules*

$$\overline{\Gamma}_\eta^{\mathbb{H}}(\lambda,\mu) : Hom_{U(\mathfrak{g})}\left(M(\lambda), M(\mu)\otimes V^{\otimes \ell}\right) \to H^0_\eta\left(\mathfrak{n}_\eta, \left(std_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right).$$

Moreover, $\overline{\Gamma}_\eta^{\mathbb{H}}(\lambda,\mu)$ is an isomorphism if λ is dominant and $W_\eta = W_\lambda$.

Proof. We will begin by constructing a natural isomorphism

$$\mathrm{Hom}_{U(\mathfrak{g})}\left(\mathrm{std}_{\mathcal{N}}(\lambda,\eta), \mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes \ell}\right) \cong H^0_\eta\left(\mathfrak{n}_\eta, \left(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right).$$

From Corollary 2.1.5, we see that $(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell})^{[\lambda]}$ has a filtration with subquotients isomorphic to $\mathrm{std}_{\mathcal{N}}(\lambda+\nu,\eta)$, where ν is a weight of $V^{\otimes \ell}$ and $\lambda+\nu \in W\bullet\lambda$. Recall that $P(V^{\otimes n})$ denotes the set of non zero $\mathfrak{h}$-weights of $V^{\otimes n}$. By Theorem 2.1.3, we have that the non zero $\mathfrak{z}$-weight spaces of $\mathrm{std}_{\mathcal{N}}(\lambda+\nu,\eta)$ have weights $w\bullet\lambda+\gamma$ for $\gamma \in P(U(\bar{\mathfrak{n}}^\eta))$. Notice that $\mu\in\mathfrak{h}^*$ and $w\bullet\mu$ are equal as $\mathfrak{z}$-weights if $w\in W_\eta$. Since λ is dominant, we have that $\mathrm{std}_{\mathcal{N}}(w\bullet\lambda,\eta)$ has a non zero $\mathfrak{z}$-weight space of weight λ if and only if $w\in W_\eta$. Moreover, since the action of $\mathfrak{n}^\eta$ takes a $\mathfrak{z}$-weight vector of weight λ to a vector of weight $\lambda+\gamma'$ for $\gamma'\in P(\mathfrak{n}^\eta)$, and $(w\bullet\lambda + P(U(\bar{\mathfrak{n}}^\eta)))\cap(\lambda+P(\mathfrak{n}^\eta)) = \emptyset$, we can conclude that $\mathfrak{n}^\eta v = 0$ for all $\mathfrak{z}$-weight vectors v of weight λ in $\mathrm{std}_{\mathcal{N}}(w\bullet\lambda,\eta)$. In conclusion, $v\in\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell}$ is contained in $H^0_\eta\left(\mathfrak{n}_\eta, \left(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right)$ if and only if v is a $\mathfrak{z}$-weight vector with weight λ, and has $xv = \eta(x)v$ for all $x\in\mathfrak{n}$. Since $\mathrm{std}_{\mathcal{N}}(\lambda,\eta)$ is isomorphic to $U(\mathfrak{g})v$ for each $v\in H^0_\eta\left(\mathfrak{n}_\eta, \left(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right)$, we can define a morphism $\phi_v : \mathrm{std}_{\mathcal{N}}(\lambda,\eta) \to \mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell}$ for each $v\in H^0_\eta\left(\mathfrak{n}_\eta, \left(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes \ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right)$. Alternatively, for each $\phi\in\mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{std}_{\mathcal{N}}(\lambda,\eta),\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell})$, we can define a vector $v_\phi := \phi(1\otimes 1\otimes 1)\in\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell}$ by considering the image of the canonical generator $1\otimes1\otimes1$ of $\mathrm{std}_{\mathcal{N}}(\lambda,\eta) = U(\mathfrak{g})\otimes_{U(\mathfrak{p}_\eta)}\left(U(\mathfrak{l}_\eta)/\xi_\eta(\lambda)U(\mathfrak{l}_\eta)\otimes_{U(\mathfrak{n}_\eta)}\mathbb{C}_\eta\right)$. Since ϕ is morphism of $U(\mathfrak{g})$-modules, it follows that v_ϕ is a $\mathfrak{z}$-weight vector of weight λ and $xv_\phi = \eta(x)v_\phi$ for all $x\in\mathfrak{n}$. Moreover, the maps $v\mapsto\phi_v$ and $\phi\mapsto v_\phi$ are inverses of each other, which proves the natural identification described above.

Consider the map induced on morphisms by the functor $\overline{\Gamma}_\eta$. Let

$$\phi\in\mathrm{Hom}_{U(\mathfrak{g})}(A,B)$$

for A, B in BGG category $\mathcal{O}$. Then $\overline{\Gamma}_\eta(\phi)\in\mathrm{Hom}_{U(\mathfrak{g})}\left(\overline{\Gamma}_\eta(A),\overline{\Gamma}_\eta(B)\right)$ is given by

$$\overline{\Gamma}_\eta(\phi)\left(\sum_{\nu\in P(A)} x_\nu\right) = \sum_{\nu\in P(A)}\phi(x_\nu),$$

