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Note

A Lower Bound on the Number of Unit Distances between the Vertices of a Convex Polygon*

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This paper proves that for every $n \geqslant 4$ there is a convex n-gon such that the vertices of $2n-7$ vertex pairs are one unit of distance apart. This improves the previously best lower bound of $\lfloor (5n-5)/3 \rfloor$ given by Erdős and Moser if $n \geqslant 17$.

1. INTRODUCTION

Γhis paper addresses an instance of a combinatorial distance problem ginally mentioned by Erdős and Moser [EM] (see also [E2]). The ›blem can be defined as follows. Call a finite set, S, of points in the plane *ıvex* if it is the set of vertices of a convex polygon, and write $f(S)$ for the mber of vertex pairs $\{p, q\}$ with $|p, q| = 1$ ($|p, q|$ being the Euclidean tance between p and q). Now define

$$f(n) = \max\{f(S) \mid S \text{ convex and } |S| = n\};$$

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that is, $f(n)$ is the maximum number of unit-distance pairs of a convex set of n points in the plane.

Almost 30 years ago, Erdős and Moser [EM] showed $f(n) \geqslant \lfloor 5n - 5/3 \rfloor$ (see also [MP]), and recently Füredi [F] proved $f(n) \leqslant c \cdot n \log n$ for some constant $c \leqslant 12$. This implies that the convexity restriction is an essential part of the problem since the maximum number of unit distance pairs for the general planar case is known to be at least $n^{1 + c/\log\log n}$ (see [E1]). The currently best upper bound for the general case is $c \cdot n^{4/3}$ (see [SST]).

This note shows that the constant factor of the lower bound of Erdős and Moser is not best possible. More specifically, we proved $f(n) \geqslant 2n - 7$ which exceeds the other bound if $n \geqslant 17$. The lower bound construction is presented in Section 2.

2. The Construction

This section describes a convex set of n points in the plane that realizes $2n - 7$ unit-distance pairs. It consists of the centers of three circles with unit radius and $n - 3$ points that lie on the three circles. An elementary geometry lemma that is used for the construction will be proved after describing the point set and counting the unit-distance pairs.

Let A, B, and C be the corners of an equilateral triangle whose sides have length 1. A, B, and C will not belong to the point set but merely aid in the construction of S. Now draw a circle arc between B and C with center at A and let a be the midpoint of this arc. Symmetrically construct b between C and A and c between A and B (see Fig. 1(a)). We finish the first step of the construction by drawing a circle c_a with center a that passes

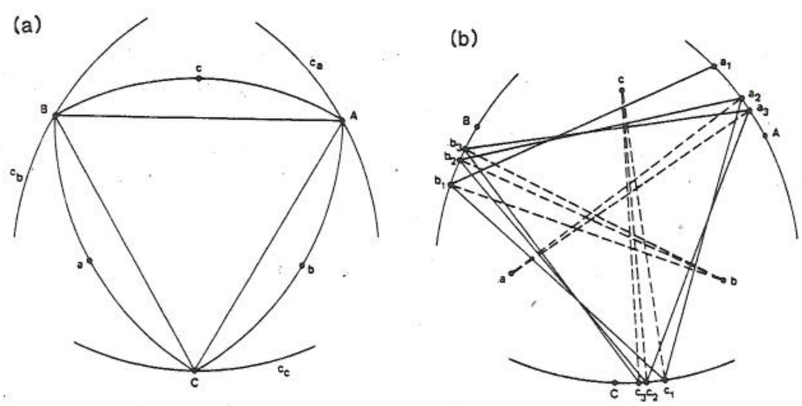

FIG. 1. (a) Drawing the circle arcs. (b) Choosing the points.

through A and symmetrically circles c_b and c_c. By construction, all three circles have radius 1. Of the three circles only points sufficiently close to A, B, or C will be used for S (see Fig. 1(a)).

For the next step of the construction assume that A, B, C is the counterclockwise sequence of vertices around the triangle they define. Choose a point a_1 on circle c_a in counterclockwise direction after A but still sufficiently close to A. If $\varepsilon = |A, a_1|$ then the rest of the construction will guarantee that all points of $S - \{a, b, c\}$ lie in ε-neighborhoods of A, B, and C. For sufficiently small $\varepsilon > 0$ this guarantees that S will be convex.

