Sheaf-Theoretic Stratification Learning

Adam Brown and Bei Wang

University of Utah

abrown@math.utah.edu

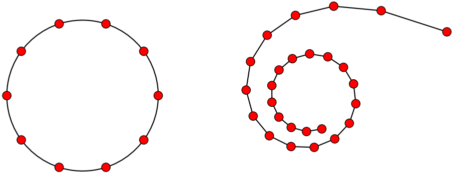
https://www.math.utah.edu/~abrown

Manifold Learning

- Infer structure from data

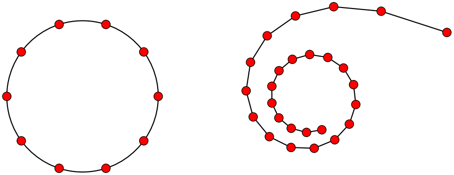
Manifold Learning

- Infer structure from data $\rightsquigarrow$ manifold learning

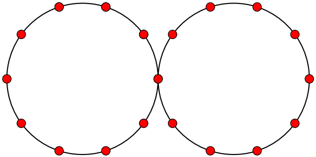


Manifold Learning

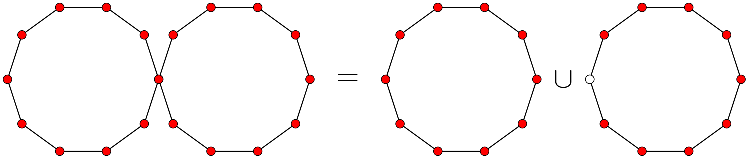
- Infer structure from data $\rightsquigarrow$ manifold learning



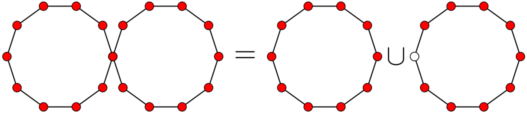
- Study the structure of singularities in data $\rightsquigarrow$ stratification learning



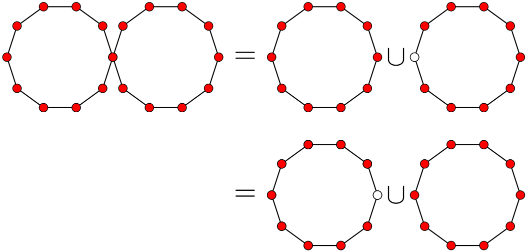
Topological Stratifications



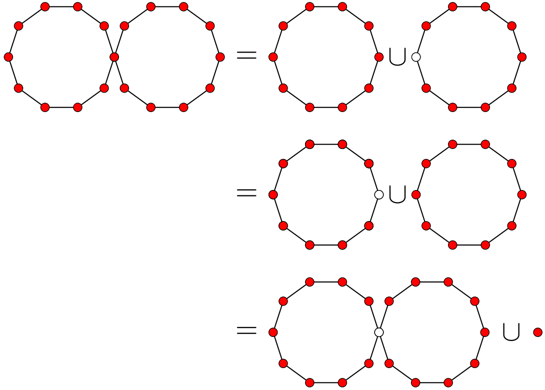
Topological Stratifications



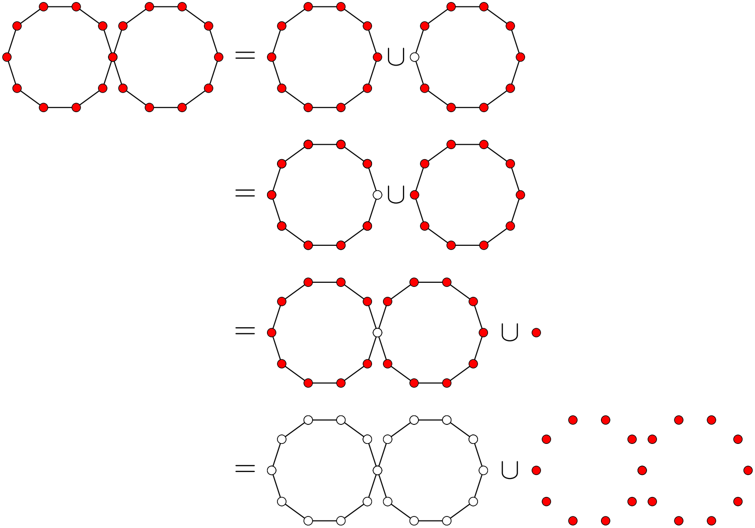
Topological Stratifications



Topological Stratifications



Topological Stratifications



Topological Stratifications

Definition

A topological stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $X_i - X_{i-1}$ is an i-dimensional (topological) manifold.

