

RESEARCH STATEMENT

Adam Brown

My research interests lie in representation theory as well as computational and applied topology. In representation theory, I use combinatorial models of geometric spaces to compute topological invariants which reveal deep structural information about representations of Lie algebras, Lie groups, and Hecke algebras. This research focuses on the theory of Whittaker modules of complex Lie algebras, and explores their functorial relationships to graded affine Hecke algebra modules. While these functorial relationships are defined algebraically, their properties are directly related to the geometric structures of flag varieties, graded nilpotent classes, and the Springer resolution. My research in computational topology explores how sheaf theory and statistics can be combined to develop new methods in data science. In what follows, I will describe my past, current, and future research in each of these areas.

1 Introduction

I study modules over the universal enveloping algebra $U(\mathfrak{g})$ of a complex semisimple Lie algebra \mathfrak{g} . A particularly well-behaved category of $U(\mathfrak{g})$ -modules, known as category \mathcal{O} , was introduced by Bernstein, Gelfand, and Gelfand [BGG71]. The objects in category \mathcal{O} are amenable to calculations and examples, while the structure of the category retains many of the rich and subtle properties exhibited by larger and more general categories of representations. In [KL79], Kazhdan and Lusztig gave a series of conjectures tying together the representation theory of category \mathcal{O} , the combinatorial theory of Hecke algebras, and the geometry of flag varieties. The conjectures and their subsequent proof [BB81, BK81] initiated the development of an extensive body of work exploring and generalizing these deep connections. Three subsequent advances in representation theory motivate and provide a basis for my research. Zelevinsky and Lusztig independently developed analogues of the Kazhdan-Lusztig conjectures for p -adic groups by studying modules over affine Hecke algebras and graded affine Hecke algebras [Zel81, BZ77, Lus89, Lus95]. Arakawa and Suzuki studied how the conjectures in complex and p -adic settings lead to functorial relationships between categories of Lie algebra representations and graded affine Hecke algebra representations [AS98, Suz98]. In [BB93], Beilinson and Bernstein gave a proof of the Jantzen conjectures (originally proposed by Jantzen in [Jan79]), illuminating the structure of filtrations of Lie algebra representations in category \mathcal{O} . My research builds on these three subjects by investigating new applications of geometric and combinatorial methods for proving various algebraic and analytic properties for Whittaker modules, Harish-Chandra modules, and graded affine Hecke algebra modules. I will outline the parallel progression of the representation theory of Whittaker modules and of graded affine Hecke algebra modules before discussing how my research illuminates the functorial and structural similarities between these seemingly unrelated topics.

Motivated by the study of Whittaker models of representations, Kostant defined a family of $U(\mathfrak{g})$ -modules and classified the irreducible modules contained in the family [Kos78]. In [MS97], Milićić and Soergel give an axiomatic construction of the category of Whittaker modules, which

contains Kostant's family of $U(\mathfrak{g})$ -modules as well as the classical Bernstein-Gelfand-Gelfand category \mathcal{O} . Suppose \mathfrak{b} is a Borel subalgebra of \mathfrak{g} with Cartan decomposition $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, and let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$. The category of Whittaker modules, denoted \mathcal{N} , is defined to be all $U(\mathfrak{g})$ -modules which are finitely generated over $U(\mathfrak{g})$, locally finite over $U(\mathfrak{n})$, and locally finite over $Z(\mathfrak{g})$. Existing techniques, developed to study category \mathcal{O} , are often insufficient to prove results for Whittaker modules, due to the irregular singularities of the \mathcal{D} -modules obtained through localization, and the loss of the semisimple action of the Cartan subalgebra \mathfrak{h} . My research builds on techniques developed by Kostant [Kos78], Backelin [Bac97], and Miličić-Soergel [MS14, MS97], which can be used to extend results from category \mathcal{O} to the category of Whittaker modules.

