# Mock Exam

Note, some questions might have no answer (i.e. *"you can't do that"*), or there is only an answer if you make some conditions (e.g. *"only if $n = m$"*, or *"only if $A$ is invertible"*). You should be able to identify these.

## Linear Algebra

- for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ $z = (z_1, \ldots, z_m) \in \mathbb{R}^m$, write using indices:

$$x + y = \qquad\qquad x - z =$$

- for $A \in \mathbb{R}^{n \times m}$ with $A = (a_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$, $x$ and $y$ as above,

  write using indices and summation symbols

$$Ax = \qquad xA = \qquad x^\top Ay = \qquad yA^\top x =$$

- compute

$$Ax = \qquad \text{for } A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \text{ and } x = \begin{pmatrix} 1 \\ 1/2 \\ 1/4 \\ 1/8 \end{pmatrix}$$

- for $A$ as above and $B \in \mathbb{R}^{m \times n}$ with $(b_{ij})_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}}$, written using indices/summation symbols:

$$\operatorname{trace}(AB) = \qquad\qquad \operatorname{trace}(BA) =$$

- for $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbb{R}^{n \times n}$, write out the following expressions

$$\det(D) = \qquad\qquad \det(D^{-1}) =$$

- compute the values of

$$\det(M) = \quad \text{for } M = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{pmatrix} \qquad \det(N) = \quad \text{for } N = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

- for $x$ and $z$ as above, write using indices (where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$ and $\| \cdot \|$ is the standard norm)

$$\langle x, z \rangle = \qquad\qquad \|x\|^2 =$$

- write using inner products of vectors (not indices)

$$\|x+z\|^2 = \qquad\qquad \|x-z\|^2 =$$

- for $u, v, w \in \mathbb{R}^n$, which of the following inequalites is true?

$$\|v+w\| \leq \|v\| + \|w\| \qquad \|v+w\|^2 \leq \|v\|^2 + \|w\|^2 \qquad \|v+w\|^2 \leq (\|v\| + \|w\|)^2$$

$$\|v-w\| \leq \|v\| - \|w\| \qquad \|v-w\| \leq \big|\,\|v\| - \|w\|\,\big| \qquad \|v-w\| \geq -\big|\,\|v\| - \|w\|\,\big|$$

$$\|v+w+u\| \leq \|v\| + \|w\| + \|u\|$$

- which properties do these matrices have: orthogonal, symmetric, positive definite, invertible

$$\begin{pmatrix} \sqrt{1/2} & -\sqrt{1/2} \\ \sqrt{1/2} & \sqrt{1/2} \end{pmatrix} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \sqrt{1/3} & -\sqrt{1/3} & 0 \\ \sqrt{1/3} & \sqrt{2/3} & -\sqrt{1/3} \\ 0 & \sqrt{1/3} & -\sqrt{1/3} \end{pmatrix} \quad \mathrm{diag}(1,-1,1,-1,1,-1)$$

- what are the eigenvalues and eigenvectors of the following matrices?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

- give a recipe for efficiently computing the exponential of a matrix? what's the result for the following matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 3 & 5 \\ 1 & -1 \end{pmatrix} \leftarrow \text{that one is a bit more work...}$$

- what's the *Cholesky* decomposition of a matrix? what 's the *singular value decomposition* of a matrix?

## Calculus

- what are the derivatives $\frac{d}{dx}$ of the following functions

$$f(x) = ax+b \qquad f(x) = x^2 \qquad f(x) = \frac{1}{x} \qquad f(x) = x \log x$$

- what are the partial derivatives $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of the following functions

$$f(x,y) = ax+by \qquad f(x,y) = (x+y)^2 \qquad f(x,y) = x^y \qquad f(x,y) = \frac{1}{(x-y)^2}$$

- what are the gradients of the following functions for $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times m}$

$$f(x) = \|x\|^2 \qquad f(x,y) = \frac{1}{2} x^\top A x \qquad f(x) = \frac{1}{\|x-y\|^2} \qquad f(x) = \frac{e^{-w^\top x}}{1+e^{-w^\top x}}$$

- derive solutions to the following ordinary differential equations

$$\frac{dx}{dt} = t \text{ with } x_0 = 1 \qquad \frac{dx}{dt} = t + a \text{ with } x_0 = 0$$

- write down the expression for the Fourier transform of a function.
- find the maximum or mimimum of the following functions. Are these global or local minima/maxima?

$$f(x) = ax^2 \qquad f(x) = x^3 - x^2 \qquad f(p) = e^{e^{e^{-x^2}}}$$

## Probability

- what's the probability density function of the following distributions: uniform on $[0,1]$, Gaussian in $\mathbb{R}$?
- what's the cumulative density function of the uniform distribution on $[-1,1]$?
- for a distributions of two random variables $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, which of the following statements are true?

$$p(x,y) = p(x)+p(y) \qquad p(x,y) = p(x)p(y) \qquad p(x|y) = \frac{p(x)}{p(y)} \qquad p(x|y)p(y) = p(y|x)p(x)$$

$$p(x) = \sum_{y \in \mathcal{Y}} p(x,y) \qquad p(x|y) = \sum_{y \in \mathcal{Y}} p(x,y) \qquad p(x,y) = \sum_{y \in \mathcal{Y}} p(x|y)$$

- write down *Bayes rule*
- give an example of two random variables that are not independent of each other
- what are the mean, median, mode and standard deviation of the standard normal distribution?
- what are the expressions for the following distribution: Bernoulli, uniform, (multivariate) Gaussian, Poisson, Exponential, Beta, Gamma. Where one example each where they occur?

## Programming

- what does the following `C` program do? what's the last line that is printed?

```
#include <stdio.h>
int main() {
  for (i=1;i<1001;i++) {
    if ((i%7)==0) { printf("yes %d \n",i); };
  }
  return 0
}
```

- what does the following `python` program do? what's the last line that is printed?

```
def preference(x):
  if x == "fresh":
    return "yummy"
  else:
    return "yucky"

fruits = ["apples", "oranges", "watermelons"]
attributes = ["fresh", "black"]
for f in fruits:
  for a in attributes:
    print "%s are %s!" % (f,preference(a))
```

- read a file that reads an arbitrary number of values from a text file, computes the squares of the values and then writes those into another file
- write a program that multiplies two matrices: 1) using loops and indices, 2) using a subroutine/functionality of the programming language of your choice
- write a program that creates a vector containing 1024 random samples from a standard normal distribution
- write a program that compute the (discrete) Fourier transform of this vector
- write a program that uses a (pseudo)random number generator with real values in $[0, 1]$ to generate an integer random integer in the range $[-10, 10]$
- how can a deterministic computer produce (pseudo)random number at all?

## Numerics

- how would you approximate the derivative of a function $f(x)$ at the position $x_0$ using only 2 function evaluation?
- how would you approximate the integral of a function $f(x)$ in the interval $[0, 1]$ using only 5 function evaluation?