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ON THE ZONE THEOREM FOR HYPERPLANE ARRANGEMENTS*

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Abstract. The zone theorem for an arrangement of n hyperplanes in d-dimensional real space says that the total number of faces bounding the cells intersected by another hyperplane is $O(n^{d-1})$. This result is the basis of a time-optimal incremental algorithm that constructs a hyperplane arrangement and has a host of other algorithmic and combinatorial applications. Unfortunately, the original proof of the zone theorem, for $d \geq 3$, turned out to contain a serious and irreparable error. This paper presents a new proof of the theorem. The proof is based on an inductive argument, which also applies in the case of pseudohyperplane arrangements. The fallacies of the old proof along with some ways of partially saving that approach are briefly discussed.

Key words. discrete and computational geometry, arrangements, hyperplanes, zones, counting faces, induction, sweep

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1. Introduction. A set H of n hyperplanes in d-dimensional space $\mathbf{R}^d$ decomposes $\mathbf{R}^d$ into open cells of dimension d (also called d-faces) and into relatively open faces of dimension k between 0 and $d-1$. These cells and faces define a cell complex which is commonly known as the *arrangement* $\mathcal{A}(H)$ of H. We define the *complexity of a cell in* $\mathcal{A}(H)$ to be the number of faces that are contained in the closure of the cell.

For a hyperplane b (not necessarily in H) the *zone of* b is defined to be the set of all cells in $\mathcal{A}(H)$ that intersect b. Define the *complexity of a zone* to be the sum of the complexities of the cells in the zone. A fundamental result on hyperplane arrangements is presented in the following theorem.

ZONE THEOREM. *Any zone in any arrangement of n hyperplanes in* $\mathbf{R}^d$ *has complexity* $O(n^{d-1})$.

Various algorithmic and combinatorial applications of this theorem appear throughout the computational and combinatorial geometry literature [6]. For the case $d = 2$ a number of different and fairly straightforward proofs are known, following paradigms such as induction [5]; sweep [3], [11]; tree construction [8]; and Davenport–Schinzel sequences [9]. Only the sweep proof was extended to three and higher dimensions. However, this generalization turned out to be too sweeping. The authors of this paper discovered an irreparable error in that proof, which left the zone theorem unproven for dimensions $d > 2$.

This paper presents a new proof of the general zone theorem. It is based on a relatively straightforward inductive argument whose simplicity fosters confidence that this time the proof is actually correct. The new proof does not exploit the "straightness" of hyperplanes and thus it applies also to arrangements of pseudohyperplanes [4]. The validity of the zone theorem for such arrangements had been considered an open question.

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Section 2 contains the new proof. Explanations of why the old sweep-based proof fails and of how it can be partially saved are presented in the appendix.

A few bibliographical remarks: After the incorrectness of the old sweep proof of the zone theorem was announced, we soon learned of two partial proofs of the theorem, one by Houle [13] dating back to 1987, and the other by Matoušek [15]. They showed that the zone theorem is correct if the complexity of a cell is defined to be just the number of its facets, i.e., the number of $(d-1)$-dimensional faces contained in the cell's boundary. In the terminology of the following section, they showed that $z_1(n,d) = O(n^{d-1})$. Interestingly, Houle and Matoušek's proofs methods are virtually identical, and in essence are also the same as the saved version of the sweep proof presented in the appendix. Attempts to adapt this proof method to the general zone theorem have failed so far.

2. The new proof. For a d-polyhedron P let $g_k(P)$ denote the number of faces of P of *codimension* k (i.e., dimension $d-k$). For a hyperplane b and a set of hyperplanes H in $\mathbf{R}^d$, let $Zone(b;H)$ denote the set of cells in the arrangement $\mathcal{A}(H)$ that intersect b, and for $0 \le k \le d$ let $z_k(b;H)$ denote $\sum_{C\in Zone(b;H)} g_k(\bar{C})$, where $\bar{C}$ denotes the topological closure of C. Finally, for $n > 0$, $d > 0$, and $0 \le k \le d$, let $z_k(n,d)$ denote the maximum of $z_k(b;H)$ over all hyperplanes b and all sets H of n hyperplanes in $\mathbf{R}^d$. Our goal is to prove the following.

THEOREM 2.1. *$z_k(n,d) = O(n^{d-1})$ for each $d > 0$ and $0 \le k \le d$. In particular, for all $n > 0$, $d > 0$, and $0 \le k < d$, we have*

$$z_k(n,d) \le c_k \binom{d-1}{k}\binom{n}{d-1} + 2^k \sum_{k\le j<d-1} \binom{j}{k}\binom{n}{j},$$

where $c_0 = 1$, and $c_k = \frac{3}{4}(6^k + 2^k)$ for $k > 0$.

As will be explained in §3, the constant can be slightly improved to $c_k = \frac{2}{3}6^k + \frac{3}{4}2^k$.

First note that, for any fixed k, $z_k(b;H)$ achieves its maximum when b and H are in generic position, i.e., every k hyperplanes in $H \cup \{b\}$ intersect in a $(d-k)$-flat, for $1 < k \le d+1$. This can be proved using a standard perturbation argument: translating b slightly can only enlarge $Zone(b;H)$ and displacing a hyperplane of H by a small amount can only increase the complexities of the cells in $Zone(b;H)$ through vertex truncation or the actions dual to vertex pulling or pushing (see [12, pp. 78–83]).