The graded affine Hecke algebra naturally arises from the study of representations of $\mathrm{GL}_n(\mathbb{Q}_p)$. Inspired by the work of Kazhdan and Lusztig, Zelevinsky developed a p -adic analogue of the Kazhdan-Lusztig conjectures for smooth representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ containing Iwahori fixed vectors [Zel81]. The representations studied by Zelevinsky are related to representations of the affine Hecke algebra by the Borel-Casselman correspondence, and to finite dimensional modules of the graded affine Hecke algebra by Lusztig [Lus89]. Let W be the Weyl group for \mathfrak{g} , $\Delta \subset R^+$ be the set of simple and positive roots, respectively, corresponding to the choice of Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Let $S(\mathfrak{h})$ be the symmetric algebra of \mathfrak{h} . The graded affine Hecke algebra \mathbb{H} is the associative algebra generated by $\mathbb{C}[W]$ and $S(\mathfrak{h})$ subject to the relations $s_\alpha \cdot h - s_\alpha(h) \cdot s_\alpha = \langle \alpha, h \rangle$ for all $\alpha \in \Delta$ and $h \in \mathfrak{h}$. In this setting, the graded affine Hecke algebra plays a role loosely analogous to that of the Lie algebra in the complex setting. To complete the p -adic analogue of the Kazhdan-Lusztig conjectures, Lusztig constructed algebra isomorphisms between graded affine Hecke algebras and higher endomorphism algebras of certain perverse sheaves [Lus95]. The multiplicity of irreducible representations in the composition series of standard representations is then directly related to the geometry of orbits of a Levi subgroup L_σ on $\mathfrak{g}_1(\sigma) = \{x \in \mathfrak{g} : \mathrm{ad}(\sigma)x = x\}$. These discoveries illustrate the subtle relationships between the combinatorial representation theory of S_n , the geometry of L_λ orbits on $\mathfrak{g}_1(\lambda)$, and the representation theory of $\mathrm{GL}_n(\mathbb{Q}_p)$.

The combinatorial and geometric classification of irreducible Whittaker modules and irreducible graded affine Hecke algebra modules provide a foundation for building functorial relationships between the respective categories. Moreover, these functorial relationship can be used to study the algebraic and analytic properties of Whittaker modules and of graded affine Hecke algebra modules. In the following sections we will explore several research projects aimed and illuminating such connections.

2 The category of Whittaker modules

My primary research focus is Whittaker modules, graded affine Hecke algebra modules, as well as the Dirac cohomology of Harish-Chandra modules. This research will be accomplished through three projects. The first, completed in [Bro18], constructs a family of Arakawa-Suzuki type functors for the category of Whittaker modules. The second, a collaboration currently in progress with Anna Romanov, constructs a Jantzen filtration for Whittaker modules. The third project will introduce an exact contravariant duality functor for the category of Whittaker modules. The completion of the third project is crucial for defining tilting modules in the category of Whittaker modules.

2.1 Arakawa-Suzuki functors for Whittaker modules

The basis for my past research begins with the work of Schur and Weyl on connections between the finite-dimensional representations of $G = \mathrm{GL}(V)$, the group of invertible linear transformations of $V = \mathbb{C}^n$, and the symmetric group S_n . These connections can be interpreted as a functor,

$F(X) = \text{Hom}_G(\mathbf{1}, X \otimes V^{\otimes n})$, from the category of finite dimensional representations of G to the category of finite dimensional representations of S_n . The remarkable Schur-Weyl duality of G and S_n implies that this functor maps irreducible G -representations to irreducible S_n -representations (or zero). Arakawa and Suzuki generalized the classical Schur-Weyl duality by constructing an action of \mathbb{H} on the tensor product representation $X \otimes V^{\otimes n}$, and defining a family of functors $F_\lambda(X) = \text{Hom}_{U(\mathfrak{g})}(M(\lambda), X \otimes V^{\otimes n})$ which map irreducible objects in category \mathcal{O} to irreducible \mathbb{H} -modules [AS98]. Their generalization revealed surprising relationships between representations of complex and p -adic groups, which is not apparent in the study of Schur-Weyl duality in its original form. The focus of my past research is on generalizing the results of Arakawa and Suzuki to the category of Whittaker modules, and will lead to several additional research projects on the structure of the category of Whittaker modules.

