Renormalisation group analysis of critical models at and below the upper critical dimension

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My current research involves the application of probabilistic and analytical tools to the study of problems in statistical physics. My main efforts have been directed at the extension and application of a rigorous renormalisation group method developed by Roland Bauerschmidt, David Brydges, and Gordon Slade [11].

This method is based on Ken Wilson's formulation of the renormalisation group [38], which has become a standard approach to understanding critical phenomena in quantum field theory and statistical physics. The essential idea involves the identification of critical models with points on a finite-dimensional stable manifold of an infinite-dimensional dynamical system in which a single iteration corresponds to the integration of small-scale fluctuations in a lattice field theory.

The method of Bauerschmidt, Brydges, and Slade, which is based on a finite-range covariance decomposition constructed in [3] (see also [12]), was initially developed for the analysis of the weakly self-avoiding walk [6] and the $|\varphi|^4$ spin model [4] in the upper critical dimension $d_c = 4$. In collaboration with Bauerschmidt, Slade, and Tomberg [8] and with Bauerschmidt and Slade [9], I have demonstrated that this method can be used to compute critical exponents for the finite-order correlation length of these models, as well as a number of critical exponents of the weakly self-avoiding walk with nearest-neighbour attraction.

The weakly self-avoiding walk with self-attraction

Given a continuous-time simple random walk X on $\mathbb{Z}^d$ conditioned to start at 0, define the *intersection local time* and *self-contact local time* of X up to time T, respectively, by

$$I_T(X) = \int_0^T ds \int_0^T dt \; \mathbb{1}_{X_s = X_t} \tag{1}$$

$$C_T(X) = \int_0^T ds \int_0^T dt \; \mathbb{1}_{X_s \sim X_t}, \tag{2}$$

where $x \sim y$ for $x, y \in \mathbb{Z}^d$ means that x and y are nearest neighbours. Given $\beta > 0$ and $\gamma \in \mathbb{R}$, define the potential

$$U_{\beta,\gamma,T} = \beta I_T(X) - \frac{\gamma}{2d} C_T(X). \tag{3}$$

The continuous-time weakly self-avoiding walk with contact self-attraction (WSAW-SA) is a model of a linear polymer in a solution, defined by the measure whose density, with respect to the distribution of X, is $c_T^{-1} e^{-U_{\beta,\gamma,T}}$. The normalizing constant c_T is given by

$c_T = E_0(e^{-U_{\beta,\gamma,T}})$, where E_0 is the expectation of X. A related quantity of interest is $c_T(x) = E_0(e^{-U_{\beta,\gamma,T}} \mathbb{1}_{X_T=x})$.

The first term in the definition of the potential $U_{\beta,\gamma,T}$ models the *excluded volume effect* by suppressing self-intersections. The second term models the solution in which the polymer is immersed: in a poor solution $\gamma > 0$ and it is energetically favourable for a polymer to be in contact with itself rather than the solution.

The case $\gamma = 0$ is simply called the WSAW and is a variation of the self-avoiding walk (SAW), which is the uniform distribution on paths with a fixed, discrete number of steps and no self-intersections. Nevertheless, both models are expected to possess the same general asymptotic properties. A simple example of such a property is the existence of $\nu_c = \nu_c(\beta, 0)$ (known as the *critical point*) such that $c_T^{1/T} \to e^{\nu_c}$; this follows by a standard subadditivty argument. It was predicted in the physics literature that, as $T \to \infty$ and with a logarithmic correction if $d = 4$,

$$c_T \sim A_{\beta,0} e^{\nu_c T} T^{\bar{\gamma}-1} \tag{4}$$

$$\langle |X_T|^2 \rangle \sim B_{\beta,0} T^{2\bar{\nu}}, \tag{5}$$

where $\langle \cdot \rangle$ is the expectation with respect to the WSAW measure, $a \sim b$ means that a/b approaches 1 in the limit, and $A_{\beta,0}, B_{\beta,0}, \bar{\gamma}, \bar{\nu}$ are constants. The constants $\bar{\gamma}$ and $\bar{\nu}$, which are of particular importance, are known as *critical exponents*.