Topological Stratifications

Definition

A topological stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $X_i - X_{i-1}$ is an i-dimensional (topological) manifold.

$+$ conditions on how the strata fit together.

Topological Stratifications

Definition

A topological stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $X_i - X_{i-1}$ is an i-dimensional (topological) manifold.

$+$ conditions on how the strata fit together.

Notation: We call $X_i - X_{i-1}$ a stratum, and denote it by S_i.

Topological Stratifications

Definition

A topological stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $X_i - X_{i-1}$ is an i-dimensional (topological) manifold.

$+$ conditions on how the strata fit together.

Notation: We call $X_i - X_{i-1}$ a stratum, and denote it by S_i.

$+$ visualize the stratification by

$$X = S_n \sqcup S_{n-1} \sqcup \cdots \sqcup S_0$$

Computing Topological Stratifications

Problem: Compute (reasonable) topological stratifications of triangulated topological spaces.

Computing Topological Stratifications

Problem: Compute (reasonable) topological stratifications of triangulated topological spaces.

I need to check if $X_i - X_{i-1}$ is a manifold for each i.

Computing Topological Stratifications

Problem: Compute (reasonable) topological stratifications of triangulated topological spaces.

I need to check if $X_i - X_{i-1}$ is a manifold for each i.

In practice, this is computationally difficult...

Computing Topological Stratifications

Problem: Compute (reasonable) topological stratifications of triangulated topological spaces.

I need to check if $X_i - X_{i-1}$ is a manifold for each i.

In practice, this is computationally difficult...

Revise our definition of stratification

Computing Topological Stratifications

Problem: Compute (reasonable) topological stratifications of triangulated topological spaces.

I need to check if $X_i - X_{i-1}$ is a manifold for each i.

In practice, this is computationally difficult...

Revise our definition of stratification

Rourke-Sanderson [4], Bendich-Wang-Mukherjee [1], Nanda [3]

Results

Our contributions are four-fold:

Results

Our contributions are four-fold:

1. Definition of $\mathcal{F}$-stratification

Results

Our contributions are four-fold:

1. Definition of $\mathcal{F}$-stratification
2. Existence and uniqueness results for certain $\mathcal{F}$-stratifications

Results

Our contributions are four-fold:

1. Definition of $\mathcal{F}$-stratification
2. Existence and uniqueness results for certain $\mathcal{F}$-stratifications
3. Algorithm for computation of $\mathcal{F}$-stratifications

Results

Our contributions are four-fold:

1. Definition of $\mathcal{F}$-stratification
2. Existence and uniqueness results for certain $\mathcal{F}$-stratifications
3. Algorithm for computation of $\mathcal{F}$-stratifications
4. Application to local homology stratifications

Results

Our contributions are four-fold:

1. Definition of $\mathcal{F}$-stratification
2. Existence and uniqueness results for certain $\mathcal{F}$-stratifications
3. Algorithm for computation of $\mathcal{F}$-stratifications
4. Application to local homology stratifications

We envision that our abstraction could give rise to a larger class of computable stratifications beyond homological stratification.

Sheaves

Suppose X is a topological space.

Sheaves

Suppose X is a topological space.
A *sheaf* $\mathcal{F}$ on X associates each open set $U \subset X$ to a vector space denoted $\mathcal{F}(U)$,

Sheaves

Suppose X is a topological space.
A *sheaf* $\mathcal{F}$ on X associates each open set $U \subset X$ to a vector space denoted $\mathcal{F}(U)$, and to each inclusion $U \subset V$ a linear map

$$\mathcal{F}(U \subset V) : \mathcal{F}(V) \to \mathcal{F}(U)$$

Sheaves

Suppose X is a topological space.
A *sheaf* $\mathcal{F}$ on X associates each open set $U \subset X$ to a vector space denoted $\mathcal{F}(U)$, and to each inclusion $U \subset V$ a linear map