Problem 1. Extend the family of functors F_λ , constructed by Arakawa and Suzuki, to the category of Whittaker modules, completing the following diagram.

$$\begin{array}{ccccc}
 \text{finite-dimensional} & & & & \\
 G\text{-representations} & \hookrightarrow & \text{category } \mathcal{O} & \hookrightarrow & \text{Whittaker modules} \\
 & & \downarrow F_\lambda & & \swarrow ? \\
 & & \text{finite-dimensional} & & \\
 \text{finite-dimensional} & \hookrightarrow & \mathbb{H}\text{-modules} & & \\
 \mathbb{C}[S_n]\text{-modules} & & & & \\
 & \downarrow F & & & \\
 & & & &
 \end{array}$$

In [Bro18], I solve this problem by introducing a family of functors which satisfy the desired properties. For Whittaker modules, the exact functors are constructed by mapping a Whittaker module X to a subspace of the tensor product $X \otimes V^{\otimes \ell}$ determined by $\eta \in \text{chn} = (\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}])^*$ and $\lambda \in \mathfrak{h}^*$. We can define the functor $F_{\lambda, \eta}$ by

$$F_{\lambda, \eta}(X) = H_\eta^0 \left(\mathfrak{n}_\eta, (X \otimes V^{\otimes n})_{\lambda_3}^{[\lambda]} \right).$$

The techniques required for such a generalization are based on associations between the geometric parametrizations of irreducible objects in each category. Following Zelevinsky [Zel85], I construct a map from geometric parameters for irreducible Whittaker modules to geometric parameters for irreducible \mathbb{H} -modules

$$\Psi : P_\eta \text{ orbits on } G/P_\lambda \rightarrow \{L_\lambda \text{ orbits on } \mathfrak{g}_1(\lambda)\} \sqcup \{\emptyset\},$$

where P_η and P_λ are parabolic subgroups, L_λ is a Levi subgroup of P_λ , and $\mathfrak{g}_1(\lambda) = \{x \in \mathfrak{g} : \text{ad}(\lambda^\vee)x = x\}$. Using the induced map between intersection cohomology complexes, I was able to reduce the problem to computing the image of standard Whittaker modules. This calculation can be accomplished algebraically, resulting in the following theorem.

Theorem 1. [Bro18] If λ is integral and dominant, and $W_\lambda = W_\eta$, then

$$F_{\lambda, \eta}(\text{irr}_{\mathcal{N}}(\mathcal{O})) \cong \begin{cases} \text{irr}_{\mathbb{H}}(\Psi(\mathcal{O})) & \text{if } \Psi(\mathcal{O}) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

This theorem illustrates how the original duality of Schur and Weyl can be extended to functorial relationships between Whittaker modules and graded affine Hecke algebra modules.

2.2 Jantzen conjecture for Whittaker modules

My second project focuses on the development of Jantzen filtrations of Whittaker modules. This project aims to generalize the theory of Jantzen filtrations of highest weight modules, which give a beautiful representation-theoretic interpretation of coefficients of Kazhdan-Lusztig polynomials. Additionally, a Jantzen-type conjecture for Whittaker modules would pave the way for future research on the properties of the Arakawa-Suzuki functors constructed in [Bro18], generalizing results of Suzuki for highest weight modules [Suz98].

In [Jan79], Jantzen introduced a filtration for a particular class of well behaved $U(\mathfrak{g})$ -modules in category \mathcal{O} known as Verma modules. For each Verma module, denoted by $M(\lambda)$, Jantzen showed that there is a filtration

$$M(\lambda)^0 \supset M(\lambda)^1 \supset \cdots \supset M(\lambda)^N = 0$$

with the property that the corresponding quotients $M(\lambda)^i/M(\lambda)^{i+1}$ admit a nondegenerate contravariant form. Using this filtration, Jantzen developed a sum formula

$$\sum_i \text{ch} M(\lambda)^i = \sum_{s_\alpha \cdot \lambda < \lambda} \text{ch} M(s_\alpha \cdot \lambda)$$

which gives a powerful tool for studying the characters of Verma modules. The sum formula can be used to prove the BGG theorem for category \mathcal{O} [Hum08], and helped lead to the Jantzen conjecture, which describes how the filtration of $M(\lambda)$ relates to the filtration of $M(\mu)$ when $M(\lambda)$ embeds as a submodule into $M(\mu)$. The Jantzen conjecture (proved in [BB93]) provides deep insight to the structure of highest weight modules, and implies the truth of the celebrated Kazhdan-Lusztig conjectures (which were originally proved with geometric techniques in [BB81] and [BK81]). Moreover, the Jantzen conjecture illustrates a beautiful relationship between the coefficients of Kazhdan-Lusztig polynomials and the filtrations of Verma modules introduced by Jantzen:

$$\sum_i [\text{gr}_i M(w \cdot \lambda) : L(y \cdot \lambda)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q).$$