where we view elements of $\overline{\Gamma}_\eta(A)$ (elements of $\overline{\Gamma}_\eta(B)$) as formal infinite linear sums of weight vectors of A (weight vectors of B, respectively). Now, the action of $\mathbb{H}_\ell$ on $\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M(\mu)\otimes V^{\otimes \ell})$ (which we will denote here by $h.\phi$) is given by the action of $\mathbb{H}_\ell$ on $M(\mu)\otimes V^{\otimes\ell}$. By Proposition 2.4.4, we have that

$$\mathrm{Hom}_{U(\mathfrak{g})}\left(\overline{\Gamma}_\eta(M(\lambda)), \overline{\Gamma}_\eta(M(\mu)\otimes V^{\otimes\ell})\right) = \mathrm{Hom}_{U(\mathfrak{g})}\left(\overline{\Gamma}_\eta(M(\lambda)), \overline{\Gamma}_\eta(M(\mu))\otimes V^{\otimes\ell}\right).$$

Since $\overline{\Gamma}_\eta(M(\lambda))$ is a $U(\mathfrak{g})$-module, we see that $\mathbb{H}_\ell$ acts on

$$\mathrm{Hom}_{U(\mathfrak{g})}\left(\overline{\Gamma}_\eta(M(\lambda)), \overline{\Gamma}_\eta(M(\mu))\otimes V^{\otimes\ell}\right),$$

which we will denote again by $h.\phi$. For $h\in\mathbb{H}_\ell$ we have

$$\begin{aligned} h.\overline{\Gamma}_\eta(\phi)\left(\sum_{\nu\in P(M(\lambda))} x_\nu\right) &= h\sum_{\nu\in P(M(\lambda))}\phi(x_\nu) \\ &= \sum_{\nu\in P(M(\lambda))} h\phi(x_\nu) \\ &= \overline{\Gamma}_\eta(h.\phi)\left(\sum_{\nu\in P(M(\lambda))} x_\nu\right). \end{aligned}$$

Therefore, $\overline{\Gamma}_\eta$ induces a morphism of $\mathbb{H}_\ell$-modules. We will now turn our focus to showing that the induced morphism of $\mathbb{H}_\ell$-modules is an isomorphism when $W_\eta = W_\lambda$, and μ is $\mathfrak{n}_\eta$ antidominant. The generating vector $1\otimes 1\otimes 1$ of $\mathrm{std}_{\mathcal{N}}(\lambda,\eta)$ has $\mathfrak{z}$-weight λ. Therefore, if we write the generating vector as a linear combination $\sum_{\nu\in P(M(\lambda))} x_\nu$, we have

$$z\sum_{\nu\in P(M(\lambda))} x_\nu = \sum_{\nu\in P(M(\lambda))} zx_\nu = \sum_{\nu\in P(M(\lambda))}\nu(z)x_\nu = \lambda(z)\sum_{\nu\in P(M(\lambda))} x_\nu,$$

for all $z\in\mathfrak{z}$. Therefore, the $\nu\in P(M(\lambda))$ for which x_ν is non zero must have the property that $\nu(z)=\lambda(z)$ for all $z\in\mathfrak{z}$. In other words, the $\nu\in P(M(\lambda))$ for which x_ν is non zero must be contained in the set $P(U(\mathfrak{l}_\eta)v_\lambda)$, where v_λ is a λ-highest weight vector in $M(\lambda)$. Since $W_\lambda = W_\eta$, we have that $U(\mathfrak{l}_\eta)v_\lambda$ is an irreducible $U(\mathfrak{l}_\eta)$-module. Therefore, if $\phi\in\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M(\mu)\otimes V^{\otimes\ell})$ and $\phi(v_\lambda)\neq 0$, then $\phi(w)\neq 0$ for all (non zero) $w\in U(\mathfrak{l}_\eta)v_\lambda$. Since ϕ is determined completely by its value on v_λ, we have that $\phi\neq 0$ implies that $\phi(v_\lambda)\neq 0$. This implies that $\phi(x_\nu)\neq 0$ for each x_ν which appears in the sum decomposition of the generating vector of $\mathrm{std}_{\mathcal{N}}(\lambda,\eta)$. Observe that since ϕ is a morphism of $U(\mathfrak{g})$-modules, it preserves weight spaces. Therefore, if $\nu_1 \neq \nu_2 \in \mathfrak{h}^*$, we have $\phi(x_{\nu_1}) + \phi(x_{\nu_2}) = 0$ if and only if $\phi(x_{\nu_1}), \phi(x_{\nu_2}) = 0$. So if $\phi \neq 0$, then

$$\overline{\Gamma}_\eta(\phi)\left(\sum_{\nu \in P(M(\lambda))} x_\nu\right) = \sum_{\nu \in P(M(\lambda))} \phi(x_\nu) \neq 0.$$

We have therefore shown that the map $\overline{\Gamma}_\eta^{\mathbb{H}}(\lambda, \mu)$ is injective. To conclude the proof, we will show that

$$\dim \mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M(\mu) \otimes V^{\otimes \ell}) = \dim \mathrm{Hom}_{U(\mathfrak{g})}\left(\mathrm{std}_{\mathcal{N}}(\lambda, \eta), \mathrm{std}_{\mathcal{N}}(\mu, \eta) \otimes V^{\otimes \ell}\right).$$

