Edge Insertion for Optimal Triangulations*

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Abstract. Edge insertion iteratively improves a triangulation of a finite point set in \( \mathbb{R}^2 \) by adding a new edge, deleting old edges crossing the new edge, and retriangulating the polygonal regions on either side of the new edge. This paper presents an abstract view of the edge insertion paradigm, and then shows that it gives polynomial-time algorithms for several types of optimal triangulations, including minimizing the maximum slope of a piecewise-linear interpolating surface.

1. Introduction

A triangulation of a finite set of points \( S \) in \( \mathbb{R}^2 \) is a maximally connected, straight-line planar graph with vertex set \( S \). Each bounded face is a triangle, and the triangulations includes the boundary of the convex hull. Triangulations find use in areas such as finite element analysis [2], [27], computational geometry [21],

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Applications typically require triangulations with “well-shaped” triangles, meaning—for example—that triangles with very small or large angles should be avoided. Taking a worst-case approach, the quality of a triangulation can be defined to be the quality of its worst triangle. Interesting algorithmic questions then arise when we ask for a triangulation of a given point set that optimizes some quality criterion. These questions take the form of minmax or maxmin problems, where the first quantifier is over all triangulations of the point set, and the second is over all triangles in the triangulation.

The problem of automatically generating optimal triangulations has been a subject for research since the 1960s (see, e.g., the discussion in [13]). In spite of this attention, very little is known about constructing optimal triangulations in polynomial time. Exhaustive search can be ruled out since a set of $n$ points has, in general, exponentially many triangulations. Greedy approaches (such as eliminating triangles from worst to best) are ruled out by the NP-completeness of the following decision problem [20]: given a collection of points and edges, decide whether a subset of the edges defines a triangulation of the points.

Most positive results are related to the Delaunay triangulation [6]. It has been shown that among all triangulations of a given finite point set, the Delaunay triangulation optimizes various criteria. The Delaunay triangulation maximizes the minimum angle [26], minimizes the maximum circumscribing circle [5], and minimizes the maximum smallest enclosing circle [5], [22]. Efficient algorithms for constructing Delaunay triangulations are abundant in the literature and are based on such diverse algorithmic paradigms as edge-flipping [17], [18], divide-and-conquer [25], [15], geometric transformation [3], plane-sweep [12], and randomized incrementation [14].

Recently, Edelsbrunner, Tan, and Waupotitsch devised a polynomial-time algorithm that minimizes the maximum angle [10]. This algorithm constructs a minmax-angle triangulation by iteratively inserting a new edge, removing old edges crossed by the new edge, and then retriangulating the polygonal “holes” on either side of the new edge.

This paper presents an abstraction of the minmax-angle algorithm, which we call the edge-insertion paradigm, and applies it to obtain polynomial-time algorithms for some other optimal triangulation problems. The specific new results are an $O(n^2 \log n)$-time algorithm that constructs a triangulation with maxmin triangle height, an $O(n^2)$-time algorithm for minmax triangle eccentricity (distance from circumcenter), and—most significantly—an $O(n^2)$-time algorithm for finding a triangulated surface, interpolating given points in $\mathbb{R}^3$, with minmax gradient. All three criteria are mentioned as open problems in a survey article on “systematic” triangulations [28].

Section 2 formulates the edge-insertion paradigm, which locally improves a triangulation according to a generic criterion. When instantiated to a specific criterion, the basic paradigm gives a local optimum in time $O(n^2)$. Section 3 states two abstract conditions for quality criteria, the first strictly weaker than the second. Section 4 proves that even the weaker condition suffices to show that the edge-insertion paradigm computes a global optimum; the argument is rather delicate. Section 5 discusses refinements of the basic paradigm with improved
running times; here we show that the weaker condition implies an $O(n^3)$-time algorithm and the stronger condition implies an $O(n^2 \log n)$-time algorithm. (We do not yet know of any quality criteria globally optimized by the $O(n^3)$ basic algorithm, but not by the $O(n^3)$ algorithm.) Sections 6, 7, and 8 prove that the three specific optimization criteria mentioned above satisfy one or the other of the two conditions. Section 9 offers some concluding remarks.