Next note that Theorem 2.1 does not state an explicit upper bound for the number of vertices in a zone. However, it is easily seen that in the generic case $z_d(b;H) \le \frac{2}{d}z_{d-1}(b;H)$ holds, since in the generic (i.e., simple) case every vertex of a d-polyhedron P is incident to d edges, and each edge of P is incident to at most two vertices. Therefore we obtain from Theorem 2.1 the bound

$$z_d(n,d) \le \frac{2}{d} z_{d-1}(n,d) \le \frac{2}{d} c_{d-1}\binom{n}{d-1}.$$

The most important ingredient for our proof of Theorem 2.1 is the following lemma.

LEMMA 2.2. *For all $d > 1$, $0 \le k < d$, and $n > k$ we have*

$$z_k(n,d) \le \frac{n}{n-k}\left(z_k(n-1,d) + z_k(n-1,d-1)\right). \tag{1}$$

Proof. Let H be a set of n hyperplanes in $\mathbf{R}^d$, and let b be some other hyperplane. Because of the remarks above we assume that the hyperplanes in $H \cup \{b\}$ are in generic position. A face f in $\mathcal{A}(H)$ of codimension k now lies in exactly k hyperplanes of H

and is part of the boundary of 2^k cells of $\mathcal{A}(H)$. More than one of these cells can lie in $Zone(b; H)$, and thus the contribution of the face f to $z_k(b; H)$ can be larger than one. In order to have entities that contribute at most one to the count $z_k(b; H)$, we define a *border* of codimension k to be a pair (f, C), where f is a face of codimension k in $\mathcal{A}(H)$ and C is a cell that has f on its boundary. Thus $z_k(b; H)$ counts all borders of codimension k in $Zone(b; H)$, i.e., borders (f, C) for which C is in $Zone(b; H)$.

Now let h be some hyperplane in H, and let H/h be $\{j \cap h | j \in H \backslash \{h\}\}$. Note that H/h forms a $(d-1)$-dimensional arrangement of $n-1$ "hyperplanes" within h. Consider the expression

$$z_k(b; H\backslash\{h\}) + z_k(b \cap h; H/h).$$

We claim that it is at least as large as the number of borders (f, C) of codimension k in $Zone(b; H)$, for which f is not contained in h. Note that every such border is equal to or contained in a border in $Zone(b; H\backslash\{h\})$. Our strategy is thus to consider borders in this latter zone, and analyze what happens to them when h is added back to H. So let (f, C) be a border of codimension k in $Zone(b; H\backslash\{h\})$.

Case 1. $h \cap C = \emptyset$. The border (f, C) gives rise to exactly one border of codimension k in $Zone(b; H)$, namely, itself.

Case 2. $h \cap C \neq \emptyset$ but $h \cap f = \emptyset$. Let h_f be the (open) halfspace bounded by h that contains f, and let $C' = C \cap h_f$. If C' intersects the base hyperplane b, then (f, C) gives rise to one border of codimension k in $Zone(b; H)$, namely, (f, C'); otherwise it gives rise to no border in $Zone(b; H)$.

Case 3. $h \cap C \neq \emptyset$ and $h \cap f \neq \emptyset$. Let h' and h'' be the two open halfspaces bounded by h and let $C' = C \cap h'$ and $C'' = C \cap h''$. If only one of C' and C'' intersect b (say, C'), then (f, C) gives rise to one border of codimension k in $Zone(b; H)$, namely, $(f \cap h', C')$. However, if both C' and C'' intersect b, then (f, C) gives rise to two borders in $Zone(b; H)$, namely, $(f \cap h', C')$ and $(f \cap h'', C'')$. But in that case $C \cap h$ is part of $Zone(b \cap h; H/h)$ and $(f \cap h, C \cap h)$ is a border of codimension k in $Zone(b \cap H; H/h)$. (Note that, in generic position, the border $(f \cap h, C \cap h)$ uniquely determines the border (f, C).)

Since all borders (f, C) of codimension k in $Zone(b; H)$ for which f is not contained in h must arise as described in the three cases, it follows that the number of such borders is at most $z_k(b; H\backslash\{h\}) + z_k(b \cap h; H/h)$, as claimed. But from this we can conclude that

$$(n-k) z_k(b; H) \leq \sum_{h \in H} \left(z_k(b; H\backslash\{h\}) + z_k(b \cap h; H/h) \right),$$

since every border (f, C) of codimension k in $Zone(b; H)$ is counted in the sum $n-k$ times, once for each hyperplane h that does not contain f. From this last inequality the statement of the lemma follows immediately. □

The recurrence of Lemma 2.2 is a bit unwieldy. However, it becomes more manageable by putting $z_k(n, d) = \binom{n}{k} w_k(n, d)$, for $n \geq k$, which transforms the recurrence (1) into

$$w_k(n, d) \leq w_k(n-1, d) + w_k(n-1, d-1),$$

for all $d > 1$, $0 \leq k < d$, and $n > k$. By iterating this new recurrence on the first summand one obtains

$$w_k(n, d) \leq w_k(k, d) + \sum_{k \leq m < n} w_k(m, d-1), \tag{2}$$

valid again for all $d > 1$, $0 \le k < d$, and $n > k$.

Proving the asymptotic version of Theorem 2.1 is now an easy induction on d. The base case $z_k(n, 2) = O(n)$ for $k = 0, 1, 2$ is proved separately using any one of the proofs offered in [3], [5], [8], [9] (or also in Lemma 2.3).

