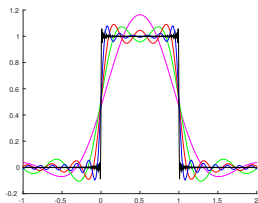


An Invitation to Harmonic Analysis

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Abstract



Fourier series are a central topic in the study of differential equations. However, it can be difficult to gain an intuition for these mysterious decompositions. We will explore how Fourier series naturally appear in representation theory, and how they can be used to solve differential equations. Generalizations of our techniques compose an extremely interesting field known as harmonic analysis.

Motivation

Gelfand's Program

Let G be a topological group acting on a topological space \mathcal{X} .

Example:

- $G = GL(n, \mathbb{R})$, $\mathcal{X} = \{X : X \text{ is a lattice in } \mathbb{R}^n\}$
- $G = U(1) = \{z \in \mathbb{C} : |z| = 1\}$, $\mathcal{X} = G$
- $G = \mathbb{R}$, $\mathcal{X} = G$

Answer questions about \mathcal{X} by understanding the group action.

Problem: this is hard

Solution: linearize

Motivation

Gelfand's Program

Find a nice Hilbert space \mathcal{H} related to \mathcal{X} . For example, a function space on \mathcal{X} . If G preserves a measure on \mathcal{X} , we can take $\mathcal{H} = L^2(\mathcal{X})$. G will now act on \mathcal{H} , and here we can make use of linear algebra and analysis.

Motivation

Gelfand's Program

If our choices are made well, G will act on \mathcal{H} by unitary operators

$$\pi : G \rightarrow U(\mathcal{H})$$

Gelfand's program says that questions about \mathcal{X} should be reformulated as questions about the representation of G on \mathcal{H} , where we can use tools from linear algebra.

Motivation

Linear Algebra

Definition

Suppose G is a topological group. A *unitary representation* of G is a pair (π, \mathcal{H}) consisting of a complex Hilbert space \mathcal{H} , and a homomorphism

$$\pi : G \rightarrow \text{Aut}_{\mathbb{C}}(\mathcal{H})$$

so that each operator $\pi(g)$ is unitary and

$$\begin{aligned} G \times \mathcal{H} &\rightarrow \mathcal{H} \\ (g, v) &\mapsto \pi(g)v \end{aligned}$$

is continuous.

Motivation

Linear Algebra

We can think about a representation as a group action of G on \mathcal{H} , $G \curvearrowright \mathcal{H}$.

Definition

An *invariant subspace* of the representation is a subspace W of \mathcal{H} which is preserved by all operators $\pi(g)$ ($\pi(g)W \subseteq W$).

Definition

A representation (π, \mathcal{H}) of a group G is *irreducible* if there are no non-trivial, proper, closed, invariant subspaces of \mathcal{H} .

Let \mathcal{X} be a Riemann manifold, G be a group, and D be a self adjoint differential operator on $W^{k,2}(\mathcal{X})$. Suppose $L^2(\mathcal{X})$ is a representation of G , and D commutes with the action of G on $W^{k,2}(\mathcal{X})$. Then $\ker(D)$ is a representation of G .

Heat Equation

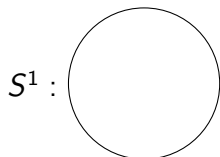
The heat equation describes the dissipation of heat through a material. Mathematically, the heat equation is

$$\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} = \frac{\partial u}{\partial t}$$

where u is a function of x_1, \dots, x_n and t .

Heat Equation on a Circle

Consider a circle.



If we want to study how heat would dissipate through a circle like this, we need to solve

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial u}{\partial t} \quad (1)$$

we will denote equation (1) by $u_{\theta\theta} = u_t$. This is equivalent to finding the kernel of the operator $u_{\theta\theta} - u_t$ on \mathcal{H} , where \mathcal{H} is a suitable function space (L^2 , differentiable, etc) on $S^1 \times \mathbb{R}$.

Heat Equation on a Circle

Let G be a group of symmetries of $S^1 \times \mathbb{R}$.

$$G = U(1) \times \mathbb{R} = \{(e^{i\theta}, x) : \theta \in [0, 2\pi), \quad x \in \mathbb{R}\}$$

G acts on S^1 by rotation and \mathbb{R} by translation. This leads to a unitary representation of G :

$$\begin{aligned}\pi & : G \rightarrow U(L^2(S^1 \times \mathbb{R})) \\ \pi(g)f(x) & = f(g^{-1}.x)\end{aligned}$$

for $x \in S^1 \times \mathbb{R}$ and $g \in G$.