2. The Edge-Insertion Paradigm

We start with some definitions. A triangulation of a finite point set $S$ in $\mathbb{R}^2$ is defined above as a maximally connected, straight-line planar graph with vertex set $S$. A constrained triangulation is a maximally connected, straight-line planar graph restricted to lie within a given connected polygonal region; the vertex set of the triangulation includes the vertices of the polygonal region along with any interior point “holes.” Thus, a triangulation of a point set $S$ is the special case in which the polygonal region is the convex hull of $S$. Another special case is a polygon triangulation in which there are no holes.

We denote by $xy$ the relatively open line segment that connects the points $x, y \in \mathbb{R}^2$. For $x, y, z \in \mathbb{R}^2$, $xyz$ is the open triangle with corners $x, y, z$. For a given finite point set $S$ in $\mathbb{R}^2$ and $x, y, z \in S$, we call $xyz$ an empty triangle if all other points of $S$ lie outside the closure of $xyz$.

Let $\mu$ be a function that maps each triangle $xyz$ to a real value $\mu(xyz)$, called the measure of $xyz$. We restrict our attention to minmax criteria, that is, for each $\mu$ we consider the construction of a triangulation that minimizes the maximum $\mu(xyz)$ over all triangles $xyz$. Maxmin criteria can be simulated by considering $-\mu$. The measures of particular interest in this paper are largest angle, height (actually, negative height, since we desire maxmin height), eccentricity, and the gradient on a triangulated (nonplanar) surface.

The measure of a triangulation $\mathcal{A}$ is defined as $\mu(\mathcal{A}) = \max\{\mu(xyz) | xyz \text{ a triangle of } \mathcal{A}\}$. If $\mathcal{A}$ and $\mathcal{B}$ are two triangulations of the same point set, then $\mathcal{B}$ is called an improvement of $\mathcal{A}$, denoted $\mathcal{B} < \mathcal{A}$, if $\mu(\mathcal{B}) < \mu(\mathcal{A})$ or $\mu(\mathcal{B}) = \mu(\mathcal{A})$ and the set of triangles $xyz$ in $\mathcal{B}$ with $\mu(xyz) = \mu(\mathcal{B})$ is a proper subset of the set of such triangles in $\mathcal{A}$. A triangulation $\mathcal{A}$ is optimal for $\mu$ if there is no improvement of $\mathcal{A}$.

The edge-insertion paradigm uses a natural local improvement operation, not surprisingly called an “edge-insertion.” Given a triangulation $\mathcal{A}$ of a point set $S$, the edge-insertion of $qs$, for $q, s \in S$, goes as follows:

Function Edge-insertion ($\mathcal{A}$, $qs$): triangulation.
1. $\mathcal{B} := \mathcal{A}$.
2. Add $qs$ to $\mathcal{B}$ and remove from $\mathcal{B}$ all edges that intersect $qs$.
3. Retriangulate the polygonal regions $P$ and $R$ constructed in step 2.
4. return $\mathcal{B}$.

For now we assume that regions $P$ and $R$ (see Fig. 2.1) are retriangulated in an optimal fashion minimizing the maximum $\mu$, e.g., by dynamic programming [16].
Fig. 2.1. Inserting $qs$ leaves two polygonal regions $P$ and $R$.

The basic, most general, version of the edge-insertion paradigm is given below; it tries all possible edge-insertions and halts when no edge-insertion improves the current triangulation.

**Input.** A set $S$ of $n$ points in $\mathbb{R}^2$.

**Output.** An optimal triangulation $T$ of $S$.

**Algorithm**

Construct an arbitrary triangulation $A$ of $S$.

repeat $T := A$;

for all pairs $q, s \in S$ do

$B :=$ EDGE-INSERTION($A$, $qs$);

if $B < A$ then $A := B$; exit the for-loop endif

endfor

until $T = A$.