$$\mathcal{F}(U \subset V) : \mathcal{F}(V) \to \mathcal{F}(U)$$

such that

1. $\mathcal{F}(\emptyset) = 0$;
2. $\mathcal{F}(U \subset U) = \mathrm{id}_U$.
3. If $U \subset V \subset W$, then $\mathcal{F}(U \subset W) = \mathcal{F}(U \subset V) \circ \mathcal{F}(V \subset W)$.
4. If $\{V_i\}$ is an open cover of U, and $s_i \in \mathcal{F}(V_i)$ has the property that $\forall i, j$, $\mathcal{F}((V_i \cap V_j) \subset V_i)(s_i) = \mathcal{F}((V_j \cap V_i) \subset V_j)(s_j)$, then there exists a unique $s \in \mathcal{F}(U)$ such that $\forall i$, $\mathcal{F}(V_i \subset U)(s) = s_i$.

~~Topological~~ Stratifications

Definition

A ~~topological~~ stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $X_i - X_{i-1}$ is ~~an i-dimensional (topological) manifold~~.

~~Topological~~ Stratifications

Definition

A ~~topological~~ stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $X_i - X_{i-1}$ is ~~an i-dimensional (topological) manifold~~.

Definition

An $\mathcal{F}$-stratification of a topological space X is a filtration by closed subsets

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_n = X$$

such that $\mathcal{F}$ is locally constant when restricted to $X_i - X_{i-1}$, for each i.

Local Homology Sheaf

Suppose X is an (abstract) simplicial complex

Local Homology Sheaf

Suppose X is an (abstract) simplicial complex, viewed as a topological space with the up-set topology:

$$U \subset X \text{ is open if } \sigma \in U \text{ implies that } \tau \in U \text{ for all } \tau \geq \sigma.$$

Local Homology Sheaf

Suppose X is an (abstract) simplicial complex, viewed as a topological space with the up-set topology:

$$U \subset X \text{ is open if } \sigma \in U \text{ implies that } \tau \in U \text{ for all } \tau \geq \sigma.$$

Local homology can be viewed as a sheaf on X defined by:

$$\mathcal{L}(U) = H_\bullet(X, X - U)$$

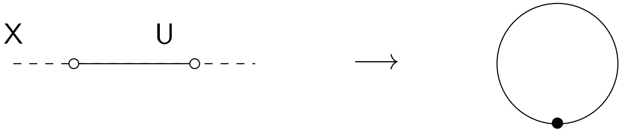
Local Homology Sheaf

Suppose X is an (abstract) simplicial complex, viewed as a topological space with the up-set topology:

$$U \subset X \text{ is open if } \sigma \in U \text{ implies that } \tau \in U \text{ for all } \tau \geq \sigma.$$

Local homology can be viewed as a sheaf on X defined by:

$$\mathcal{L}(U) = H_\bullet(X, X - U)$$



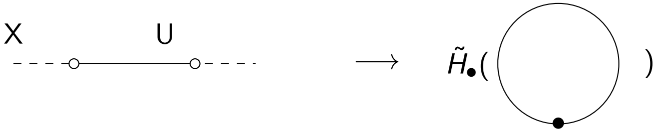
Local Homology Sheaf

Suppose X is an (abstract) simplicial complex, viewed as a topological space with the up-set topology:

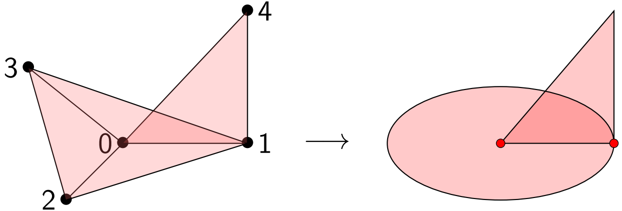
$$U \subset X \text{ is open if } \sigma \in U \text{ implies that } \tau \in U \text{ for all } \tau \geq \sigma.$$

Local homology can be viewed as a sheaf on X defined by:

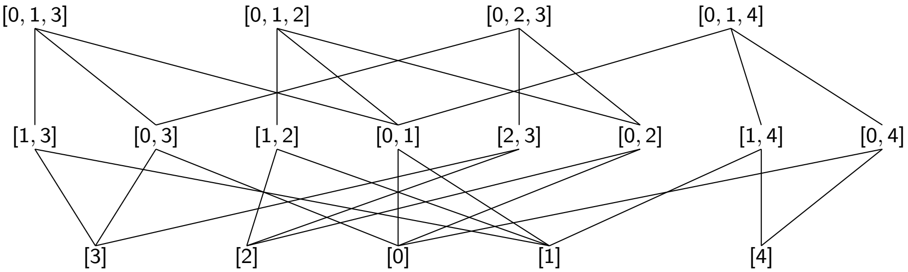
$$\mathcal{L}(U) = H_\bullet(X, X - U)$$