Representation Theory

Theorem (Peter-Weyl)

$$L^2(S^1) \cong \widehat{\bigoplus_{n \in \mathbb{Z}} \mathbb{C} e^{in\theta}}$$

as representations of $U(1)$. Therefore, any $f \in L^2(S^1)$ can be written as an infinite sum

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$$

This is called the Fourier Series of f .

Representation Theory

Question

How do we find the constants c_n ?

Projection formula:

$$\begin{aligned}\text{Proj}_{e^{in\theta}}(f) &= \frac{\langle f, e^{in\theta} \rangle}{\|e^{in\theta}\|^2} e^{in\theta} \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} f(\phi) \overline{e^{in\phi}} d\phi \right) e^{in\theta} \\ &= \left(\frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi \right) e^{in\theta}\end{aligned}$$

So we have a formula for the constants:

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) e^{-in\phi} d\phi$$

Representation Theory

Theorem (Peter-Weyl)

$$L^2(\mathbb{R}) \cong \widehat{\int_{\xi \in \mathbb{R}} \mathbb{C} e^{i\xi t}}$$

as representations of \mathbb{R} . Therefore, any $f \in L^2(\mathbb{R})$ can be written as a direct integral

$$f(x) = \int_{\xi \in \mathbb{R}} c_{\xi} e^{i\xi x}$$

This is called the Fourier Transform of f .

Spectral Theory

Heat Equation on a Circle

Representation theory tells us we can get a basis for this space:

$$L^2(S^1 \times \mathbb{R}) \cong \widehat{\int}_{(n,\xi) \in \mathbb{Z} \times \mathbb{R}} \mathbb{C} e^{in\theta} e^{i\xi t}$$

Since our operator $u_{\theta\theta} - u_t$ is self adjoint and commutes with the action of G , the eigenspaces of $u_{\theta\theta} - u_t$ will be preserved by G . So finding the zero eigenspace of $u_{\theta\theta} - u_t$ reduces to a simple calculation.

Spectral Theory

Heat Equation on a Circle

$$\begin{aligned}(u_{\theta\theta} - u_t)e^{in\theta}e^{i\xi t} &= 0 \\ -n^2e^{in\theta}e^{i\xi t} - i\xi e^{in\theta}e^{i\xi t} &= 0 \\ -n^2 - i\xi &= 0 \\ \implies \xi &= in^2\end{aligned}$$

Therefore, solutions of the heat equation on the circle can be written as

$$u(\theta, t) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta} e^{-n^2 t}$$

Motivation

Gelfand's Program

For example, let \mathcal{X} be a pseudo-Riemannian manifold, and G be a group of isometries of \mathcal{X} . Under suitable conditions, the Laplace-Beltrami operator $-\nabla^2$ on \mathcal{X} is self adjoint. In this case, G will preserve the spectral decomposition of $-\nabla^2$. Conversely, if the action of G is transitive, then any G -invariant subspace of $L^2(\mathcal{X})$ will be preserved by $-\nabla^2$. Therefore the problem of finding the G -invariant subspaces of $L^2(\mathcal{X})$ is a refinement of the spectral problem of $-\nabla^2$ on \mathcal{X} .

Representation Theory

Question

What functions, f , have the property that

$$f_\phi(\theta) = g(\phi)f(\theta)$$

for some $g : S^1 \rightarrow \mathbb{C}$.

Answer:

If $f(\theta) = e^{in\theta}$ for an integer $n \in \mathbb{Z}$, then

$$f_\phi(\theta) = e^{in(\theta+\phi)} = e^{in\phi} e^{in\theta} = g(\phi)f(\theta)$$

Representation Theory

Theorem (Peter-Weyl)

$$\mathcal{H} \cong \widehat{\bigoplus_{n \in \mathbb{Z}} \mathbb{C} e^{in\theta}}$$

We can write a generic initial condition as a linear combination of basis elements $e^{in\theta}$:

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$$

This is called the Fourier Series of f .

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Now we want to solve $u_{\theta\theta} = u_t$ with initial condition $u(\theta, 0) = e^{in\theta}$. Let us take a guess that the answer will look something like

$$u(\theta, t) = e^{in\theta} g(t)$$

where we need to determine the function g .