Next, choose a point b_1 on circle c_b such that $|a_1, b_1| = 1$. It is fairly clear that b_1 lies after B, in counterclockwise direction, and we will establish below that $0 < |B, b_1| < |A, a_1|$. Now just repeat the last construction step until we have n points (together with a, b, and c) and let this be set S. Thus, from b_1 we construct point c_1 on circle c_c such that $|b_1, c_1| = 1$, from c_1 we construct a_2, from a_2 we construct b_2, and so on and so forth (see Fig. 1(b)). Since the construction is symmetric and we have $0 < |B, b_1| < |A, a_1|$ if a_1 is sufficiently close to A, we obtain

$$|A, a_1| > |B, b_1| > |C, c_1| > |A, a_2| > |B, b_2| > \cdots > 0.$$

In fact, the points a_i converge to A, the b_i converge to B, and the c_i converge to C.

Let us now compute $f(S)$, the number of unit–distance pairs. First note that a is at distance 1 from every point a_i, $i \geqslant 1$, and the symmetric statements hold for b and c. This gives $n-3$ pairs. Second, a_1 is at distance 1 from b_1, b_1 is at distance 1 from c_1, c_1 from a_2, a_2 from b_2, and so on. This give $n-4$ pairs which adds up to $2n-7$ unit–distance pairs as claimed.

Finally, we prove the elementary geometry lemma used for the construction.

LEMMA. *$0 < |B, b_1| < |A, a_1|$ if $|A, a_1|$ is sufficiently small.*

Proof. Consider a trapezoid W, X, Y, Z with horizontal sides WX and YZ that is symmetric with respect to a vertical axis. Assume $|W, X| > |Y, Z|$ as in Fig. 2a. Let c be the circumcircle of the trapezoid and let U be the midpoint of the circle arc that connects X and Y. Symmetrically define V between Z and W. Observe that UV can be rotated around the center of c so that it coincides with WY which implies

$$|U, V| = |W, Y|.$$

Now replace the circle arcs connecting X and Y and Z and W by circle arcs

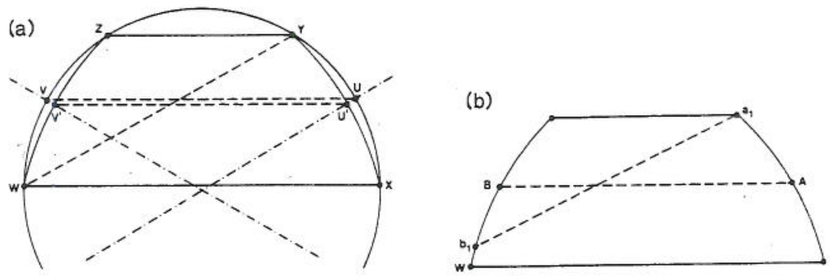

FIG. 2. (a) Trapezoid construction. (b) $A = U'$, $B = V'$, $a_1 = Y$.

with larger radius than c. The edge connecting the new midpoints U' and V' is shorter than UV which implies

$$|U', V'| < |W, Y|.$$

We remark that by the same argument a diagonal of the trapezoid is longer than the edge connecting the midpoints of XY and of ZW. This, however, is trivial since the total length of the diagonals of a convex quadrilateral is always larger than the total length of two opposite sides.

We finish the proof of the lemma by explaining how the trapezoid relates to the construction of S. As shown in Fig. 2b, we can identify U' with A, V' with B, and Y with a_1. Point W is after B in counterclockwise direction and the distance between B and W is the same as between A and a_1. By the above analysis, we have $|a_1, W| = |Y, W| > |A, B| = 1$ which implies that b_1, which is at distance 1 from a_1, must fall between B and W, exclusive. ∎

Remarks. (1) The reader might notice that the lemma is not necessary for the construction because $|B, b_1|$ varies continuously with $|A, a_1|$. It is, however, necessary for a complete understanding of the point set and helps in eliminating an otherwise necessary case-analysis in constructing the set.

(2) Consider the graph whose nodes are the first 12 points of S and whose edges are the unit-distance pairs among those points. If we contract nodes a_i, b_i, c_i for $i = 1, 2, 3$ we get a $K_{3,3}$ which shows that the graph is not planar (of course, that it is not plane as drawn in Fig. 1b is plain). This excludes the possibility to prove a linear upper bound on $f(n)$ by showing that the induced graph is always planar.

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