Since λ is dominant, $M(\lambda)$ is a projective object in $\mathcal{O}$, and the left hand side is equal to $[M(\mu) \otimes V^{\otimes \ell} : L(\lambda)]$, cf. [22]. Again, since λ is dominant, $[M(w \bullet \lambda) : L(\lambda)]$ is zero unless $w \bullet \lambda = \lambda$, in which case the multiplicity is 1. Therefore, the left hand side is equal to $\dim(V^{\otimes \ell})_{\lambda - \mu}$. On the right hand side, we have that $\mathrm{std}_{\mathcal{N}}(\mu, \eta) \otimes V^{\otimes \ell}$ has a filtration with quotients isomorphic to $\mathrm{std}_{\mathcal{N}}(\mu + \tau, \eta)$ for $\tau \in P(V^{\otimes \ell})$. So

$$\dim \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{std}_{\mathcal{N}}(\lambda, \eta), \mathrm{std}_{\mathcal{N}}(\mu, \eta) \otimes V^{\otimes \ell}) \leq \sum_{\tau \in W_\eta \bullet \lambda - \mu} \dim V_\tau^{\otimes \ell}.$$

Since $W_\eta \bullet \lambda = \lambda$, we have that

$$\dim \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{std}_{\mathcal{N}}(\lambda, \eta), \mathrm{std}_{\mathcal{N}}(\mu, \eta) \otimes V^{\otimes \ell}) \leq \dim(V^{\otimes \ell})_{\lambda - \mu}.$$

Since $\overline{\Gamma}_\eta^{\mathbb{H}}(\lambda, \mu)$ is injective, the inequality must be an equality, and the map must be an isomorphism. □

Theorem 5.2.3. *Let λ be dominant, integral, and suppose $W_\eta = W_\lambda$. Then*

$$F_{\ell,\eta,\lambda}(std_{\mathcal{N}}(y \bullet \lambda, \eta)) \cong \begin{cases} std_{\mathbb{H}}(\lambda, y \bullet \lambda) & \text{if } \lambda - y \bullet \lambda \in P(V^{\otimes \ell}) \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, under the geometric parametrization of standard modules (with the notation of Section 4), we have

$$F_{\ell,\eta,\lambda}(std_{\mathcal{N}}(\mathcal{O})) \cong \begin{cases} std_{\mathbb{H}}(\Psi(\mathcal{O})) & \text{if } \Psi(\mathcal{O}) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\mathcal{O}$ is a P orbit on G/P.

Proof. By Theorem 5.2.2,

$$\overline{\Gamma}^{\mathbb{H}}_{\eta}(\lambda,\mu) : \mathrm{Hom}_{U(\mathfrak{g})}\left(M(\lambda), M(\mu)\otimes V^{\otimes\ell}\right) \to H^0_{\eta}\left(\mathfrak{n}_{\eta}, \left(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right)$$

is an isomorphism of $\mathbb{H}_{\ell}$-modules. By Theorem 5.1.3,

$$\left(M(\mu)\otimes V^{\otimes\ell}\right)^{[\lambda]}_{\lambda} \cong \begin{cases} \mathrm{std}_{\mathbb{H}}(\lambda,\mu) & \text{if } \lambda-\mu\in P(V^{\otimes\ell}) \\ 0 & \text{otherwise} \end{cases}$$

as $\mathbb{H}_{\ell}$-modules. There is a canonical bijection [1, Remark 1.4.3]

$$\mathrm{Hom}_{U(\mathfrak{g})}\left(M(\lambda), M(\mu)\otimes V^{\otimes\ell}\right) \to \left(M(\mu)\otimes V^{\otimes\ell}\right)^{[\lambda]}_{\lambda}.$$

Therefore

$$\begin{aligned} H^0_{\eta}\left(\mathfrak{n}_{\eta}, \left(\mathrm{std}_{\mathcal{N}}(\mu,\eta)\otimes V^{\otimes\ell}\right)^{[\lambda]}_{\lambda_{\mathfrak{z}}}\right) &\cong \mathrm{Hom}_{U(\mathfrak{g})}\left(M(\lambda), M(\mu)\otimes V^{\otimes\ell}\right) \\ &\cong \left(M(\mu)\otimes V^{\otimes\ell}\right)^{[\lambda]}_{\lambda} \\ &\cong \begin{cases} \mathrm{std}_{\mathbb{H}}(\lambda,\mu) & \text{if } \lambda-\mu\in P(V^{\otimes\ell}) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

as $\mathbb{H}_{\ell}$-modules.

□

5.2.2 Images of irreducible modules

Fix a character η of $\mathfrak{n}$, and let $\ell = n$. The following theorem is the main result of the paper, and gives an algebraic relationship between simple Whittaker modules and simple modules for the graded affine Hecke algebra. Moreover, this algebraic relationship agrees with the geometric relationship between the corresponding multiplicity formulas and intersection homologies observed by Zelevinsky [54].