The edge-insertion paradigm can be viewed as a generalization of the edge-flipping paradigm that computes a Delaunay triangulation [17], [18]. An edge-flip inserts the diagonal of a convex quadrilateral formed by two neighboring triangles; the process halts when no edge-flip improves the current triangulation. The simpler edge-flipping paradigm, however, fails to compute global optima for maximum angle, height, eccentricity, and slope, as we show in later sections of this paper.

We now argue that the basic algorithm above terminates after time $O(n^5)$. A single edge-insertion operation takes time $O(n^3)$ when retriangulation is done by dynamic programming [16], assuming the measures of any two triangles can be compared in constant time. The for loop thus takes time $O(n^5)$ per iteration of the repeat loop. Finally, the repeat loop is iterated at most $O(n^3)$ times, because there are only $\binom{n}{3}$ triangles spanned by $S$, and each iteration permanently discards at least one of them when it finds an improvement of the current triangulation.

**Remark.** The edge-insertion paradigm can be extended to constrained triangulations by limiting the edge-insertion operation to edges $ab$ that lie in the interior of the restricting polygonal region. As a consequence, a triangulation that lexicographically minimizes the decreasing vector of triangle measures can be constructed in the nondegenerate case, that is, when $\mu(abc) \neq \mu(xyz)$ unless $abc = xyz$. Details can be found in [10].
3. Two Sufficient Conditions

We now formulate two conditions on measures $\mu$, sufficient to show that the edge-insertion paradigm computes a global optimum (i.e., minmax $\mu$). They are also sufficient to imply algorithms much faster than $O(n^3)$; these are given in Section 5.

Let $S$ be a set of $n$ points in $\mathbb{R}^2$, let $\mathcal{B}$ be a triangulation of $S$, and let $xyz$ be an empty triangle in $S$. We say that $\mathcal{B}$ breaks $xyz$ at $y$ if it contains an edge $yt$ with $yt \cap xz \neq \emptyset$. Note that if $\mathcal{B}$ breaks $xyz$ at $y$, then it cannot break $xyz$ at $x$ or $z$.

We call vertex $y$ an anchor of an empty triangle $xyz$ in point set $S$, if every triangulation $\mathcal{B}$ of $S$, with $\mu(\mathcal{B}) \leq \mu(\mathcal{B})$, either contains $xyz$ or breaks $xyz$ at $y$. For example, if $\mu(\mathcal{B})$ is the measure of the largest angle in $\mathcal{B}$, and the largest angle has vertex $y$, then $y$ is an anchor. Intuitively speaking, if a triangle has an anchor, it will be the triangle's “worst vertex.” We can now give the two conditions on quality measures $\mu$.

I (Weak Anchor Condition) For each triangulation $\mathcal{A}$, and each triangle $xyz$ of $\mathcal{A}$ with $\mu(xyz) = \mu(\mathcal{A})$, there is an anchor vertex of $xyz$.

In other words, $\mathcal{B}$ can be an improvement of $\mathcal{A}$ only if it breaks a worst triangle of $\mathcal{A}$ at its anchor. Since $\mathcal{B}$ cannot break a triangle at two vertices, a triangle's anchor is unique in triangulations $\mathcal{A}$ with $\mu(\mathcal{A})$ larger than the minimum. Thus, if $xyz$ is an empty isosceles triangle with two largest angles, then no triangulation can have minmax angle less than this largest angle.

II (Strong Anchor Condition) For each triangulation $\mathcal{A}$ and each triangle $xyz$ of $\mathcal{A}$, there is an anchor vertex of $xyz$.