Now let $d > 2$ and assume inductively that $z_k(m, d-1) = O(m^{d-2})$, for all $k \le d-1$ (where the constant of proportionality depends on k and d). Then $w_k(m, d-1) = O(m^{d-2-k})$, and thus by (2)

$$w_k(n,d) = w_k(k,d) + \sum_{k \le m < n} O(m^{d-2-k}),$$

which implies $w_k(n,d) = O(n^{d-1-k})$, thereby showing that $z_k(n,d) = O(n^{d-1})$, provided $k \le d-2$. When $k = d-1$ this approach yields only an $O(n^{d-1}\log n)$ bound. However, one can now establish the desired $z_k(n,d) = O(n^{d-1})$ for $k = d-1$ and $k = d$ as follows: Euler's relation states that for any cell C in an arrangement the sum $\sum_{0 \le k \le d}(-1)^{d-k} g_k(\overline{C})$ evaluates to 0 or 1, depending on whether C is bounded (see [12, pp. 130–140]). Thus it follows that in $\mathbf{R}^d$ for any set H of $n \ge d$ hyperplanes and any hyperplane b in generic position

$$\sum_{0 \le k \le d} (-1)^{d-k} z_k(b; H) \ge 0.$$

Recalling that $z_d(b; H) \le \frac{2}{d} z_{d-1}(b; H)$, we obtain the relation

$$(3) \qquad \left(1 - \frac{2}{d}\right) z_{d-1}(b;H) \le z_{d-1}(b;H) - z_d(b;H) \le \sum_{0 \le k \le d-2} (-1)^{d-k} z_k(b;H).$$

But as $d > 2$ and we have proven already that $z_k(b; H) \le z_k(n,d) = O(n^{d-1})$ for $0 \le k \le d-2$, and since $z_d(b;H) \le \frac{2}{d} z_{d-1}(b;H)$, relation (3) yields that $z_k(b;H) = O(n^{d-1})$ holds for all k. As this is true for any H and b in generic position and since it suffices to consider only generic position we can conclude that $z_k(n,d) = O(n^{d-1})$ for $0 \le k \le d$. This completes the proof of the asymptotic version of Theorem 2.1.

The proof for the more exact, nonasymptotic version follows the same inductive scheme, except that we will use a slightly different method for dealing with the case $k > d-2$. First we need a few small lemmas. Recall that when arguing about $z_k(n,d)$ we need only to consider simple arrangements with the zone-producing hyperplane b in generic position.

LEMMA 2.3. $z_1(n, 2) \le 6n$.

Proof. Of course this is just the Zone Theorem for arrangements of lines in the plane, and we could refer to a number of different proofs ([3], [5], [8], [9]; in fact, [3] gives a slightly better bound—see §3). For the sake of completeness, however, we include here yet a different proof.

Let H be a set of n lines in the plane, and let b be some other line. Without loss of generality we assume that $H \cup \{b\}$ is in generic position (no three lines intersect, but every two do), and we assume that b is "horizontal." We need to show that the sum of the edges of the cells in *Zone*$(b; H)$ is at most $6n$.

Since no line in H is parallel to the horizontal line b, it makes sense to talk about the left bounding edges and the right bounding edges of a cell. It suffices to show that the total number of all left bounding edges of the cells in *Zone*$(b; H)$ is at most $3n$. This is clearly true when H is empty. So let h be the line in H that intersects b furthest to the right. By induction the total number of left bounding edges of the cells in

Zone$(b; H\backslash\{h\})$ is at most $3n - 3$. The addition of h to the arrangement formed by $H\backslash\{h\}$ can increase this number at most by 3. □

LEMMA 2.4. *For any set H of n hyperplanes and any hyperplane b in generic position in $\mathbf{R}^d$ with $d > 2$ we have*

$$z_{d-3}(b; H) \geq 2^{d-3}\binom{d-1}{2}\binom{n}{d-1} + 2^{d-3}\binom{d-2}{1}\binom{n}{d-2}.$$

Proof. Assuming generic position of a set H of n hyperplanes and an additional hyperplane b, every 2-face in the $(d-1)$-dimensional arrangement induced by H in b derives from a 3-face in $\mathcal{A}(H)$ that is in the boundary of 2^{d-3} cells of $\mathcal{A}(H)$, all of which are in *Zone*$(b; H)$. It now suffices to observe that in a simple $(d-1)$-dimensional arrangement of n hyperplanes the number of 2-faces is

$$\sum_{0\leq i\leq 2}\binom{d-1-i}{2-i}\binom{n}{d-1-i} \geq \binom{d-1}{2}\binom{n}{d-1} + \binom{d-2}{1}\binom{n}{d-2}$$

(see [6, p. 7]). □

LEMMA 2.5. *For all $d > 2$ and for all $n > d-1$ we have*

$$\binom{d-1}{2}z_{d-1}(n,d) \leq 3(d-2)z_{d-2}(n,d) - 6\cdot 2^{d-3}\left[\binom{d-1}{2}\binom{n}{d-1} + \binom{d-2}{1}\binom{n}{d-2}\right].$$

Proof. Lemma 2.4 implies that it suffices to show the validity of the inequality

$$\binom{d-1}{2}z_{d-1}(b; H) \leq 3(d-2)z_{d-2}(b; H) - 6z_{d-3}(b; H),$$

for any set H of n hyperplanes in $\mathbf{R}^d$ and any hyperplane b in generic position (recall that it suffices to consider only generic position).