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where we need to determine the function g . Solving a differential equation with this method is called separation of variables. If we plug our guess into the differential equation we get

$$\begin{aligned} -n^2 e^{in\theta} g(t) = u_{\theta\theta} &= u_t = e^{in\theta} g'(t) \\ -n^2 e^{in\theta} g(t) &= e^{in\theta} g'(t) \\ -n^2 g(t) &= g'(t) \end{aligned}$$

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So in order to find $g(t)$, we just need to solve the ordinary differential equation.

Heat Equation on a Circle

$$g'(t) = -n^2 g(t)$$

$$\frac{dg}{dt} = -n^2 g$$

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$$\int \frac{1}{g} dg = \int -n^2 dt$$

$$\ln(g) = -n^2 t + c$$

$$g(t) = Ce^{-n^2 t}$$

From our separation of variables assumption, we know that $u(\theta, 0) = e^{in\theta} g(0) = e^{in\theta}$, so $g(0) = 1$ implies that $C = 1$.

Heat Equation on a Circle

Now we can state the solution to the heat equation on a circle with initial condition $u(\theta, 0) = e^{in\theta}$:

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We can extend this solution to a solution for any initial condition $f(\theta)$ with the Fourier series. If

$$f(\theta) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta}$$

then the solution to the heat equation with initial condition $f(\theta)$ will be a linear combination of solutions of the heat equation with initial conditions $e^{in\theta}$

Heat Equation on a Circle

We can apply the principle of superposition to get the solution to the heat equation with initial condition $f(\theta)$

$$u(\theta, t) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta} e^{-n^2 t}$$

Heat Equation on a Circle

We can apply the principle of superposition to get the solution to the heat equation with initial condition $f(\theta)$

$$u(\theta, t) = \sum_{n \in \mathbb{Z}} c_n e^{in\theta} e^{-n^2 t}$$

Now we need to note that the right hand side of the equation has real and imaginary values. The real values are what we are interested in, because they tell us what the temperature is at a given location and time.

Heat Equation on a Circle

A small calculation gives us

$$\mathcal{R}(u(\theta, t)) = c_0 + \sum_{n=1}^{\infty} a_n \sin(2\pi n\theta) + b_n \cos(2\pi n\theta)$$

$$c_0 = \int_0^{2\pi} f(\phi) d\phi$$

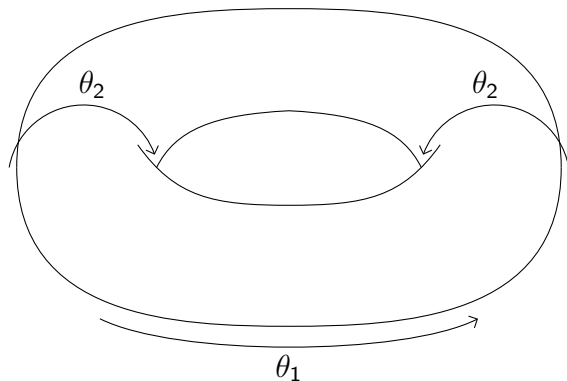
$$a_n = 2 \int_0^{2\pi} f(\phi) \sin(2\pi n\phi) d\phi$$

$$b_n = 2 \int_0^{2\pi} f(\phi) \cos(2\pi n\phi) d\phi$$

This is called the general solution to the heat equation.

Heat Equation on the Torus

$$u_{\theta_1\theta_1} + u_{\theta_2\theta_2} = u_t$$



Now we can use two different rotational symmetries to find a basis for the space of functions on the torus.

$$f_{(\phi_1, \phi_2)}(\theta_1, \theta_2) = f(\theta_1 + \phi_1, \theta_2 + \phi_2)$$

Representation Theory of T^2

Find a basis of $L^2(T^2)$

Which functions have the property that

$$f_{(\phi_1, \phi_2)}(\theta_1, \theta_2) = f(\theta_1 + \phi_1, \theta_2 + \phi_2) = g(\phi_1, \phi_2)f(\theta_1, \theta_2)$$

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Which functions have the property that

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Answer:

If $n, m \in \mathbb{Z}$, then

$$f(\theta_1, \theta_2) = e^{in\theta_1} e^{im\theta_2}$$

satisfies (4). So we want to solve our PDE with initial conditions $u(\theta_1, \theta_2, 0) = e^{in\theta_1} e^{im\theta_2}$