Notice that $\mu$ equal to the measure of the largest angle satisfies (II), since the largest angle in any triangle $xyz$—not just a worst triangle—must either appear in a triangulation $\mathcal{A}$ with $\mu(\mathcal{A}) \leq \mu(xyz)$, or be subdivided by it. An important difference between the weak and strong conditions is that in (I) the triangulation $\mathcal{A}$ that contains $xyz$ plays an important role, while in (II) $\mathcal{A}$ is insignificant.

4. Proof of Correctness

The Cake Cutting Lemma (below) asserts that if $\mathcal{A}$ is not yet optimal for measure $\mu$ satisfying condition (I), then there is an edge whose insertion leads to an improvement, specifically an edge breaking a worst triangle at its anchor. In [10] this lemma is proved for the maximum angle measure using an argument that rotates edges of an optimal triangulation of $S$. While this argument works for angles, we need a different argument for the general class of measures that satisfy (I).

Before continuing, we remark that the regions $P$ and $R$ (created in step 2 of an edge-insertion) are not necessarily simple polygons in the usual meaning of the term. Although their interiors are always simply connected, there can be edges contained in the interiors of their closures, as shown in Fig. 2.1. Nevertheless, each
such edge can be treated as if it consisted of two edges, one for each side, which then allows us to treat \( P \) and \( R \) as if they were simple polygons.

As usual, a diagonal of a simple polygon is a line segment that connects two vertices and—except at its endpoints—lies interior to the polygon. An ear is a triangle bounded by two polygon edges and one diagonal.

**Lemma 4.1 (Cake Cutting).** Assume \( \mu \) satisfies condition (I). Let \( \mathcal{F} \) be two triangulations of point set \( S \). Let \( pqr \) be a triangle in \( \mathcal{A} \) but not in \( \mathcal{F} \) with \( \mu(pqr) = \mu(\mathcal{A}) \); let \( q \) be an anchor of \( pqr \); and let \( qr \) be an edge in \( \mathcal{F} \) that intersects \( p \). Let \( P \) and \( R \) be the polygons generated by adding \( qr \) to \( \mathcal{A} \) and removing all edges that intersect \( qr \). Then there are triangulations \( \mathcal{P} \) and \( \mathcal{R} \) of \( P \) and \( R \) with \( \mu(\mathcal{P}) < \mu(pqr) \) and \( \mu(\mathcal{R}) < \mu(pqr) \).

**Proof.** We prove the assertion for \( P \), and by symmetry it follows for \( R \). The plan is to use the edges of \( \mathcal{F} \) to locate ears of \( P \) with a small \( \mu \) value, thereby obtaining \( \mathcal{P} \). Each connected component of an edge of \( \mathcal{F} \) intersected with \( P \) (that is, a segment seen through the “window” \( P \) is called a clipped edge. As \( P \) is not necessarily convex, several clipped edges can belong to the same edge of \( \mathcal{F} \). A clipped edge partitions \( P \) into two polygons, the near side supported by \( q \) and the far side not supported by \( q \).

If no clipped edge exists in the window, then \( P \) has only three vertices and therefore must be a triangle of \( \mathcal{F} \). This triangle is not in \( \mathcal{A} \), which implies that its measure is less than \( \mu(\mathcal{A}) \), because any triangle of \( \mathcal{F} \) with measure \( \mu(\mathcal{A}) \) is also a triangle of \( \mathcal{A} \). So assume the existence of at least one clipped edge. Denote by \( q = p_0, p_1, \ldots, p_k, p_{k+1} = s \) the sequence of vertices of \( P \).

**Claim 1.** For \( 1 \leq j \leq k \), if \( \angle p_{j-1}p_jp_{j+1} < \pi \), then \( p_{j-1}p_{j+1} \) is a diagonal of \( P \).

**Proof of Claim 1.** By construction of \( P \), it is possible to find nonintersecting line segments \( p_{j-1}x \) and \( p_{j+1}y \), both inside \( P \), so that \( x \) and \( y \) lie on \( qr \). (If \( j = 1 \), then \( x = p_{j-1} = q \); if \( j = k \), then \( y = p_{j+1} = s \).) The (possibly degenerate) pentagon \( xp_{j-1}p_jp_{j+1}y \) is part of \( P \), and, because the interior angles at \( p_{j-1} \) and \( p_{j+1} \) measure less than \( \pi \), edge \( p_{j-1}p_{j+1} \) is a diagonal of the pentagon and therefore also of \( P \).