Theorem 5.2.4. *Let λ be integral and dominant. Assume $W_{\lambda} = W_{\eta}$. Let ${}_{\eta}y_{\lambda}$ denote the longest element in the double coset $W_{\eta}yW_{\lambda}$. If $\lambda - {}_{\eta}y_{\lambda}\bullet\lambda \in P(V^{\otimes n})$, then*

$$F_{n,\eta,\lambda}\left(irr_{\mathcal{N}}(y\bullet\lambda,\eta)\right) = irr_{\mathbb{H}}(\lambda, {}_{\eta}y_{\lambda}\bullet\lambda).$$

If $\lambda - {}_{\eta}y_{\lambda}\bullet\lambda \notin P(V^{\otimes n})$, then $F_{n,\eta,\lambda}(irr_{\mathcal{N}}(y\bullet\lambda,\eta)) = 0$. Again, we will restate the theorem in the more natural setting of geometric parameters:

$$F_{n,\eta,\lambda}\left(irr_{\mathcal{N}}(\mathcal{O})\right) \cong \begin{cases} irr_{\mathbb{H}}(\Psi(\mathcal{O})) & \text{if } \Psi(\mathcal{O})\neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows directly from the statements following Theorem 4.0.6. □

Corollary 5.2.5. Every simple $\mathbb{H}_n$-module (with central character given by $\lambda + \rho$) appears as the image of the irreducible Whittaker module $\mathrm{irr}_{\mathcal{N}}(\mathcal{O})$ for some choice of $\eta \in \mathrm{ch}\mathfrak{n}$ and $\mathcal{O}$ a P_η orbit on G/P_λ.

Proof. For any give parameter $\mathcal{O} \in \mathcal{P}^g_{\mathbb{H}}(\lambda + \rho)$, choose η such that $W_\eta = W_\lambda$. Recall the map Φ from Section 4

$$\Phi(\mathcal{O}) = P \cdot (1 + N^t) \cdot \mathfrak{p}$$

for some $N \in \mathcal{O}$. We have that $\Phi(\mathcal{O})$ is a single P (hence P_η) orbit on G/P. Therefore $\Psi(\Phi(\mathcal{O})) = \mathcal{O}$, and

$$F_{n,\eta,\lambda}(\mathrm{irr}_{\mathcal{N}}(\Phi(\mathcal{O}))) = \mathrm{irr}_{\mathbb{H}}(\mathcal{O}).$$

□

Chapter 6

Contravariant Forms

In this chapter we will classify contravariant forms on Whittaker modules. The content of this chapter is part of an ongoing collaboration with Anna Romanov.

Definition. Let τ be the antiautomorphism of $U(\mathfrak{g})$ induced by the transpose map given by $x_\alpha \mapsto x_{-\alpha}$ for a Chevalley basis $\{x_\alpha : x_\alpha \in \mathfrak{g}_\alpha\}$ of $\mathfrak{n} \oplus \bar{\mathfrak{n}}$ and $h \mapsto h$ for $h \in \mathfrak{h}$. A *contravariant* form on a $U(\mathfrak{g})$-module X is a $\mathbb{C}$-bilinear symmetric form $\langle , \rangle$ on X such that

$$\langle uv, w \rangle = \langle v, \tau(u) w \rangle$$

for all $u \in U(\mathfrak{g})$ and $v, w \in X$.

Proposition 6.0.6. The set of contravariant forms on a cyclic $U(\mathfrak{g})$-module $X = U(\mathfrak{g})v$ are in bijection with linear functionals $\gamma : U(\mathfrak{g}) \to \mathbb{C}$ satisfying the following conditions:

(a) $\gamma\left(\mathrm{Ann}_{U(\mathfrak{g})} v\right) = 0$, and

(b) $\gamma(u) = \gamma(\tau(u))$ for all $u \in U(\mathfrak{g})$.

Proof. Given a contravariant form $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{C}$, define $\gamma : U(\mathfrak{g}) \to \mathbb{C}$ by

$$\gamma(u) = \langle uv, v \rangle.$$

An easy computation shows that γ is a linear functional satisfying conditions (a) and (b). Alternatively, given $\gamma : U(\mathfrak{g}) \to \mathbb{C}$ satisfying (a) and (b), define a bilinear form on X by

$$\langle x, y \rangle = \langle uv, u'v \rangle = \gamma(\tau(u')u)$$

for $x, y \in X$. Here $x = uv$ and $y = u'v$ for $u, u' \in U(\mathfrak{g})$. Notice that the choices of $u, u' \in U(\mathfrak{g})$ such that $x = uv$ and $y = u'v$ are not necessarily unique. It is straightforward to check that this form is symmetric, bilinear, and τ-contravariant. To see this form is well defined, suppose that $x = tv$ and $y = t'v$, with $t \neq u \in U(\mathfrak{g})$ and $t' \neq u' \in U(\mathfrak{g})$. Then

$$\begin{aligned}\langle uv, u'v\rangle - \langle tv, t'v\rangle &= \langle (u-t)v, u'v\rangle + \langle tv, (u'-t')v\rangle \\ &= \gamma(\tau(u')(u-t)) + \gamma(\tau(t)(u'-t')) \\ &= 0.\end{aligned}$$

Here the second equality follows from condition (b) and the third equality follows from (a), since $u - t$ and $u' - t'$ are in the annihilator of v. □

Let $\mathrm{std}_{\mathcal{N}}(\lambda, \eta)$ be a standard Whittaker module with Whittaker vector $w = 1 \otimes 1 \otimes 1$. Using the PBW basis theorem, we have the decomposition

$$U(\mathfrak{g}) = U(\mathfrak{l}_\eta) \oplus (U(\mathfrak{g})\mathfrak{n}^\eta + \bar{\mathfrak{n}}^\eta U(\mathfrak{g})).$$

Let π_η denote the corresponding projection map from $U(\mathfrak{g})$ to $U(\mathfrak{l}_\eta)$.