A well-known approach to the analysis of the $T \to \infty$ asymptotics of a function $f(T)$ is to study the behaviour of its Laplace transform near its dominant singularity. We define the *two-point function* $G_x(\beta, \gamma, \nu)$ and the *susceptibility* $\chi(\beta, \gamma, \nu)$ to be the Laplace transforms of $c_T(x)$ and c_T, respectively:

$$G_x(\beta, \gamma, \nu) = \int_0^\infty c_T(x) e^{-\nu T} \, dT, \tag{6}$$

$$\chi(\beta, \gamma, \nu) = \int_0^\infty c_T e^{-\nu T} \, dT. \tag{7}$$

We also define the *correlation length of order p*

$$\xi_p(\beta, \gamma, \nu) = \left(\frac{1}{\chi(\beta, \gamma, \nu)} \sum_{x \in \mathbb{Z}^d} |x|^p G_{\beta,\gamma,\nu}(x) \right)^{1/p} \tag{8}$$

$$= \left(\frac{1}{\chi(\beta, \gamma, \nu)} \int_0^\infty \langle |X_T|^p \rangle c_T e^{-\nu T} \, dT \right)^{1/p}, \tag{9}$$

which is a normalization of the Laplace transform of $E_0(|X_T|^p e^{-U_{\beta,\gamma,T}})$.

By a standard *Abelian theorem* [36], the relations (4)–(5) (if they hold) imply that, as $\nu \downarrow \nu_c(\beta, 0)$,

$$\chi(\beta, 0, \nu) \sim C_{\beta,0} (\nu - \nu_c)^{-\bar{\gamma}} \tag{10}$$

$$\xi_p(\beta, 0, \nu) \sim D_{\beta,0} (\nu - \nu_c)^{-\bar{\nu}}, \tag{11}$$

again with logarithmic corrections if $d = 4$. The two-point function is not divergent at the critical point and it is predicted that the *critical* two-point function satisfies

$$G_x(\beta, 0, \nu_c) \sim F_{\beta,0}|x|^{-(d-2+\eta)} \tag{12}$$

with no logarithmic correction.

The critical exponents $\bar{\gamma}, \bar{\nu}, \eta$ are expected to be rather insensitive to changes in the specification of the model. For instance, they should remain unchanged for small $\gamma \neq 0$ and for the SAW. This is a manifestation of *universality*, which is the notion that the large-scale properties of critical models should be largely independent of their fine-scale structure (the simplest example of universality is given by the central limit theorem). For the WSAW (or SAW) on $\mathbb{Z}^d$, the hypothesis of universality suggests that the critical exponents should only depend on the underlying spatial dimension d.

Although the most interesting case from a physical standpoint, the values of the critical exponents in $d = 3$ have only been approximated numerically and rigorous results are lacking. In $d = 2$ dimensions, the critical exponents are expected to take on rational values; this case has received much attention in recent years in connection with the development of the Schramm-Loewner evolution [34, 21].

Above the *upper critical dimension* $d_c = 4$, the critical exponents are predicted to equal the corresponding values for the simple random walk, namely $\bar{\gamma} = 1, \bar{\nu} = 1/2$, and $\eta = 0$. In particular, they become independent of even the dimension d when $d > d_c$. This phenomenon is known as *mean-field behaviour*. For a discrete-time version of the WSAW, mean-field behaviour was first established for the mean squared displacement $\langle |X_T|^2 \rangle$ in [20] using the *lace expansion* technique introduced in the same paper. In [26, 25], the lace expansion was extended and used to prove mean-field behaviour of the SAW.

However, the lace expansion has two primary limitations: it cannot be applied in dimensions $d \leq d_c$, and it relies on the repulsive nature of the self-avoidance constraint (although a unique exception to the latter limitation is provided by [35]). In particular, it cannot be used to study the critical behaviour of the SAW or WSAW in dimension 4 or the WSAW-SA with $\gamma \neq 0$ in *any* dimension.

In a recent series of papers [16, 17, 7, 18, 19], Bauerschmidt, Brydges, and Slade have developed a rigorous renormalisation group method to tackle the case of the critical dimension 4. This method was applied in [6, 5] to study the susceptibility and the two-point function of the WSAW. The following theorem, obtained in collaboration with Bauerschmidt and Slade, extends these result to the WSAW-SA, i.e. to (small) $\gamma \neq 0$.

Theorem 1 (Bauerschmidt, Slade, Wallace [9])**.** *Let $d = 4$, let $\beta > 0$ be sufficiently small, and let $\gamma \in \mathbb{R}$ be small depending on β. Then there exists $\nu_c = \nu_c(\beta, \gamma) < 0$ such that,*

$$\chi(\beta, \gamma, \nu) \sim C_{\beta,\gamma}(\nu - \nu_c)^{-1}(\log(\nu - \nu_c)^{-1})^{1/4}, \quad \nu \downarrow \nu_c \tag{13}$$

$$G_x(\beta, \gamma, \nu_c) \sim F_{\beta,\gamma}|x|^{-(d-2)}, \qquad |x| \to \infty. \tag{14}$$

Asymptotics for the WSAW-SA correlation length of order p were also obtained in [9] using the main result of [8], which will be discussed in the next section.