Since in a simple arrangement of $n > d-1$ hyperplanes in $\mathbf{R}^d$ every face is pointed (i.e., has a vertex) it suffices to show that for any pointed simple d-polyhedron P with $d > 2$, the inequality $\binom{d-1}{2}f_1 \leq 3(d-2)f_2 - 6f_3$ holds, where $f_i = g_{d-i}$ is the number of i-dimensional faces of P.

Let F_3 be the set of three-dimensional faces of P, and for a 3-face $c \in F_3$ let $e(c)$ denote the number of edges of c and let $s(c)$ denote the number of 2-faces of c. For $j = 1, 2$ let $I_{j,3}$ denote the number of pairs (X, Y) so that Y is a 3-face of P, and X is a j-face of Y.

Because of the simplicity of P every 1-face is contained in $\binom{d-1}{2}$ faces of dimension 3, and thus $I_{1,3} = \binom{d-1}{2}f_1$. Similarly $I_{2,3} = (d-2)f_2$. On the other hand $I_{1,3}$ can also be expressed as $\sum_{c\in F_3} e(c)$, and $I_{2,3}$ as $\sum_{c\in F_3} s(c)$. Euler's relation in any three-dimensional pointed polyhedron with e edges and s facets implies the inequality $e \leq 3s - 6$. Hence,

$$\binom{d-1}{2}f_1 = \sum_{c\in F_3} e(c) \leq \sum_{c\in F_3}(3s(c) - 6) = 3I_{2,3} - 6f_3 = 3(d-2)f_2 - 6f_3,$$

as asserted. □

We now have everything ready to give a complete inductive proof of our main Theorem 2.1, which claims that for each $d > 0$ the following holds for all $n > 0$ and for each $0 \leq k < d$:

$$z_k(n,d) \le Z_k(n,d) \equiv c_k \binom{d-1}{k}\binom{n}{d-1} + 2^k \sum_{k\le j<d-1} \binom{j}{k}\binom{n}{j}, \tag{4}$$

where $c_0 = 1$, and $c_k = \frac{3}{4}(6^k + 2^k)$ for $k > 0$. For $k > 1$ it is easy to check that $c_k = 6(c_{k-1} - 2^{k-2})$ with $c_1 = 6$.

We first dispose of a few easy cases. The bound (4) is trivially correct when $d = 1$. So assume $d > 1$ and consider $n < k < d$. In this case the bound $Z_k(n,d)$ in (4) evaluates to 0, which is correct since there cannot be any face with codimension k when there are fewer than k hyperplanes.

Finally we can dispose of the case $n = k < d$ since in an arrangement of $k < d$ hyperplanes in generic position there is exactly one face of codimension k and it is in the boundary of 2^k cells, all of which are intersected by the zone hyperplane. Thus $z_k(k,d) = 2^k \le Z_k(k,d)$, as desired. From now on we will thus assume that $d > 1$ and $n > k$.

Applying the binomial product identity $\binom{A}{B}\binom{C}{A} = \binom{C}{B}\binom{C-B}{A-B}$ to each term on the right side of (4), and using the substitution $z_k(n,d) = \binom{n}{k}w_k(n,d)$ as before, we can rewrite (4) equivalently as

$$w_k(n,d) \le W_k(n,d) \equiv c_k \binom{n-k}{d-1-k} + 2^k \sum_{k\le j<d-1} \binom{n-k}{j-k}.$$

We can now prove the bounds (4) by induction on d. For the base case $d = 2$ Lemma 2.3 implies that indeed $z_1(n,2) \le 6n = Z_1(n,d)$; the fact that $z_0(n,d) \le n+1 = Z_0(n,d)$ is trivial.

Now let $d > 2$ and assume inductively that for $0 \le k < d-1$ and for all $m \ge k$ the bounds $z_k(m,d-1) \le Z_k(m,d-1)$ and therefore also the equivalent bounds $w_k(m,d-1) \le W_k(m,d-1)$ hold.

Recall that for $0 \le k < d$, and $n > k$ Lemma 2.2 implies the inequality (2)

$$w_k(n,d) \le w_k(k,d) + \sum_{k\le m<n} w_k(m,d-1).$$

Clearly $w_k(k,d) = z_k(k,d)$ and hence $w_k(k,d) = 2^k$. Thus employing our inductive assumption and exploiting the binomial identities $\binom{A}{0} = 1$ and $\sum_{0\le i<A}\binom{i}{B} = \binom{A}{B+1}$ if $B \ge 0$, we obtain

$$\begin{aligned} w_k(n,d) &\le 2^k + \sum_{k\le m<n} W_k(m,d-1) \\ &= 2^k + \sum_{k\le m<n}\left[c_k\binom{m-k}{d-2-k} + 2^k \sum_{k\le j<d-2}\binom{m-k}{j-k}\right] \\ &= 2^k + \left[c_k\binom{n-k}{d-1-k} + 2^k \sum_{k\le j<d-2}\binom{n-k}{j+1-k}\right] \\ &= c_k\binom{n-k}{d-1-k} + 2^k \sum_{k\le j<d-1}\binom{n-k}{j-k} = W_k(n,d), \end{aligned}$$

as desired.

Thus we have established for all $n > k$ the desired $w_k(n,d) \leq W_k(n,d)$ and equivalently $z_k(n,d) \leq Z_k(n,d)$—however, only for $0 \leq k < d-1$. For $k = d-1$ our inductive assumption does not hold.