**Claim 2.** There is at least one clipped edge whose far side is a triangle.

**Proof of Claim 2.** Let \( xy \) be a clipped edge so that its far side, \( F \), contains no further clipped edge. Consider the triangle in \( \mathcal{F} \) that lies on the same side of \( xy \) as \( F \). Polygon \( F \) must be a subset of this triangle, and since all vertices of \( F \)—except possibly \( x \) and \( y \)—are points in \( S \), \( F \) must be a triangle \( xp_1y \).

The clipped edges \( xy \) that satisfy Claim 2 fall into four classes as illustrated in Fig. 4.1. An ear \( p_{i-1}p_ip_{i+1} \), so that \( xy \) is a clipped edge with far side \( xp_1y \) can now be removed from \( P \), leaving a polygon \( P' \) with one less vertex. Claims 1 and 2 remain
true for $P'$ because the removed ear is not supported by $qs$. Hence we can iterate and compute a triangulation $\mathcal{P}$ of $P$. Symmetrically, we get a triangulation $\mathcal{R}$ of $R$. Let $\mathcal{A}$ be the triangulation of $S$ thus obtained.

Claim 3. $\mu(abc) < \mu(pqr)$ for all triangles $abc$ in $\mathcal{P}$ and $\mathcal{R}$.

Proof of Claim 3. Let $abc$ be a triangle in $\mathcal{P}$ or $\mathcal{R}$ with maximum $\mu$. Assume without loss of generality that $abc$ is a triangle of $\mathcal{P}$ and that $a = p_i, b = p_j, c = p_k$ with $i < j < k$. At the time immediately before $abc$ was removed by adding the edge $ac$ there was a clipped edge $xy$ with far side $xhy$, as shown in Fig. 4.2. Hence, $\mathcal{T}$ does not break $abc$ at $b$, and, by construction, $\mathcal{A}$ breaks $abc$ at $b$ and therefore neither at $a$ nor at $c$.

If $xy = ac$ (as in the leftmost picture in Fig. 4.1), then $abc$ is a triangle in $\mathcal{T}$ that is not in $\mathcal{A}$, and therefore $\mu(abc) < \mu(pqr)$. So assume $xy \neq ac$, and assume for the sake of contradiction that $\mu(abc) \geq \mu(pqr) = \mu(\mathcal{A}) \geq \mu(\mathcal{T})$. Since we chose $abc$ to have maximum $\mu$ in $\mathcal{P}$ or $\mathcal{R}$, this means that $\mu(abc) = \mu(\mathcal{A})$. Then condition (I) requires $abc$ to have an anchor. However, $b$ cannot be the anchor of $abc$, because $\mathcal{T}$ neither contains $abc$ nor breaks $abc$ at $b$. Similarly, neither $a$ nor $c$ can be an anchor of $abc$ because $\mathcal{A}$ neither contains $abc$ nor breaks $abc$ at $a$ or $c$. This contradiction completes the proofs of Claim 3 and Lemma 4.1. □

The Cake Cutting Lemma now shows that the basic edge-insertion paradigm cannot get stuck in a local optimum for $\mu$ satisfying condition (I).

Lemma 4.2. Assume $\mu$ satisfies condition (I). Let $\mathcal{A}$ be a nonoptimal triangulation of point set $S$. Then there is an edge-insertion operation that improves $\mathcal{A}$.