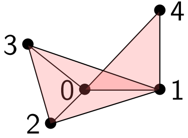


Triangulation of the Sundial

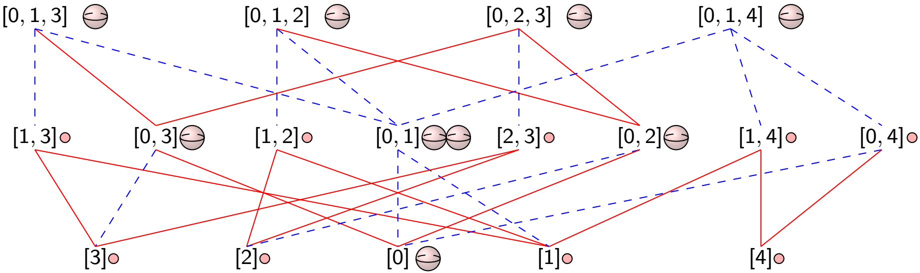


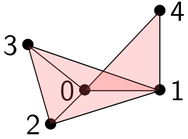
Hasse Diagram





Labeled Hasse Diagram





Inductively defined stratification

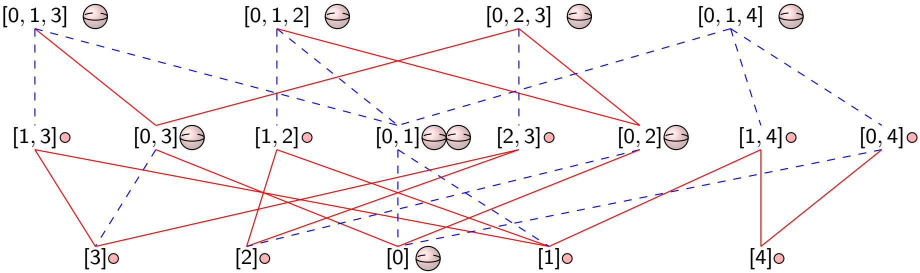
Following Goresky-MacPherson [2], Rourke-Sanderson [4], Nanda [3], we define a stratification of X inductively:

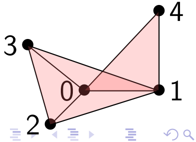
Inductively defined stratification

Following Goresky-MacPherson [2], Rourke-Sanderson [4], Nanda [3], we define a stratification of X inductively:

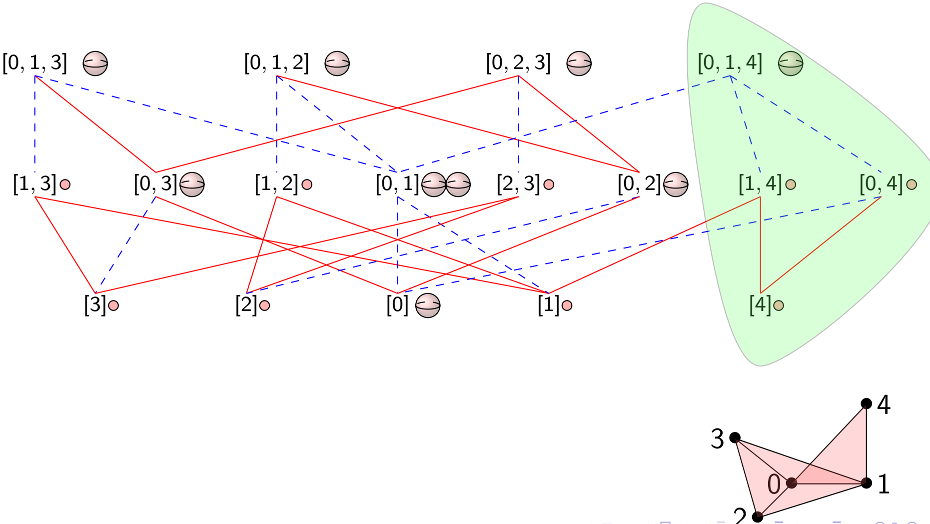
$$S_n = \{\sigma \in X : \mathcal{L}|_{\mathrm{St}\sigma} \text{ is constant}\}$$