To prove the desired bound for $k = d-1$ we proceed as follows. We need only consider the case $n > k = d-1$. We just proved that

$$z_{d-2}(n,d) \leq Z_{d-2}(n,d) = c_{d-2}(d-1)\binom{n}{d-1} + 2^{d-2}\binom{n}{d-2}.$$

Plugging this in the inequality

$$z_{d-1}(n,d) \leq \frac{1}{\binom{d-1}{2}}\left[3(d-2)z_{d-2}(n,d) \; -6\cdot 2^{d-3}\binom{d-1}{2}\binom{n}{d-1} -6\cdot 2^{d-3}\binom{d-2}{1}\binom{n}{d-2}\right]$$

of Lemma 2.5 yields the desired

$$\begin{aligned} z_{d-1}(n,d) &\leq \frac{1}{\binom{d-1}{2}}\left[3(d-2)c_{d-2}(d-1)\binom{n}{d-1} + 3(d-2)2^{d-2}\binom{n}{d-2}\right. \\ &\qquad \left. -6\cdot 2^{d-3}\binom{d-1}{2}\binom{n}{d-1} - 6\cdot 2^{d-3}(d-2)\binom{n}{d-2}\right] \\ &= 6(c_{d-2} - 2^{d-3})\binom{n}{d-1} = c_{d-1}\binom{n}{d-1} = Z_{d-1}(n,d). \end{aligned}$$

This completes the induction on d and the proof of Theorem 2.1.

3. Remarks. First let us point out that our proof of the zone theorem does not exploit the "straightness" of hyperplanes per se, but only the restricted kinds of intersection patterns that are possible amongst hyperplanes. Thus the proof applies equally well to arrangements of pseudohyperplanes which can be modeled combinatorially by oriented matroids (see [4]).

For the two-dimensional case Bern et al. [3] prove the slightly stronger bound of $z_1(n,2) \leq \frac{11}{2}n$, which is tight up to an additive constant in the worst case. Using this bound as an induction basis in our proof yields a slightly better value for the constants c_k, namely, $c_k = \frac{2}{3}6^k + \frac{3}{4}2^k$ for $k > 0$.

The definition of a zone that we use in this paper is slightly different from definitions used before. We define $Zone(b; H)$ to be the set of all cells in the arrangement $\mathcal{A}(H)$ that intersect the hyperplane b, and the cells were defined to be open sets. Previously the zone used to be defined as the set of all cells *whose closure intersects* b. Let us call this set $Czone(b;H)$. Note that in degenerate cases $Czone(b;H)$ can be substantially more extensive than $Zone(b,H)$. Nevertheless the $O(n^{d-1})$ upper bound also applies to the sum of the complexities of the cells in $Czone(b;H)$. This can most easily be seen by observing that $Czone(b;H) = Zone(b^+;H) \cup Zone(b^-;H)$, where b^+ and b^- are two hyperplanes parallel to b, one on each side of b and sufficiently close to b.

Deriving the exact value of $z_k(n,d)$ looks like a fairly challenging problem. For a start, tight lower bounds on $z_k(n,3)$ are desired.

Finally, we remark that the proof techniques of this paper, in particular Lemma 2.2, bear some similarity to the so-called combination lemma techniques; see [7], [10], [16].

It would be interesting to see to what extent it can be generalized and applied to related discrete geometry problems. Recent successful applications have been achieved by Aronov et al. [1], who derive bounds on the sum of squares of cell complexities in a hyperplane arrangement, by Aronov and Sharir [2], who prove bounds on the complexity of the zone of a convex or fixed-degree algebraic hypersurface in a hyperplane arrangement, and by Houle and Tokuyama [14], who give bounds on the complexity of the zone of a flat in a hyperplane arrangement.

4. Appendix: Why the sweep proof fails in the general case and how it can be partially saved. In this appendix we first show what is wrong with the old sweep-based proof of the zone theorem as presented in [11] or [6], and then we show how the sweep approach can be used to still prove the zone theorem in the three-dimensional case, and a weak version of the theorem in the general d-dimensional case, which counts only facets.

Before we get to the details of the old proof let us change the problem slightly: Firstly we rename the base plane b of the previous sections h, and secondly we could the number of faces bounding the cells in $\mathcal{A}(H \cup \{h\})$ that lie on one side of h and have a facet in h. In the generic case (which we can again assume without loss of generality), even if we ignored the faces contained in h, this number, for at least one side of h, would be at least half the complexity of the zone of h in $\mathcal{A}(H)$ as defined in §1.

Choose a coordinate system so that h is the hyperplane $x_d = 0$ and consider the cells above h (in the half-space $x_d > 0$) that have a facet in h. Call these cells and their faces *active*. As above, when we count the active faces we will count a face once for each active cell it bounds. We therefore continue to use the notion of borders, and define 2^{d-k} *k-borders* (f, C) for each k-face f, one for each cell C it bounds. Call a k-border (f, C) *active* if and only if cell C is active.

4.1. The notion of a chain. The basic idea of the sweep proof is to move a hyperplane h_t continuously over the active cells. Think of $t \geq 0$ as the time and define $h_t : x_d = t$. So $h_0 = h$ and h_t moves upwards as t increases. At any point in time t the cross section of $\mathcal{A} = \mathcal{A}(H)$ within h_t is an arrangement defined by n hyperplanes in $\mathbf{R}^{d-1}$, which we denote as $\mathcal{A}_t$.