Lemma 6.0.7. If $a \in \mathrm{Ann}_{U(\mathfrak{g})} w$, then $\pi_\eta(a) \in \mathrm{Ann}_{U(\mathfrak{g})} w$.

Proof. Using the above decomposition, write $a = l + m + n$ for $l \in U(\mathfrak{l}_\eta)$, $m \in U(\mathfrak{g})\mathfrak{n}^\eta$, and $n \in \bar{\mathfrak{n}}^\eta U(\mathfrak{g})$. Since m annihilates w, we have that $a - m = l + n$ annihilates w. Since $n = yu$ for $y \in \bar{\mathfrak{n}}^\eta$, $u \in U(\mathfrak{g})$, [36, Proposition 2.4] implies that nw is either zero or is not contained in $U(\mathfrak{l}_\eta)w$. If $nw \notin U(\mathfrak{l}_\eta)w$, then $lw + nw \neq 0$ (since $lw \in U(\mathfrak{l}_\eta)w$). This contradicts the fact that $l + n$ annihilates w. Therefore $nw = 0$. We conclude that $l \in \mathrm{Ann}_{U(\mathfrak{l}_\eta)} w$. □

Let Γ denote the space of linear functionals on $U(\mathfrak{g})$ satisfying the conditions of Proposition 6.0.6 for the $U(\mathfrak{g})$-module $\mathrm{std}_{\mathcal{N}}(\lambda, \eta)$ with Whittaker vector w. Let Γ_η denote the space of linear functions on $U(\mathfrak{l}_\eta)$ satisfying the conditions of Proposition 6.0.6 for the $U(\mathfrak{l}_\eta)$-module $\mathrm{std}_{\mathcal{N}}(\lambda, \eta)_{\lambda_\mathfrak{z}} \cong U(\mathfrak{l}_\eta)w \cong Y(\xi_\eta(\lambda), \eta)$ with Whittaker vector w.

Proposition 6.0.8. The restriction map from $U(\mathfrak{g})$ to $U(\mathfrak{l}_\eta)$ of linear functionals induces an isomorphism

$$\mathrm{res} : \Gamma \xrightarrow{\sim} \Gamma_\eta.$$

Proof. If $\gamma \in \Gamma$, then clearly

$$\gamma\left(\mathrm{Ann}_{U(\mathfrak{l}_\eta)} w\right) = 0$$

since $\mathrm{Ann}_{U(\mathfrak{l}_\eta)} w \subset \mathrm{Ann}_{U(\mathfrak{g})} w$. Moreover,

$$\gamma(u) = \gamma(\tau(u)) \text{ for all } u \in U(\mathfrak{l}_\eta)$$

since $U(\mathfrak{l}_\eta) \subset U(\mathfrak{g})$. Therefore, the restriction map $\mathrm{res} : \Gamma \to \Gamma_\eta$ is well-defined. We will construct an inverse of this restriction map by extending a linear functional $\gamma_\eta \in \Gamma_\eta$ to a linear functional γ on $U(\mathfrak{g})$ by setting $\gamma(n) = 0$ for all $n \in (U(\mathfrak{g})\mathfrak{n}^\eta + \bar{\mathfrak{n}}^\eta U(\mathfrak{g}))$. This is clearly an inverse of the restriction map, and all that is left to show is that $\gamma \in \Gamma$. For $a \in \mathrm{Ann}_{U(\mathfrak{g})} w$, we can decompose a as

$$a = \pi_\eta(a) + n$$

for $n \in (U(\mathfrak{g})\mathfrak{n}^\eta + \bar{\mathfrak{n}}^\eta U(\mathfrak{g}))$. By Lemma 6.0.7, $\pi_\eta(a) \in \mathrm{Ann}_{U(\mathfrak{l}_\eta)} w$. Therefore

$$\gamma(a) = \gamma(\pi_\eta(a) + n) = \gamma_\eta(\pi_\eta(a)) = 0.$$

So $\gamma(\mathrm{Ann}_{U(\mathfrak{g})} w) = 0$. We can see that γ is τ-invariant by noticing that the decomposition $U(\mathfrak{g}) = U(\mathfrak{l}_\eta) \oplus (U(\mathfrak{g})\mathfrak{n}^\eta + \bar{\mathfrak{n}}^\eta U(\mathfrak{g}))$ is τ-invariant (i.e. $\pi_\eta(\tau(u)) = \tau(\pi_\eta(u))$). $\square$

Therefore, we have reduced the problem of computing the dimension of Γ to the case when η is a nondegenerate character of $\mathfrak{n}$. Suppose η is a nondegenerate character of $\mathfrak{n}$. Recall that $U_\eta(\mathfrak{n})$ denotes the kernel of $\eta : U(\mathfrak{n}) \to \mathbb{C}$. Let $A = \mathrm{Ann}_{U(\mathfrak{g})} w$, $\mathfrak{N} = U(\mathfrak{g})U_\eta(\mathfrak{n})$, and $\mathfrak{I} = U(\mathfrak{g})\xi(\lambda)$.