Problem 2. Define a Jantzen filtration for Whittaker modules and prove a corresponding Jantzen conjecture for standard Whittaker modules, relating Jantzen filtrations to the coefficients of parabolic Kazhdan-Lusztig polynomials.

For longest $W_\eta \backslash W/W_\lambda$ -coset representatives w and y , we aim to prove the equality

$$\sum_i [\text{gr}_i M(w \cdot \lambda, \eta) : L(y \cdot \lambda, \eta)] q^{(l(y)-l(w)-i)/2} = P_{w,y}(q).$$

The family of standard Whittaker modules contain Verma modules as a proper subset, and leads to a generalization of the Kazhdan-Lusztig theory for Whittaker modules [Rom18]. The successful completion of this research project would result in an interpretation of the coefficients of parabolic Kazhdan-Lusztig polynomials in terms of filtrations of standard Whittaker modules. The Jantzen conjecture for category \mathcal{O} was first proved by Beilinson and Bernstein using weight filtrations of perverse sheaves [BB93], and later proved by Williamson using the local Hodge theory of Soergel bimodules [Wil16]. In extending these results to the category of Whittaker modules, I will explore techniques involving weight filtrations of twisted \mathcal{D} -modules, and methods using the local Hodge theory of Soergel bimodules.

2.3 Duality and tilting modules in the category of Whittaker modules

Modules in category \mathcal{O} admit a natural exact contravariant duality functor, mapping modules M to $M^\vee := \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda^*$. This notion of duality has many interesting applications. For example, it plays a central role in the study of tilting modules. A tilting module M is a module such that M and M^\vee admit a filtration with quotients consisting of standard Whittaker modules. One of the primary obstructions to the study of Whittaker modules is the fact that there is not currently a notion of duality, and hence, no notion of tilting modules. The duality functor for category \mathcal{O} does not extend to the category of Whittaker modules in a straightforward way because, in general, Whittaker modules do not decompose into \mathfrak{h} -weight spaces. It is therefore necessary to develop a more general method for defining duality in the category of Whittaker modules.

Problem 3. Construct an exact contravariant functor $(-)^\vee : \mathcal{N} \rightarrow \mathcal{N}$ which preserves infinitesimal character, and agrees with the classical highest weight duality when restricted to \mathcal{O} .

Duality in the category of Whittaker modules relates to the contravariant forms appearing in my project on a Jantzen conjecture for Whittaker modules. These contravariant forms should induce natural maps from a Whittaker module J to the dual object J^\vee . The interplay between contravariant forms and duality has been exploited for proving many results in category \mathcal{O} , and would advance the theory of Whittaker modules. Once duality is defined in the category of Whittaker modules, I will continue by studying the corresponding tilting modules.

Problem 4. Classify the indecomposable tilting modules in the category of Whittaker modules and compute their formal characters.

The study of formal characters of indecomposable tilting modules will be a major step toward a Whittaker module generalization of results of Soergel in [Soe08], which relate characters of tilting modules in category \mathcal{O} to Kazhdan-Lusztig polynomials.

3 Dirac cohomology

This project focuses on how the Arakawa-Suzuki functors constructed by Trapa and Ciubotaru for Harish-Chandra modules relate the Dirac cohomology of (\mathfrak{g}, K) -modules to the Dirac cohomology of graded affine Hecke algebra modules [CT11, CT12, BCT12].