Let us index the vertices of $\mathcal{A}$ above h as $v_1, v_2, \ldots, v_m$ and define points in time u_i so that $v_i \in h_{u_i}$ for $1 \leq i \leq m$. We assume that $0 = u_0 < u_1 < u_2 < \ldots < u_m < u_{m+1} = \infty$. Unless $t = u_i$ for some i, $\mathcal{A}_t$ is a simple arrangement. If $t = u_i$ then exactly d of the hyperplanes defining $\mathcal{A}_t$ meet in a common point, and otherwise the hyperplanes are in a generic position. Let t and t' be so that $u_{i-1} < t < u_i < t' < u_{i+1}$ for some $1 \leq i \leq m$ and call the transition from $\mathcal{A}_t$ to $\mathcal{A}_{t'}$ an *elementary step*. Ignoring the positional differences of the hyperplanes in $\mathcal{A}_t$ and $\mathcal{A}_{t'}$, the only combinatorial difference between the two arrangements is that a $(d-1)$-simplex in $\mathcal{A}_t$ reverses its orientation in $\mathcal{A}_{t'}$. This is illustrated in Fig. 4.1, which shows three cross sections of an arrangement of three planes in $\mathbf{R}^3$.

Let $h_1, h_2, \ldots, h_d$ be the hyperplanes (in $\mathbf{R}^d$) that intersect at v_i. The faces of $\mathcal{A}$ that intersect h_t are the same as the faces that intersect $h_{t'}$, except for a group of faces that have v_i as their topmost vertex (they intersect h_t but not $h_{t'}$) and another group of faces that have v_i as their bottommost vertex (they intersect $h_{t'}$ but not h_t). For each $1 \leq k \leq d$, we *identify* a k-face f in the first group with a k-face f' in the second group if f and f' span the same k-flat. Similarly, we identify the k-borders (f, C) and (f', C') if f and f' are identified and C and C' lie on the same side of each hyperplane that contains f and f'. The identification effectively defines equivalence classes of faces and borders. We call an equivalence class of k-borders a *k-chain*, for $1 \leq k \leq d$. For example, a

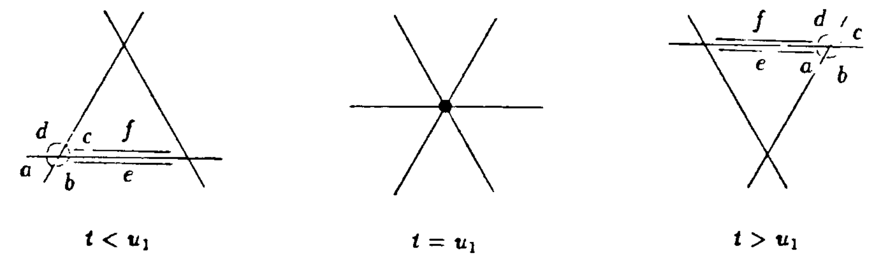

FIG. 4.1. *The triangle defined by three lines (the cross sections of three planes) changes orientation as h_t sweeps through the vertex where the three planes meet.*

1-chain is a sequence of 1-borders (sided edges) on a line, a 2-chain is a sequence of 2-borders in a common 2-flat, and so on and so forth. Figure 4.1 shows the cross sections of four 1-chains (the four-sided versions of a sequence of edges) and of two 2-chains in a three-dimensional arrangement.

4.2. 2-chains do not necessarily die. The sweep proof of the zone theorem hinges on the claim that whenever we sweep through an active vertex there is at least one chain that dies. A chain is said to be *dead* at time t if it contains no further active borders, that is, all borders of the chain that intersect some $h_{t'}$ with $t' > t$ are inactive. Since the number of chains is $O(n^{d-1})$, this claim implies the Zone Theorem. We show below that, unfortunately, this claim is incorrect starting in dimension $d = 3$.

For a vertex v call the cell for which v is the topmost vertex the *cell below* v, and consider the three types of elementary steps shown in Fig. 4.2. In type 1 the 3-chain whose first cell is the cell below v dies. Indeed, every 3-chain has only one active cell, namely, its first cell. In type 2 the 1-chain labeled a dies. This is because all future 1-borders of this 1-chain lie on an edge of a triangular cone disjoint from h and their associated cells lie inside this cone. We now demonstrate that in type 3 the 2-chain labeled ϵ does not die, contrary to the claim in [11].

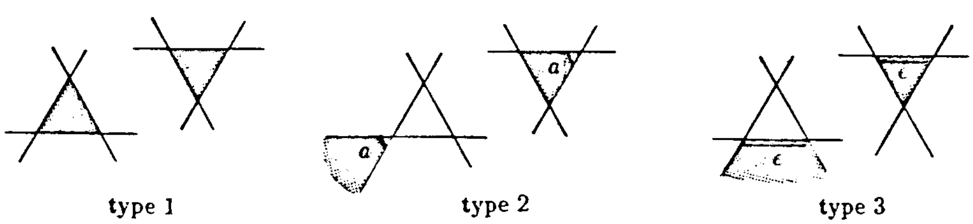

FIG. 4.2. *We distinguish three types of elementary steps associated with a vertex v in a three-dimensional arrangement. For each type the cross section of the active cell is shaded. In type 1 the cell below v is active, in type 2 the cell below v shares an edge with the active cell, and in type 3 the cell below v shares a 2-face with the active cell. Elementary steps where more than one cell bounded by v are active are decomposed into instances of the three basic types.*