Proposition 6.0.9. [32, Theorem 3.1]

$$A = \mathfrak{N} + \mathfrak{I}.$$

Let $A^\tau = A + \tau(A)$ and $\mathfrak{N}^\tau = \mathfrak{N} + \tau(\mathfrak{N})$. Consider the decomposition $U(\mathfrak{g}) = \mathfrak{N}^\tau \oplus U(\mathfrak{h})$ and the short exact sequence:

$$0 \to \mathfrak{N}^\tau \to U(\mathfrak{g}) \xrightarrow{p_\eta} U(\mathfrak{h}) \to 0.$$

Let $Q = U(\mathfrak{h})/p_\eta(\mathfrak{I})$.

Lemma 6.0.10. As vector spaces,

$$\Gamma \cong Q^*.$$

Proof. First we will note that Q^* is canonically isomorphic to the space of linear functionals on $U(\mathfrak{h})$ which vanish on $p_\eta(\mathfrak{I})$. We will show that the restriction map from $U(\mathfrak{g})^*$ to $U(\mathfrak{h})^*$ defines an isomorphism from Γ to Q^*:

$$\begin{aligned} \mathrm{res}_{U(\mathfrak{h})} : \Gamma &\to Q^* \\ \gamma &\mapsto \gamma|_{U(\mathfrak{h})}. \end{aligned}$$

First we will show that this map is well defined. Since $\gamma(A) = 0$, and $\gamma(u) = \gamma(\tau(u))$, we have that $\gamma(A^\tau) = 0$. Using the above short exact sequence, each element $u \in \mathfrak{I}$ can be written as $u = n + p_\eta(u)$, where $n \in \mathfrak{N}^\tau$. Since $\mathfrak{I}$ and $\mathfrak{N}^\tau$ are contained in A^τ, we have that

$$0 = \gamma(u) = \gamma(n) + \gamma(p_\eta(u)) = \gamma(p_\eta(u)).$$

Therefore γ vanishes on $p_\eta(\mathfrak{I})$, and the above restriction map is well defined. It is easy to check that the inverse of the restriction map is given by

$$\begin{aligned} Q^* &\to \Gamma \\ \phi &\mapsto \phi \circ p_\eta. \end{aligned}$$

If $a \in A$, then $a = n + u$ with $n \in \mathfrak{N}$ and $u \in \mathfrak{I}$, and

$$\phi \circ p_\eta(a) = \phi(p_\eta(n)) + \phi(p_\eta(u)) = 0$$

since ϕ is assumed to vanish on $p_\eta(\mathfrak{I})$ and $p_\eta(n) = 0$. Moreover, if $u \in U(\mathfrak{g})$, then $u = m + h$, where $m \in \mathfrak{N}^\tau$ and $h \in U(\mathfrak{h})$ (using the short exact sequence above). Therefore

$$\phi\circ p_\eta(u) = \phi\circ p_\eta(m+h) = \phi\circ p_\eta(h) = \phi\circ p_\eta(\tau(h)) = \phi\circ p_\eta(\tau(m)+\tau(h)) = \phi\circ p_\eta(\tau(u))$$

since $\tau(m) \in \mathfrak{N}^\tau$ (hence $p_\eta(\tau(m)) = 0$). Composition with p_η and $\mathrm{res}_{U(\mathfrak{h})}$ are inverses because $p_\eta|_{U(\mathfrak{h})}$ is the identity map. □

Let $S = \langle S(\mathfrak{h})^W_+ \rangle$ be the ideal in $S(\mathfrak{h})$ generated by the W-invariant polynomials with positive degree. There is a graded, W-invariant, decomposition

$$S(\mathfrak{h}) = C \oplus S,$$

where C is isomorphic to $\mathbb{C}[W]$ as a representation of W, see [9, Chapter 5 Section 5 Subsection 2 page 112]. In what follows, we will identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$.

Lemma 6.0.11. If $s \in S$, then there exists an $r \in p_\eta(\mathfrak{I})$, and an $e \in U(\mathfrak{h})$ such that

$$s = r + e \text{ and } \deg(e) < \deg(s).$$

Additionally, if $r' \in p_\eta(\mathfrak{I})$, then there exists an $s' \in S$, and an $f \in U(\mathfrak{h})$ such that

$$r' = s' + f \text{ and } \deg(f) < \deg(r').$$

Proof. Let t_ρ be the ρ-twist map from $U(\mathfrak{h})$ to $U(\mathfrak{h})$, $H \mapsto H - \rho(H)$. If $s \in S$, then

$$s = h \cdot t_\rho(p_0(z))$$

where p_0 is the Harish-Chandra homomorphism, $z \in Z(\mathfrak{g})_+$, and $h \in U(\mathfrak{h})$. Writing z in the PBW basis of $U(\mathfrak{g})$, we get

$$z = p_0(z) + \sum_i y_i h_i x_i$$

where $y_i \in U(\bar{\mathfrak{n}})_+$, $h_i \in U(\mathfrak{h})$, $x_i \in U(\mathfrak{n})_+$.