Fig. 4.2. Triangle $abc$ cannot have an anchor.
Proof. Let $B$ be an improvement of $A$ and consider a triangle $pqr$ in $A$ with $\mu(pqr) = \mu(A)$ that is not in $B$. Condition (I) requires $pqr$ to have an anchor, say $q$, so $B$ must contain an edge $qs$ with $qs \cap pr \neq \emptyset$. Let $P$ and $R$ be the polygonal regions generated by adding $qs$ and deleting the edges that intersect $qs$. The Cake Cutting Lemma implies that there are polygon triangulations $P$ and $R$ of $P$ and $R$ with $\mu(P)$ and $\mu(R)$ both smaller than $\mu(pqr)$. □

Remark. Lemmas 4.1 and 4.2 remain true for constrained triangulations provided the optimization criterion satisfies (I) or (II) in this more general setting. This is indeed the case for all criteria considered in this paper.

5. Refinements of the Paradigm

The refined versions of edge-insertion differ from the basic paradigm in two major ways. First, edge-insertions are restricted to candidate edges $qs$ that break a worst triangle $pqr$ at its anchor $q$. Second, the two polygonal regions created by adding edge $qs$ are retriangulated by repeatedly removing ears (as in the proof of the Cake Cutting Lemma), rather than by dynamic programming.

Outline of Refinements

Let $A$ be a triangulation with worst triangle $pqr$, that is, $\mu(pqr) = \mu(A)$, and let $q$ be the anchor of $pqr$. We denote by $q_1 s_1, q_2 s_2, \ldots$ the sequence of candidate edges. This order may be arbitrary for the $O(n^3)$ refinement, but, for criteria satisfying condition (II), a carefully chosen order speeds up the running time to $O(n^2 \log n)$. Both refinements are specializations of the algorithm given below. We use the notation $s_{i+1} = \text{NEXT}(s_i)$.

Algorithm

Construct an arbitrary triangulation $A$ of $S$.
repeat $\mathcal{F} := A$;
  find a worst triangle $pqr$ in $A$, let $q$ be its anchor, and set $s_1 := s_1$;
  while $s$ is defined do
    $\mathcal{B} := A$, add $qs$ to $\mathcal{B}$, and remove all edges that intersect $qs$;
    (partially) triangulate the two polygonal regions $P$ and $R$
    by cutting off ears $xyz$ with $\mu(xyz) < \mu(pqr)$;
    if $P$ and $R$ are completely triangulated then $A := \mathcal{B}$; exit the while-loop
    else $s := \text{NEXT}(s)$
  endif
until $\mathcal{F} = A$ and all worst triangles $pqr$ in $A$ have been tried.

In an implementation of the algorithm we would not really copy entire triangulations. Instead of the assignment $\mathcal{F} := A$, we would use a flag to check whether an iteration of the repeat-loop produced an improved triangulation. The
assignment $\mathcal{S} := \mathcal{A}$ can be avoided by making changes directly in $\mathcal{A}$ and undoing them to the extent necessary. The remainder of this section explains some of the steps in greater detail and analyzes the complexity of the two refinements.

**Triangulating by Ear Cutting**

Suppose an edge $qs$ has been added to $\mathcal{S}$ and the edges that intersect $qs$ have been removed, thus creating two polygonal regions $P$ and $R$. Let $q = p_0$, $p_1, \ldots, p_k, p_{k+1} = s$ be the sequence of vertices of $P$ and let $q = r_0, r_1, \ldots, r_m$, $r_{m+1} = s$ be the corresponding sequence for $R$. As in the proof of the Cake Cutting Lemma, the two regions are (partially) triangulated by repeatedly removing ears with measures less than $\mu(pqr)$. As implied by the proof, the sequence in which the ears are removed is immaterial as long as only the last is supported by $qs$. This method may be implemented using a stack for the vertices of $P(R)$, so that it runs in time linear in the size of $P(R)$. In the case of $P$, the stack is initialized by pushing $p_0$ and $p_1$. After that, for $i = 2$ to $k+1$ we push vertex $p_i$, and whenever the three topmost vertices, $z = p_n, y, x$, define a triangle with $\mu(xyz) < \mu(pqr)$ we pop $y$, the second vertex from the top. The triangulation is complete if, at the end of the process, $p_{k+1} = s$ and $p_0 = q$ are the only two vertices on the stack.