The example that we use consists of four planes, h_1, h_2, h_3, h_4. We choose $h_1 : x_1 = 0$, $h_2 : x_2 = 0$, and $h_3 : x_1 + x_2 + x_3 = 1$. Now choose h_4 so that it meets the line $h \cap h_3$ at a point with negative x_2-coordinate and so that the central triangle (the 2-face that lies in h_3 and in front of h, h_1, and h_2) lies just slightly behind h_4. In Fig. 4.3 only the lines of intersection of h_4 with h and h_3 are shown. Of the three shaded 2-borders, which belong to a 2-chain, the first and the third are active and the middle one is inactive. Indeed, the middle 2-border is within the dead cone defined by h_1, h_2, h_3, but the third 2-border is not. When we sweep through vertex $v = h_1 \cap h_2 \cap h_3$ we have type 3 as shown in Fig. 4.2, but the 2-chain labeled e does not die.

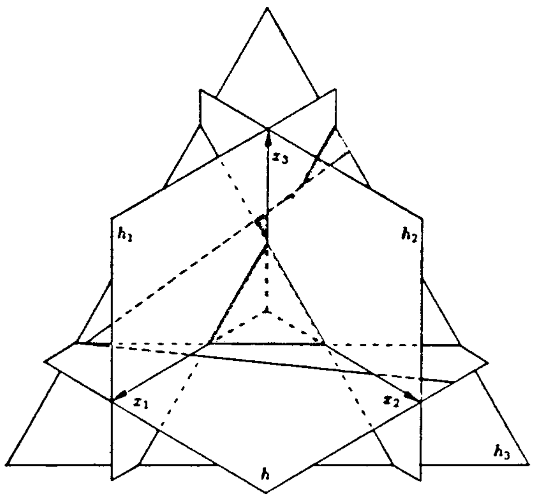

FIG. 4.3. *The shaded 2-borders (they face the observer) belong to a 2-chain. The first and the third 2-borders are active, the middle one is inactive.*

4.3. Fixing the sweep proof in three dimensions. In spite of the fact that in type 3 no chain dies we can still argue that the zone theorem is correct for $d = 3$ using the sweep approach and the type classification illustrated in Fig. 4.2. The reason is that 1-chains still die and 2-chains are linked to 1-chains in elementary steps of the sweep.

At time $t = 0$. $\mathcal{A}_0$ is a two-dimensional arrangement defined by n lines. Because we assume generic position it consists of $\nu_0 = \binom{n}{2}$ vertices, $\epsilon_0 = n^2$ edges, and $\phi_0 = \binom{n}{2} + n + 1$ 2-faces (see, e.g., [6, p. 7]). So we have $4\nu_0$ 1-borders each starting a 1-chain, $2\epsilon_0$ 2-borders each starting a 2-chain, and ϕ_0 cells each starting a 3-chain. All these 1-borders, 2-borders, and cells are active.

Type 1. Recall that in this case the cell below the vertex v that is swept over is active. The corresponding 3-chain dies. So type 1 can occur at most ϕ_0 times. Indeed, it occurs exactly once for each bounded 2-face in $\mathcal{A}_0$ and therefore exactly $\gamma_1 = \phi_0 - 2n = \binom{n}{2} - n+1$ times. Each time an elementary step is of type 1 we encounter a new active 0-border, and, respectively, three active 1-borders and 2-borders no longer intersect h_t.

Type 2. Here, the cell below v shares an edge with the active cell. One 1-chain on the corresponding line dies and a new active 2-border is encountered. Since there are only $4\nu_0$ 1-chains, this type can occur only $\gamma_2 \leq 4\nu_0$ times. Besides the new active 2-border we also encounter two new active 1-borders and one new 0-border at v.

Type 3. Here, the cell below v shares a 2-face with the active cell. The number of active 2-borders that intersect h_t decreases by one. Type 3 thus occurs $\gamma_3 \leq 2\epsilon_0 + \gamma_2 - 3\gamma_1 \leq 2n^2 + \binom{n}{2} + 3n - 3$ times because there are $2\epsilon_0$ active 2-borders initially, we get one more at each occurrence of type 2, and $3\gamma_1$ 2-faces disappear in elementary steps of type 1. Notice that we encounter one new active 1-border and one new active 0-border.

We count the active borders when they are encountered by h_t. Initially, we have $4\nu_0$ 0-borders, $2\epsilon_0$ 1-borders, and ϕ_0 2-borders, all active and all in h. We also have $4\nu_0$ 1-chains each starting with an active 1-border, and $2\epsilon_0$ 2-chains each starting with an active 2-border. So we get

$$4\nu_0+\gamma_1+\gamma_2+\gamma_3 \le 4\binom{n}{2}+\binom{n}{2}-n+1+4\binom{n}{2}+2n^2+\binom{n}{2}+3n-3 = 7n^2+O(n)$$

active 0-borders,

$$2\epsilon_0+4\nu_0+2\gamma_2+\gamma_3 \le 2n^2+4\binom{n}{2}+8\binom{n}{2}+2n^2+\binom{n}{2}+3n-3 = \frac{21n^2}{2}+O(n)$$

active 1-borders, and

$$\phi_0+2\epsilon_0+\gamma_2 \le \binom{n}{2}+n+1+2n^2+4\binom{n}{2} = \frac{9n^2}{2}+O(n)$$

active 2-borders. This proves the zone theorem for $d=3$.