Labeled Hasse Diagram, St[4]

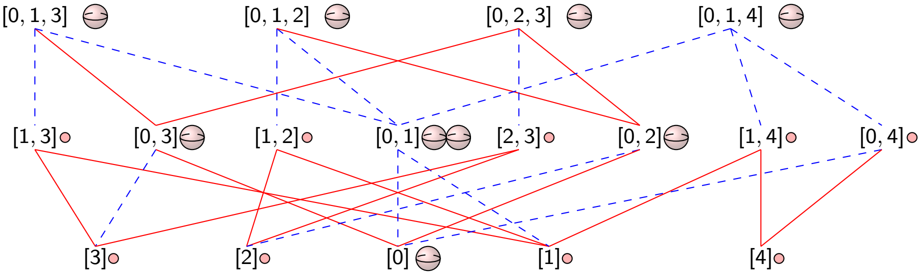


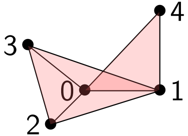


Labeled Hasse Diagram, St[4]

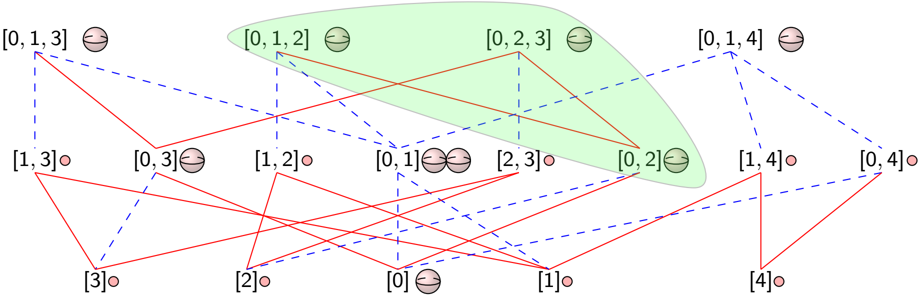


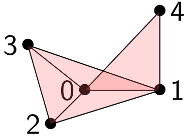
Labeled Hasse Diagram, St[0, 2]



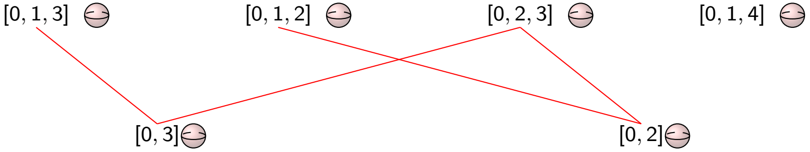


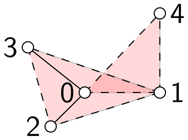
Labeled Hasse Diagram, St[0, 2]





Labeled Hasse Diagram of S_2



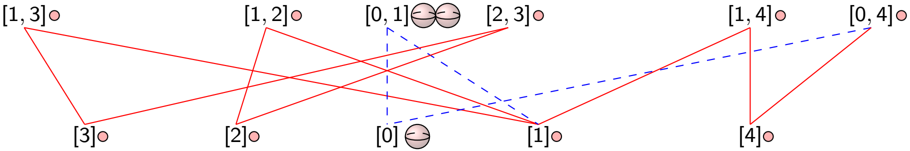


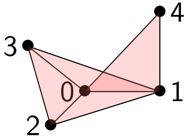
Inductively defined stratification

Inductive step: Restrict the local homology sheaf from X to the complement of S_n and repeat:

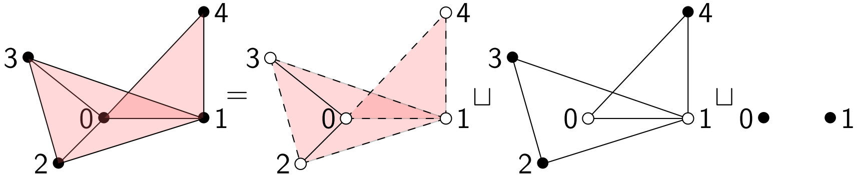
$$S_{n-1} = \{\sigma \in X - S_n : \mathcal{L}|_{\mathrm{St}_{(X-S_n)}\sigma} \text{ is constant}\}$$