**Theorem 5.1.** Let $S$ be a set of $n$ points in $\mathbb{R}^2$, and let $\mu$ be a measure that satisfies (I) so that given a worst triangle its anchor can be computed in constant time.

1. A constrained or unconstrained triangulation of $S$ that minimizes the maximum triangle measure can be constructed in time $O(n^3)$ and storage $O(n^2)$.
2. In the nondegenerate case (i.e., when $\mu(xyz) \neq \mu(abc)$ unless $xyz = abc$) the (unique) triangulation that lexicographically minimizes the decreasing vector of triangle measures can be constructed in the same amount of time and storage.

**Proof.** To achieve the claimed bounds, we use the algorithm above, along with two data structures requiring a total of $O(n^2)$ storage. First, the quad-edge data structure of Guibas and Stolfi [15] stores the triangulation in $O(n)$ memory and admits common operations, such as removing an edge, adding an edge, and walking from one edge to the next in constant time each.

Second, to record the status of candidate edges, we use an $n$-by-$n$ bit array whose elements correspond to the edges defined by $S$. If the insertion of a candidate edge $qs$ is unsuccessful, that is, the triangulation of $P$ or $R$ cannot be completed, then we know by the Cake Cutting Lemma that $qs$ cannot be in any improvement of the current triangulation. We then set the bit for $qs$, so that we do not attempt the insertion of $qs$ again. If the insertion of $qs$ is successful, we set the bit for the edge $pr$; because every improvement breaks $pr$ (by condition (I)), it cannot be in any later improvement. The bit array can also be used to compute the sequence of candidate edges $qs_1, qs_2, \ldots$: scan the row corresponding to $q$ and take all edges $qs$ that intersect $pr$ and whose flag has not yet been set.
Each edge-insertion, whether successful or not, causes a new flag set for one of the \( \binom{n}{2} \) edges defined by \( S \). Therefore, at most \( \binom{n}{2} \) edge-insertions are carried out taking a total of \( O(n^2) \) time. Part (1) of the claim follows because an initial triangulation can be constructed in time \( O(n \log n) \), most straightforwardly by plane-sweep (see Section 8.3.1 of [8]).

To obtain a triangulation that lexicographically minimizes the entire vector of triangle measures we solve a sequence of constrained triangulation problems as in [10]. The first constraining region is defined by the points and edges on the boundary of the convex hull of \( S \) with the other points forming holes. After computing an optimal triangulation as in (1), we remove the worst triangle (which is unique by nondegeneracy assumption) from the constraining region and iterate until the region is empty. The time is still \( O(n^3) \) because each edge needs to be inserted at most once during the entire process.

\[ \square \]

A Special Order of Insertions for Condition (II)

For measures \( \mu \) that satisfy (II) we define a special sequence \( qs_1, qs_2, \ldots, qs_t \) of edge-insertions, as in [10]. The first edge, \( qs_1 \), has the property that it intersects \( pr \), but otherwise it intersects no edges as possible. As we explain below, each subsequent \( s_{i+1} = \text{next}(s_i) \) lies on a particular side of \( qs_i \) and, on this side, the set of edges in the current triangulation \( A \) that intersect \( qs_{i+1} \) is the smallest proper superset of the edges that intersect \( qs_i \). The index \( l \) is the smallest integer for which \( qs_l \) leads to an improvement or \( s_{l+1} \) is undefined.

On the insertion of \( qs_q \), the retriangulation process either completes its task or it gets stuck because all ears of the remaining regions have measure at least \( \mu(pqr) \).