3.1 Dirac cohomology and Arakawa-Suzuki functors

In [CT12] and [CT11], Ciubotaru and Trapa construct a functor from the category of (\mathfrak{g}, K) -modules of the real group $\mathrm{GL}_n(\mathbb{R})$ to the category of modules over the graded affine Hecke algebra. This functor maps a (\mathfrak{g}, K) -module X to $F(X) = \mathrm{Hom}_K(\mathrm{sgn}, X \otimes V^{\otimes n})$, viewed as a module over the graded affine Hecke algebra. Ciubotaru and Trapa use the geometry of the Langlands parameters developed in [ABV92] to show that irreducible (\mathfrak{g}, K) -modules are mapped to irreducible \mathbb{H} -modules (or zero). The goal of this project is to describe how the above functor relates the Dirac cohomology of (\mathfrak{g}, K) -modules [HP06] to that of \mathbb{H} -modules [BCT12].

Dirac operators find many uses throughout mathematics, most notably in the proof of the Atiyah-Singer index theorem. Their application in representation theory began with the construction of discrete series representations by Parthasarathy and Atiyah-Schmid. Suppose G is a connected real reductive Lie group with a chosen maximal compact subgroup K , real Lie algebra

$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$, and complexified Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The Dirac operator D of \mathfrak{g} is defined to be

$$D = \sum_i Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p}),$$

where $\{Z_i\}$ is an orthonormal basis of \mathfrak{p} , and $C(\mathfrak{p})$ is the Clifford algebra of \mathfrak{p} . For a (\mathfrak{g}, K) -module X , the Dirac operator acts on the space $X \otimes S$, where S is the spin representation of $C(\mathfrak{p}_0)$. The Dirac cohomology of X is defined to be

$$H_D^{\mathfrak{g}}(X) = \ker D / (\text{im } D \cap \ker D),$$

and is naturally a representation of \tilde{K} , a double cover of K . The first phase of this project will focus on producing algorithms for explicit computations of Dirac cohomology, based on the atlas software developed by the Atlas of Lie Groups and Representations project [Ada08].

Problem 5. Impliment an algorithm for the explicit computation of Dirac cohomology in the atlas software [Ada08].

The second phase of this project is concerned with the analogous theory of Dirac cohomology for graded affine Hecke algebra modules, developed by Barbasch, Ciubotaru, and Trapa in [BCT12]. In this setting, $H_D^{\mathbb{H}}(X)$ is naturally a representation of a double cover of the Weyl group, \tilde{W} . For $G = \text{GL}_n(\mathbb{R})$, we have a natural embedding $\tilde{W} \hookrightarrow \tilde{K}$, which leads to the diagram

$$\begin{array}{ccc} (\mathfrak{g}, K)\text{-modules} & \xrightarrow{F} & \mathbb{H}\text{-modules} \\ H_D^{\mathfrak{g}}(-) \downarrow & & \downarrow H_D^{\mathbb{H}}(-) \\ \tilde{K} \text{ modules} & \xrightarrow{\text{Res}_{\tilde{W}}^{\tilde{K}}} & \mathbb{C}[\tilde{W}]\text{-modules} \end{array}$$

Problem 6. Describe when the Dirac cohomology of a (\mathfrak{g}, K) -module coincides with the Dirac cohomology of an \mathbb{H} -module, giving the equality $\text{Res}_{\tilde{W}}^{\tilde{K}}(H_D^{\mathfrak{g}}(X)) = H_D^{\mathbb{H}}(F(X))$.

By combining the two phases of this project with the results in [CT12], we will be able to use the software created to compute the Dirac cohomology of (\mathfrak{g}, K) -modules to compute the Dirac cohomology of graded affine Hecke algebra modules. Since \mathbb{H} -modules are related to representations of $\text{GL}_n(\mathbb{Q}_p)$ by the Borel-Casselman equivalence and the grading of the affine Hecke algebra developed by Lusztig, the commutativity of the above diagram would illuminate the connections between Dirac cohomology for real and p -adic groups in type A . This research project will build a foundation for understanding how Arawaka-Suzuki functors relate to Dirac cohomology in the setting of (\mathfrak{g}, K) -modules, highest weight modules, and Whittaker modules.

4 Computational topology and topological data analysis

My research on computational topology uses the framework of sheaf theory and category theory to develop algorithms and study convergence and stability in topological data analysis.

4.1 Computational stratification theory

My primary research on computation topology is concerned with recovering the structure of a stratified topological space through studying triangulations and finite point-sets sampled from the

underlying space.

Problem 7. Develop an algorithm for computing stratifications of finite simplicial complexes.