Example

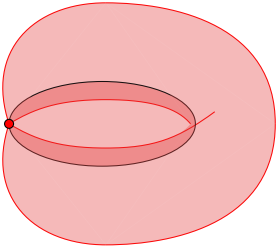




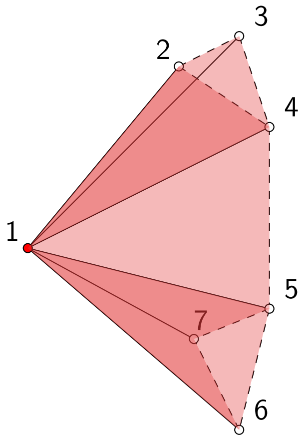
Example



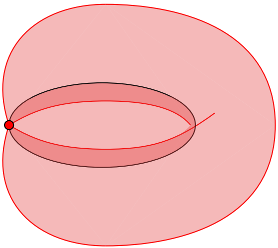
Example



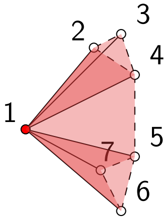
Example

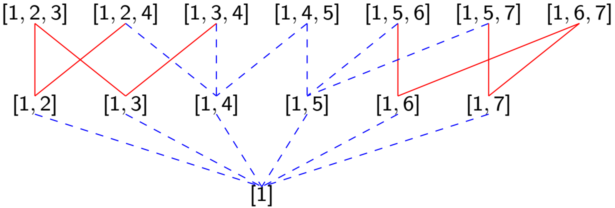


$\longrightarrow$

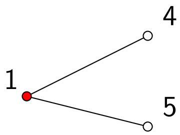


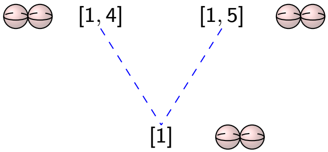
Example



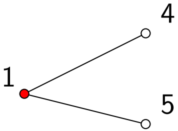


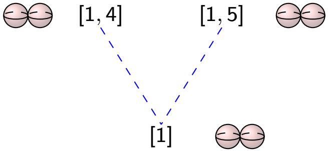
Example



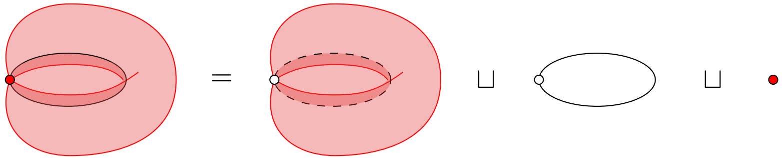


Example





Example



computational topology

computational geometry

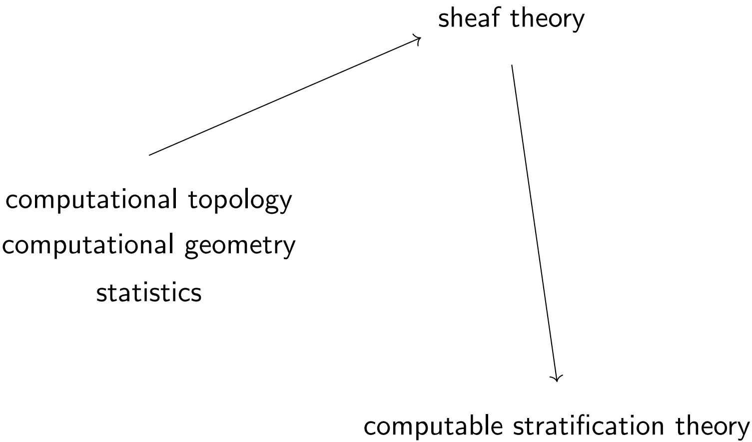
statistics

sheaf theory

computational topology

computational geometry

statistics


sheaf theory
computational topology
computational geometry
statistics
computable stratification theory

Paul Bendich, Bei Wang, and Sayan Mukherjee.
Local homology transfer and stratification learning.
ACM-SIAM Symposium on Discrete Algorithms (SODA), pages 1355–1370, 2012.

Mark Goresky and Robert MacPherson.
Intersection homology II.
Inventiones Mathematicae, 71:77–129, 1983.

Vidit Nanda.
Local cohomology and stratification.
ArXiv:1707.00354, 2017.

Colin Rourke and Brian Sanderson.
Homology stratifications and intersection homology.
Geometry and Topology Monographs, 2:455–472, 1999.

Thank you for listening.