The $|\varphi|^4$ spin model
Let $\Lambda_N = \mathbb{Z}^d/L^N\mathbb{Z}^d$ be the discrete torus of side L^N. The $|\varphi|^4$ spin model (also called the Landau-Ginzburg-Wilson model) on $\Lambda = \Lambda_N$ is defined by the probability measure

$$P_{\beta,\nu,N}(d\varphi) = \frac{1}{Z_{\beta,\nu,N}} e^{-\sum_{x\in\Lambda}\left(\frac{1}{4}\beta|\varphi_x|^4+\frac{1}{2}\nu|\varphi_x|^2+\frac{1}{2}\varphi_x\cdot(-\Delta\varphi)_x\right)} d\varphi, \qquad \varphi \in (\mathbb{R}^n)^\Lambda \tag{15}$$

where Δ is the lattice Laplacian on Λ and $\beta > 0, \nu \in \mathbb{R}$ are parameters. The $|\varphi|^4$ model has been studied extensively in quantum field theory and can be also be viewed as a version of the n-vector model of ferromagnetism (the case $n = 1$ being the celebrated Ising model) in which the spins φ_x are merely concentrated near the unit sphere (for $\nu < 0$) rather than restricted to it.

The two-point function, susceptibility, and correlation length of order p for this model are defined respectively by

$$G_x(\beta,\nu;n) = \frac{1}{n}\lim_{N\to\infty}\langle\varphi_0\cdot\varphi_x\rangle_{\beta,\nu,N} \tag{16}$$

$$\chi(\beta,\nu;n) = \lim_{N\to\infty}\sum_{x\in\Lambda} n^{-1}\langle\varphi_0\cdot\varphi_x\rangle_{\beta,\nu,N} \tag{17}$$

$$\xi_p(\beta,\nu;n) = \left(\frac{1}{\chi(\beta,\nu;n)}\sum_{x\in\mathbb{Z}^d}|x|^p G_x(\beta,\nu;n)\right)^{1/p}, \tag{18}$$

when these limits exist, where $\langle\cdot\rangle_{\beta,\nu,N}$ is the expectation with respect to $P_{\beta,\nu,N}$. Although this model is not a priori related to the WSAW, it was observed in [24] that the WSAW can be regarded *formally* as the $n \to 0$ limit of the n-vector model. This was later given a rigorous explanation via supersymmetry [28, 30]. For this reason, we regard the cases $n = 0$ of the above quantities as denoting the two-point function, susceptibility, and finite-order correlation length for the WSAW.

As with the WSAW, the quantities (16)–(18) are expected to undergo critical scaling of the form (10)–(12) (with n-dependent critical exponents). Mean-field behaviour (in $d > 4$) of the two-point function has recently been proven by Sakai [31] using the lace expansion. Earlier results, including mean-field behaviour of the two-point function and logarithmic corrections for the susceptibility, were obtained for the case $n = 1$ in $d = 4$ using alternative implementations of the renormalisation group in [23, 22, 27].

The renormalisation group method of Bauerschmidt, Brydges, and Slade has made possible the analysis of the $|\varphi|^4$ model in $d = 4$ with any $n \geq 0$. Using this method, the susceptibility and two-point function of the $|\varphi|^4$ model were analysed in [4, 33]. By an improvement to the method of [33] for $n \geq 1$ (and also [5] for $n = 0$), the asymptotics of the finite-order correlation length for both the $|\varphi|^4$ model and the WSAW were computed in [8], in collaboration with Bauerschmidt, Slade, and Tomberg.

Theorem 2 (Bauerschmidt, Slade, Tomberg, Wallace [8])**.** *Let $d = 4$, $n \geq 0$, and $p \geq 1$. For L sufficiently large and $\beta > 0$ sufficiently small (depending on p, n),*

$$\xi_p(\beta,\nu;n) \sim D_\beta^{(n)}(\nu-\nu_c)^{-1/2}(\log(\nu-\nu_c)^{-1})^{\frac{1}{2}\frac{n+2}{n+8}}. \tag{19}$$

This result was extended to the WSAW-SA with small $\gamma \in \mathbb{R}$ in [9].