Since the Harish-Chandra isomorphism induces an isomorphism between the corresponding graded objects [8, Chapter 7 Section 8 Subsection 5, page 145], we have

$$\deg(h_i) < \deg(p_0(z)) \text{ for all } i.$$

Moreover,

$$p_\eta(z) = p_0(z) + \sum_i \eta(\tau(y_i)x_i)h_i.$$

For all $h \in U(\mathfrak{h})$, $t_\rho(h) = h + e'$ for some $e' \in U(\mathfrak{h})$ with $\deg(e') < \deg(h)$. Therefore

$$\begin{aligned} s &= h \cdot t_\rho(p_0(z)) \\ &= h \cdot t_\rho\left(p_\eta(z) - \sum_i \eta(\tau(y_i)x_i)h_i\right) \\ &= h \cdot (p_\eta(z) + e'') = hp_\eta(z) + he'' \end{aligned}$$

where $\deg(e'') < \deg(p_0(z)) = \deg(p_\eta(z))$. The last thing we need is to rewrite $hp_\eta(z)$ as $u + e'''$ for $u \in p_\eta(\mathfrak{I})$ and $\deg(e''') < \deg(hp_\eta(z))$. For each $z \in Z(\mathfrak{g})$, $z - \chi_\lambda(z) \in \xi(\lambda)$, where χ_λ is the infinitesimal character corresponding to λ. So

$$hp_\eta(z) = hp_\eta(z - \chi_\lambda(z)) + \chi_\lambda(z)h = p_\eta(h(z - \chi_\lambda(z))) + \chi_\lambda(z)h.$$

Finally,

$$s = hp_\eta(z) + he'' = p_\eta(h(z - \chi_\lambda(z))) + \chi_\lambda(z)h + he''.$$

Therefore, there exists a $r \in p_\eta(\mathfrak{I})$, and an $e \in U(\mathfrak{h})$ such that

$$s = r + e \text{ and } \deg(e) < \deg(s).$$

The second statement of the lemma is proved by a similar argument. If $r \in p_\eta(\mathfrak{I})$, then

$$r = hp_\eta(z - \chi_\lambda(z)) = h\left(p_0(z) + \sum_i \eta(\tau(y_i)x_i)h_i - \chi_\lambda(z)\right).$$

Twisting by t_ρ^{-1} gives

$$r = s + f$$

where $s \in S$ and $\deg(f) < \deg(r)$. □

Lemma 6.0.12.

$$U(\mathfrak{h}) = C \oplus p_\eta(\mathfrak{I}).$$

Proof. We will begin with the graded decomposition

$$U(\mathfrak{h}) = C \oplus S$$

and proceed by induction on degree. The base case is trivial, as $p_\eta(\mathfrak{I})$ contains no constant polynomials. Let $U(\mathfrak{h})_i$ denote the set of polynomials with degree less than or equal to i. Assume $U(\mathfrak{h})_i = C_i \oplus p_\eta(\mathfrak{I})_i$. Let $h \in U(\mathfrak{h})_{i+1}$. Then

$$h = c + s$$

where $c \in C_{i+1}$ and $s \in S_{i+1}$. By the previous lemma, we can write s as

$$s = r + e$$

with $r \in p_\eta(\mathfrak{I})$ and $\deg(e) < \deg(s) \leq i + 1$. Therefore

$$h = c + r + e.$$

By induction, e can be written uniquely as $e = c' + r'$, with $c' \in C_i$ and $r' \in p_\eta(\mathfrak{I})_i$. So finally, we have a decomposition of h given by

$$h = (c + c') + (r + r').$$

Moreover, since $\deg(c'), \deg(r') < i+1$, we have that $(c+c') \in C_{i+1}$ and $(r+r') \in p_\eta(\mathfrak{I})_{i+1}$. To complete the proof, it remains to show that $C_{i+1} \cap p_\eta(\mathfrak{I})_{i+1} = 0$. Assume $x \in C_{i+1} \cap p_\eta(\mathfrak{I})_{i+1}$. By the previous lemma, $x = s + f$, where $f \in U(\mathfrak{h})$ has lower degree than x and $s \in S$. We can decompose f as $f = c + s'$ with $c \in C_i$ and $s \in S_i$. So $x = s + c + s'$. Therefore $s + s' \in S \cap C$, which implies that $s + s' = 0$. So $x = c$, but $\deg c \leq i$. So $x \in C_i \cap p_\eta(\mathfrak{I})_i$, which implies $x = 0$ by the induction hypothesis. □

Corollary 6.0.13. Assuming still that η is nondegenerate,

$$\dim \Gamma = |W|.$$

Proof. We have that $\dim C = |W|$ since C is isomorphic to $\mathbb{C}[W]$, see [9, Chapter 5 Section 5 Subsection 2 page 112]. Both Q^* and C^* are isomorphic to the space of linear functionals on $U(\mathfrak{h})$ which vanish on $p_\eta(\mathfrak{I})$. Therefore $Q^* \cong C^*$. By Lemma 8, we have that

$$\Gamma \cong C^*.$$

□

Corollary 6.0.14. We can then use the reduction at the beginning of this chapter to conclude that for general $\eta \in \operatorname{ch}\mathfrak{n}$, if Γ is the space of linear functionals on $U(\mathfrak{g})$ satisfying the conditions of Proposition 6.0.6 for the $U(\mathfrak{g})$-module $\operatorname{std}_{\mathcal{N}}(\lambda, \eta)$ and Whittaker vector w, then

$$\dim \Gamma = |W_\eta|.$$

6.1 Future directions

There are several natural extensions of this thesis which the author will study in future work.