In my publication with Bei Wang [BW18], we solve this problem through the theory of cellular sheaves, and show how local homology can be used to compute a coarsening of the skeletal stratification of simplicial complexes. Motivated by the proof of the topological invariance of intersection homology [GM83], we develop an algorithm for computing a coarsest stratification of a simplicial complex for which a given sheaf is constructible. In [BW18], we introduce the following definition for sheaf-based stratifications.

Definition ([BW18]). An \mathcal{F} -stratification of a finite topological space X is a filtration of X by closed subsets X_i

$$\emptyset \subset X_0 \subset \cdots \subset X_n = X$$

such that \mathcal{F} is locally constant when restricted to each $X_i - X_{i-1}$.

Once we introduce this notion of stratification, we prove certain uniqueness results and give example computations using the local homology sheaf on a finite simplicial complex.

Theorem ([BW18]). Let K be a finite simplicial complex, and X be a finite T_0 -space consisting of the simplices of K endowed with the Alexandroff topology. Let \mathcal{F} be a sheaf on X . There exists a unique minimal homogeneous \mathcal{F} -stratification of X . Moreover, the unique minimal homogeneous \mathcal{F} -stratification is a coarsest homogeneous \mathcal{F} -stratification.

As an example, following [BW18], we can use local homology to compute a stratification of the simplicial complex illustrated below, Figure 1. Myself and Bei Wang have extended and built on the results of [BW18] in our upcoming paper *Sheaf-Theoretic Stratification Learning from Geometric and Topological Perspectives* (submitted). There we show how the approach of [BW18] can be used to study the geometric structure of stratified spaces, rather than just the topological structure. As a second example, we can stratify the same simplicial complex (contrasting the previous example) using polynomial functions and geometry, illustrated in Figure 2.

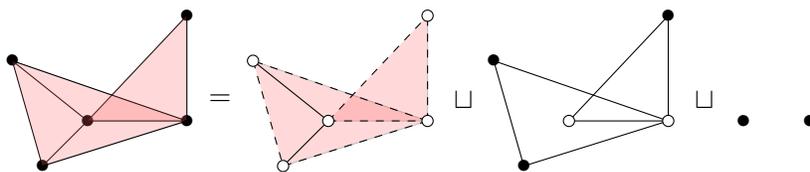


Figure 1: The minimal homogeneous stratification of a simplicial complex with respect to the local homology sheaf.

As a natural extension of this approach, we will replace simplicial complexes with finite point-sets, and local simplicial homology with local persistent homology.

Problem 8. Develop a local homology stratification algorithm for point cloud data sets.

Additionally, given a topologically stratified space which can be triangulated with a finite simplicial complex, we would like to show that if the Hausdorff distance between a finite point-set

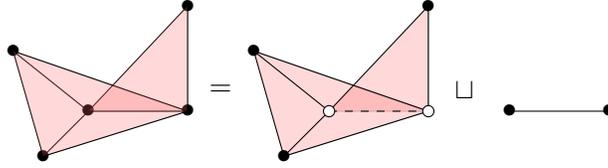


Figure 2: The minimal homogeneous stratification of a simplicial complex with respect to the pre-sheaf of vanishing polynomials.

and the underlying stratified space is sufficiently small, we can determine to which stratum a particular point in our finite point-set belongs. As a technique for solving this problem, we plan to combine the microlocal sheaf theory perspective of persistent homology developed by Kashiwara and Schapira in [KS17] and the convergence analysis of local homology in [SW14]. Specifically, we aim to construct a metric on the space of constructible functions obtained through the local Euler-Poincaré index of a sheaf \mathcal{F} in the bounded derived category of sheaves with constructible cohomology:

$$\chi(\mathcal{F})(x) = \sum_i (-1)^i H^i(\mathcal{F}_x).$$

Problem 9. Prove convergence between the constructible functions obtained through the local Euler-Poincaré index of the local homology sheaf and of the persistent homology sheaf. In other words, under a suitable metric and base topological space, prove that $\chi(\mathcal{L})|_\Omega$ converges to $\chi(\mathcal{P})|_\Omega$, where \mathcal{L} is the local homology sheaf, \mathcal{P} is the local persistent homology sheaf, and Ω is a finite point-set sampled from the base topological space.