Heat Equation on the Torus

Again, let us assume that our solution will have the form
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$$\begin{aligned}u_{\theta_1\theta_1} + u_{\theta_2\theta_2} &= u_t \\(-n^2 e^{in\theta_1} e^{im\theta_2} - m^2 e^{in\theta_1} e^{im\theta_2})g(t) &= e^{in\theta_1} e^{im\theta_2} g'(t) = u_t \\(-n^2 - m^2)g(t) &= g'(t) \\g(t) &= e^{-n^2 t} e^{-m^2 t}\end{aligned}$$

Heat Equation on the Torus

Again, let us assume that our solution will have the form

$u(\theta_1, \theta_2, t) = e^{in\theta_1} e^{im\theta_2} g(t)$. Then

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So our general solution has the form

$$\begin{aligned}u(\theta_1, \theta_2, t) &= \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} c_{n,m} e^{in\theta_1} e^{im\theta_2} e^{-n^2 t} e^{-m^2 t} \\c_{n,m} &= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(\theta_1, \theta_2) e^{-in\theta_1} e^{-im\theta_2} d\theta_1 d\theta_2\end{aligned}$$

Representation Theory

Fourier Transform

Instead of the circle, consider functions on the line \mathbb{R} :



Now we have the symmetry

$$f_c(x) = f(x - c)$$

Representation Theory

Fourier Transform



Question: When does f have the property

$$f_c(x) = g(c)f(x)$$

Representation Theory

Fourier Transform



Question: When does f have the property

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Answer: If $f(x) = e^{i\xi x}$, then

$$f_c(x) = e^{i\xi(x-c)} = e^{-ic\xi} e^{icx} = g(c)f(x)$$

for any $\xi \in \mathbb{R}$.

Representation Theory

Fourier Transform

So if $f(x) \in L^2(\mathbb{R})$, then

$$f(x) = \sum_{\xi \in \mathbb{R}} c(\xi) e^{i\xi x} = \int c(\xi) e^{i\xi x} d\xi$$

where $c(\xi) e^{i\xi x}$ is the projection of $f(x)$ onto $e^{i\xi x}$:

$$c(\xi) = \int f(x) e^{-i\xi x} dx$$

for any $c \in \mathbb{R}$.

Spectral Theory

Heat Equation on a Circle

Instead of looking at the space of initial conditions, let us think about the space of all functions whose domain is a point on the circle and a time (square integrable, differentiable, ...).

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Instead of looking at the space of initial conditions, let us think about the space of all functions whose domain is a point on the circle and a time (square integrable, differentiable, ...). We will call this space of functions \mathcal{H} .

Spectral Theory

Heat Equation on a Circle

Instead of looking at the space of initial conditions, let us think about the space of all functions whose domain is a point on the circle and a time (square integrable, differentiable, ...). We will call this space of functions \mathcal{H} . So if u solves our differential equation, then $u(\theta, t) \in \mathcal{H}$.

Spectral Theory

Heat Equation on a Circle

Representation theory tells us we can get a basis for this space:

$$\mathcal{H} \cong \widehat{\bigoplus_{\xi \in \mathbb{R}, n \in \mathbb{Z}} \mathbb{C} e^{in\theta} e^{i\xi t}}$$

Question: Which of these basis elements actually satisfy our differential equation?

Spectral Theory

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Question: Which of these basis elements actually satisfy our differential equation?

Answer:

$$\begin{aligned} -n^2 e^{in\theta} e^{i\xi t} &= i\xi e^{in\theta} e^{i\xi t} \\ -n^2 &= i\xi \\ \implies \xi &= in^2 \end{aligned}$$

Spectral Theory

So any solution to the differential equation can be written as a linear combination of basis elements which satisfy the differential equation.

$$u(\theta, t) = \bigoplus_{n \in \mathbb{Z}} c_n e^{in\theta} e^{-n^2 t}$$

This is the answer we got before! But we never used separation of variables.

Dirac Equation

Dirac equation on $\mathbb{R}^3 \times \mathbb{R}$:

$$i\hbar\gamma_0\frac{\partial}{\partial t} - i\hbar c\sum_{j=1}^3\gamma_j\frac{\partial}{\partial x_j} = mc^2I$$

We can generalize this to a differential equation on a general Lie group and study the space of solutions in terms of representation theory.