Future work: long-range models below d_c
A recent and exciting new development is the work of Slade [32], in which the renormalisation group is used to compute critical exponents *below* the upper critical dimension.

The analysis of [32] is achieved by considering long-range versions of the WSAW and $|\varphi|^4$ model; this idea has precedents in [10, 15, 1, 29]. For the WSAW this means that instead of restricting to nearest-neighbour steps, one allows the underlying random walk X to jump from 0 to x with probability decaying like $|x|^{-(d+\alpha)}$, where $d = 1, 2, 3$ is the spatial dimension and $\alpha \in (0, 2)$. For the $|\varphi|^4$ model, the Laplacian is replaced by a fractional Laplacian whose matrix entries undergo similar decay. With the choice $\alpha = \frac{1}{2}(d + \epsilon)$, the upper critical dimension of these models is given by $2\alpha = d + \epsilon$ so that one is working slightly below the upper critical dimension when ϵ is small. This makes rigorous the idea introduced in [37] of working in fractional dimensions.

Part of the main result of [32] is that (for L large, ϵ small, and β in an ϵ-dependent interval), there exists $C > 0$ such that the susceptibility $\chi(\beta, \nu; n)$ satisfies

$$C^{-1}(\nu - \nu_c)^{-\left(1+\frac{n+2}{n+8}\cdot\frac{\alpha}{\epsilon}-C\epsilon^2\right)} \leq \chi(\beta, \nu; n) \leq C(\nu - \nu_c)^{-\left(1+\frac{n+2}{n+8}\cdot\frac{\alpha}{\epsilon}+C\epsilon^2\right)}. \tag{20}$$

This is consistent (to order ϵ) with the prediction (10), and provides a rigorous demonstration of the *absence* of mean-field behaviour below the upper critical dimension; this is also known as *anomalous* scaling behaviour.

The methods used to obtain (20) open up the possibility of studying the two-point function and finite-order correlation length for these long-range models. This would involve an extension of the ideas in [5, 33, 8] regarding the renormalisation group flow of *observable coupling constants*.

Intriguingly, the critical two-point function is expected to obey (12) with

$$\eta = 2 - \alpha = 2 - \frac{1}{2}(d + \epsilon). \tag{21}$$

That is, the critical exponent η is not expected to include higher-order corrections in ϵ. In fact, the *mean-field* ($d > 2\alpha$) value of η for long-range models with $|x|^{-(d+\alpha)}$ decay is also given by $\eta = 2 - \alpha$. Thus, (21) indicates that anomalous scaling below the upper critical dimension *cannot* be detected via the critical two-point function.

Another problem is to determine the scaling limits of the long-range $|\varphi|^4$ model. These should be given by α-stable random fields. The super-critical ($\nu > \nu_c$) and near-critical ($\nu \to \nu_c$) scaling limits of the nearest-neighbour $|\varphi|^4$ model were identified in [4] and it is possible that a similar approach could be used for the long-range model. However, additional extensions would be required to identify the critical ($\nu = \nu_c$) scaling limit.

Other problems
Due to the relative novelty of the renormalisation group method discussed above, there are a number of other interesting problems to which it may be applicable. For instance, a very intriguing model, and one for which rigorous results are hard to find, is the $|\varphi|^6$ model in $d = 3$, where *tricritical* behaviour is expected to occur. The analysis of this model should bear some relation to the analysis of the WSAW-SA in [9]. It would also be of interest to

extend the recent results of Adams, Kotecký, and Müller [2], who apply the renormalisation group to the study of gradient models with non-convex interactions.

Even for the ordinary WSAW and $|\varphi|^4$ model, there remain a number of observables that are expected to undergo critical scaling at or near the critical point. Given sufficient information about $\chi(\beta, \gamma, \nu)$ and $\xi_p(\beta, \gamma, \nu)$ for ν in a sector of the complex plane, it may be possible to prove (4)–(5) using Theorems 1–2 by inversion of the Laplace transforms; this was achieved for the mean squared displacement of a hierarchical model in [13]. Other quantities of interest include the "true" correlation length (the rate of exponential decay of the two-point function) as well as the magnetization of the $|\varphi|^4$ model (the expectation of φ_0) in an external field; the analysis of either of these would require significant extensions to current methods.

Broadly speaking, the renormalisation group can potentially be applied to problems that can be formulated in terms of perturbations of Gaussian measures. The *Kac-Siegert representation* provides such a formulation of the n-vector model and in [14] it was shown that the SAW two-point function can be represented as such a perturbation as well. This suggests that the renormalisation group could also be used to study models with hard constraints.

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