6.1.1 Jantzen conjecture for Whittaker modules

Jantzen filtrations and the Jantzen conjecture are central to the study of highest weight modules and give a representation-theoretic interpretation of coefficients of Kazhdan-Lusztig polynomials. A natural question is how one might develop an analogous theory of Jantzen filtrations for Whittaker modules. Additionally, a

Jantzen-type conjecture for Whittaker modules would pave the way for future research on the properties of the Arakawa-Suzuki functors constructed in this thesis (and in [10]), generalizing results of Suzuki for highest weight modules [45].

In [23], Jantzen introduced a filtration for a particular class of well behaved $U(\mathfrak{g})$-modules in category $\mathcal{O}$ known as Verma modules. For each Verma module, denoted by $M(\lambda)$, Jantzen showed that there is a filtration

$$M(\lambda)^0 \supset M(\lambda)^1 \supset \cdots \supset M(\lambda)^N = 0$$

with the property that the corresponding quotients $M(\lambda)^i/M(\lambda)^{i+1}$ admit a nondegenerate contravariant form. Using this filtration, Jantzen developed a sum formula

$$\sum_i \mathrm{ch} M(\lambda)^i = \sum_{s_\alpha \cdot \lambda < \lambda} \mathrm{ch} M(s_\alpha \cdot \lambda)$$

which gives a powerful tool for studying the characters of Verma modules. The sum formula can be used to prove the BGG theorem for category $\mathcal{O}$ [22], and helped lead to the Jantzen conjecture, which describes how the filtration of $M(\lambda)$ relates to the filtration of $M(\mu)$ when $M(\lambda)$ embeds as a submodule into $M(\mu)$. The Jantzen conjecture (proved in [7]) provides deep insight to the structure of highest weight modules, and implies the truth of the celebrated Kazhdan-Lusztig conjectures (which were originally proved with geometric techniques in [6] and [11]). Moreover, the Jantzen conjecture illustrates a relationship between the coefficients of Kazhdan-Lusztig polynomials and the filtrations of Verma modules introduced by Jantzen:

$$\sum_i [\mathrm{gr}_i M(w \cdot \lambda) : L(y \cdot \lambda)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q).$$

As part of ongoing research, we aim to define a Jantzen filtration for Whittaker modules and prove a corresponding Jantzen conjecture, relating Jantzen filtrations to the coefficients of parabolic Kazhdan-Lusztig polynomials. For longest coset representatives w and y of $W_\eta \backslash W / W_\lambda$, we aim to prove the equality

$$\sum_i [\mathrm{gr}_i M(w \cdot \lambda, \eta) : L(y \cdot \lambda, \eta)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q)$$

where $P_{w,y}(q)$ is a parabolic Kazhdan-Lusztig polynomial. The successful completion of this research project would result in an interpretation of the coefficients of parabolic Kazhdan-Lusztig polynomials in terms of filtrations of standard Whittaker modules. The Jantzen conjecture for category $\mathcal{O}$ was first proved by Beilinson and Bernstein using weight filtrations of perverse sheaves [7], and later proved by Williamson using the local Hodge theory of Soergel bimodules [52]. In extending these results to the category of Whittaker modules, we will explore techniques involving weight filtrations of twisted $\mathcal{D}$-modules, and methods using the local Hodge theory of Soergel bimodules.

6.1.2 Duality and tilting modules

Modules in category $\mathcal{O}$ admit a natural exact contravariant duality functor, mapping modules M to $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^*$ [22, Chapter 3.2]. This notion of duality has many interesting applications. For example, it plays a central role in the study of tilting modules. A tilting module M in $\mathcal{O}$ is a module such that M and $M^\vee$ admit a filtration with quotients consisting of Verma modules. One of the primary obstructions to the study of tilting modules in $\mathcal{N}$ is the fact that there is not currently a notion of duality. The duality functor for category $\mathcal{O}$ does not extend to the category of Whittaker modules in a straightforward way because, in general, Whittaker modules do not decompose into $\mathfrak{h}$-weight spaces. It is, therefore, necessary to develop a more general method for defining duality in the category of Whittaker modules. An additional research direction extending from this thesis aims to construct an exact contravariant functor $(-)^\vee : \mathcal{N} \to \mathcal{N}$ which preserves infinitesimal character, and agrees with the classical highest weight duality when restricted to $\mathcal{O}$.

Duality in the category of Whittaker modules relates to the contravariant forms appearing in my project on a Jantzen conjecture for Whittaker modules. These contravariant forms should induce natural maps from a Whittaker module J to the dual object $J^\vee$. The interplay between contravariant forms and duality has been exploited for proving many results in category $\mathcal{O}$ and would advance the theory of Whittaker modules. Once duality is defined in the category of Whittaker modules, we will continue by studying the corresponding tilting modules and computing their characters.

The study of formal characters of indecomposable tilting modules will be a major step toward a Whittaker module generalization of results of Soergel in [44], which relate characters of tilting modules in category $\mathcal{O}$ to Kazhdan-Lusztig polynomials.

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