What kind of upper bounds does this yield for $z_k(n,3)$? Note that here we are counting the borders just on one side of the plane h, but we include the borders contained in h. Thus to get good bounds for $z_k(n,3)$ from the above bounds, we need to subtract the borders contained in h, multiply by two, and subtract the borders that intersect h. Doing this yields the bounds $z_1(n,3) \le 6n^2+O(n)$, $z_2(n,3) \le 15n^2+O(n)$, and $z_3(n,3) \le 10n^2+O(n)$. Note that the coefficients of the n^2 terms agree with the corresponding coefficients of Theorem 2.1 as proved in §2. Thus, with the improved coefficients mentioned in §3, we have better bounds than those derived above.

4.4. The sweep proof for facets. The fix of the sweep proof described in §4.3 does not extend beyond three dimensions. For example, in dimension $d=4$ the sweep links 1-chains with 3-chains and 2-chains with other 2-chains. Since 1-chains still die we can bound the number of active 3-borders, but currently we are not able to bound the number of active k-borders, $k \le 2$, using the sweep approach. Below we show how to bound the number of active $(d-1)$-borders for arbitrary dimension d.

As before, let H be a set of n hyperplanes in generic position and let $h \notin H$ be the hyperplane $x_d = 0$. There are $\sum_{i=0}^{d-1}\binom{n}{i}$ active $(d-1)$-borders in h, and initially, there are $2\sum_{i=1}^{d-1} i\binom{n}{i}$ $(d-1)$-chains, each starting with an active $(d-1)$-border. We encounter a new active $(d-1)$-border at an elementary step if and only if it is of the type that the cell below the passed vertex shares (only) an edge with the active cell. However, in this case a 1-chain on the shared edge dies. So this type of elementary step can occur at most $2^{d-1}\binom{n}{d-1}$ times, once for each 1-chain. It follows that there are at most

$$\sum_{i=0}^{d-1}\binom{n}{i}+2\sum_{i=1}^{d-1}\binom{n}{i}+2^{d-1}\binom{n}{d-1} = (2^{d-1}+2d-1)\binom{n}{d-1}+O(n^{d-2})$$

active $(d-1)$-borders.

The bound given here implies the bound $z_1(n,d) \le (2^d+2d-2)\binom{n}{d-1}+O(n^{d-2})$, which in terms of its leading coefficient is substantially worse than the bound for $z_1(n,d)$ given in Theorem 2.1.

REFERENCES

[1] B. Aronov, J. Matoušek, and M. Sharir, *On the sum of squares of cell complexities in hyperplane arrangements*, in Proceedings of the 7th ACM Symposium on Computational Geometry, Association for Computing Machinery, New York, 1991, pp. 307–313.

[2] B. Aronov and M. Sharir, *On the zone of a surface in a hyperplane arrangement*, in Proceedings of the 2nd Workshop on Algorithms and Data Structures, 1991, pp. 13–19; Springer-Verlag, Lecture Notes in Computer Science 519, Berlin, New York, 1991.

[3] M. Bern, D. Eppstein, P. Plassmann, and F. Yao, *Horizon theorems for lines and polygons*, Discrete and Computational Geometry: Papers from the DIMACS Special Year, J. Goodman, R. Pollack and W. Steiger, eds., American Mathematical Society, Providence, RI, to appear.

[4] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G. Ziegler, *Oriented Matroids*, in preparation.

[5] B. Chazelle, L. J. Guibas, and D. T. Lee, *The power of geometric duality*, BIT, 25 (1985), pp. 76–90.

[6] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*, Springer-Verlag, Heidelberg, Germany, 1987.

[7] ———, *The upper envelope of piecewise linear functions: Tight bounds on the number of faces*, Discrete Comput. Geom., 4 (1989), pp. 337–343.

[8] H. Edelsbrunner and L. J. Guibas, *Topologically sweeping an arrangement*, J. Comput. System Sci., 38 (1989), pp. 165–194.

[9] H. Edelsbrunner, L. J. Guibas, J. Pach, R. Pollack, R. Seidel, and M. Sharir, *Arrangements of curves in the plane: Topology, combinatorics and algorithms*, in Proceedings of the 15th International Colloquium on Automata, Languages and Programming, 1988, pp. 214–229.

[10] H. Edelsbrunner, L. J. Guibas, and M. Sharir, *The complexity and construction of many faces in an arrangement of lines or of segments*, Discrete Comput. Geom., 5 (1990), pp. 161–196.

[11] H. Edelsbrunner, J. O'Rourke, and R. Seidel, *Constructing arrangements of lines and hyperplanes with applications*, SIAM J. Comput., 15 (1986), pp. 341–363.

[12] B. Grünbaum, *Convex Polytopes*, John Wiley & Sons, London, 1967.

[13] M. Houle, *A note on hyperplane arrangements*, University of Tokyo, Tokyo, Japan, 1987, manuscript.

[14] M. Houle and T. Tokuyama, *On zones of flats in hyperplane arrangements*, University of Tokyo, Tokyo, Japan, 1991, manuscript.

[15] J. Matoušek, *A simple proof of the weak zone theorem*, Charles University, Prague, Czechoslovakia, 1990, manuscript.

[16] J. Pach and M. Sharir, *The upper envelope of piecewise linear functions and the boundary of a region enclosed by convex plates: Combinatorial analysis*, Discrete Comput. Geom., 4 (1989), pp. 291–309.