Computational stratification theory has seen a burst of recent activity [Nan17, BW18, AS18]. However, research generalizing these techniques to the realm of point cloud data sets has yet to be fully developed. Convergence and stability results for this area of research will provide an important theoretical guarantee for a technique that analyzes the topological and geometric structure of point cloud data sets, a key aspect of unsupervised machine learning.

4.2 Convergence between point cloud mapper and the Reeb space

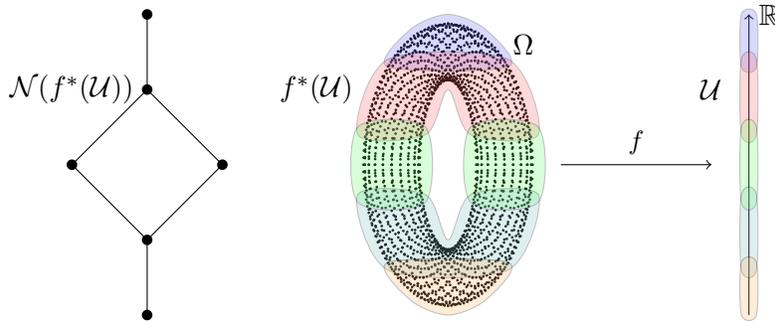


Figure 3: The mapper construction applied to the height function on the torus.

This project, a collaboration with Bei Wang and Elizabeth Munch, studies the convergence of

the point cloud mapper algorithm for a finite set of points sampled from a topological space to the Reeb space of the underlying topological space. By proving convergence for point cloud mapper, we will extend the results of Wang and Munch in [MW16].

The mapper algorithm, originally developed in [SMC07], gives a topological description of the fibers of a continuous function. For example, suppose Ω is an ϵ -net of points on the torus \mathbb{T} in \mathbb{R}^3 illustrated in Figure 3,

$$\Omega_\epsilon = \{x \in \mathbb{R}^3 : \min_{\omega \in \Omega} \|x - \omega\| < \epsilon\}$$

the ϵ -thickening of Ω , and f a continuous map from Ω_ϵ to \mathbb{R} . Let \mathcal{U} be an open cover of $f(\Omega_\epsilon)$. The mapper of \mathcal{U} and f , denoted $\mathcal{M}(\mathcal{U}, f) = \mathcal{N}(f^*(\mathcal{U}))$, is the nerve of the pull back of the open cover,

$$f^*(\mathcal{U}) = \{U \subset \Omega_\epsilon : U \text{ is a connected component of } f^{-1}(V) \text{ for some } V \in \mathcal{U}\}.$$

The Reeb space, denoted $\mathcal{R}(\mathbb{T}, f)$, is defined as the quotient \mathbb{T} / \sim_f , where $x \sim_f y$ if $f(x) = f(y)$ and x and y are contained in the same path connected component of $f^{-1}(f(x)) = f^{-1}(f(y))$. The Reeb space for our example is illustrated in Figure 4.

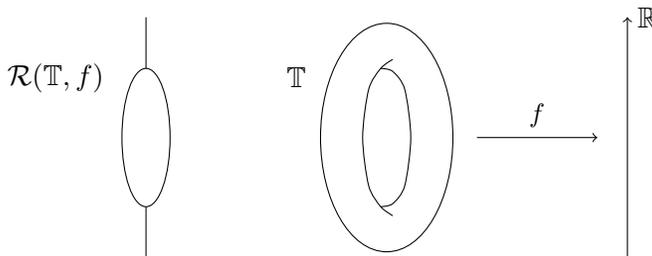


Figure 4: The Reeb space of the height function on the torus.

Problem 10. Prove that the mapper $\mathcal{M}(\mathcal{U}, f)$ of a finite set of points sampled from a topological space converges to the Reeb space $\mathcal{R}(X, f)$ as the number of points sampled increases and the resolution of the open cover decreases.

We rely on categorical and sheaf-theoretic representations of the Reeb space as well as mapper to answer this problem. We are also interested in how constructible functions can be used to define metrics which are amenable to convergence proofs, as well as the utility of describing Reeb spaces as the display space of a constructible co-sheaf (following the perspective of [Woo09, CP16]).

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