Let us now consider the case where the triangulation of \( P \) cannot be completed, as this is the case for which we need to define \( \text{next}(s_i) \). In this case the stack contains \( k + 2 \) vertices \( q = p_0, p_1, \ldots, p_k, p_{k+1} = s_l \) and the remaining region \( P' \subseteq P \); each ear \( p_{j-1}p_{j}p_{j+1} \) of \( P' \) has measure at least \( \mu(pqr) \).

**Lemma 5.2.** Let \( \mathcal{F} \) be an improvement of \( A \) for \( \mu \) satisfying condition (II), and let \( P' \) be the uncompleted part of \( P \) as above. Then all edges of \( \mathcal{F} \) that intersect \( P' \) also intersect \( qs_q \). In particular, all edges of \( \mathcal{F} \) incident to \( q \) avoid \( P' \).

**Proof.** As in the proof of the Cake Cutting Lemma we consider \( P' \) as a “window” through which we see clipped edges of \( \mathcal{F} \). Now suppose the claim is not true, that is, there is a clipped edge that does not have one of its endpoints on \( qs_q \). Then, as in the proof of the Cake Cutting Lemma, we can find a clipped edge \( xy \) whose far side is a triangle \( xp_jy \). However, now condition (II) implies \( \mu(\mathcal{F}) > \mu(p_{j-1}p_jp_{j+1}) \) if \( p_j \) is an anchor of the ear \( p_{j-1}p_jp_{j+1} \), and \( \mu(A) > \mu(p_{j-1}p_jp_{j+1}) \) if \( p_{j-1} \) or \( p_{j+1} \) is an anchor. This contradicts the assumption that \( P' \) has no such ear. \[ \square \]
It is interesting to observe that the proof of Lemma 5.2 breaks down if we assume that \( \mu \) satisfies only (I), since \( p_{j-1}p_jp_{j+1} \) need not be a worst triangle.

As we search for an insertion, we maintain an open wedge \( W \) containing all the remaining candidate edges. Initially, \( W \) is the wedge between the ray \( \overrightarrow{pq} \) (starting at \( q \) and passing through \( p \)) and the ray \( \overrightarrow{qr} \). If the edge-insertion of \( qs_i \) turns out to be unsuccessful because the triangulation of \( P \) cannot be completed, then Lemma 5.2 allows us to redefine \( W \) as the part of the old \( W \) on \( R \)'s side of \( \overrightarrow{qs_i} \).

Similarly, if the triangulation of \( R \) cannot be completed, then \( W \) can be narrowed down to \( P \)'s side of \( \overrightarrow{qs_i} \). (As a consequence, if neither \( P \) nor \( R \) can be completely triangulated, then it is impossible to improve the current triangulation by breaking \( pqr \) at \( q \).)

As soon as one of \( P \) or \( R \) has been found to be noncompletable, wedge \( W \) is updated and an edge-insertion is attempted with \( s_{i+1} = \text{next}(s_i) \). If it is \( P \) that could not be completed (the \( R \) case is symmetrical), then we choose \( s_{i+1} \) by looking first at the triangle on the far side of \( r_{m,r_{m+1}} \) (the last edge of \( R \)) from \( q \). If the third vertex \( s \) of this triangle lies in wedge \( W \), then we choose \( s_{i+1} \) to be \( s \). If this is not the case, then we move on to the next triangle sharing an edge with \( r_{m,r_{m+1},s} \), and test whether its far vertex \( z \) lies in the wedge. We eventually either run out of triangles (then no edge-insertion at \( q \) is possible) or we find a vertex \( s_{i+1} \) such that the set of edges in \( \mathcal{F} \) that intersect \( qs_{i+1} \) is the smallest proper superset of the edges that intersect \( qs_i \). See Fig. 5.1.

When we move from \( qs_i \) to \( qs_{i+1} \), most of the work done to triangulate \( P \) and \( R \) can be saved. Assume that \( qs_i \) has failed because \( P \) could not be completely triangulated. Because \( qs_{i+1} \) intersects \( r_{m,r_{m+1}} \) all ears cut off \( P \) remain the same and do not have to be reconsidered. On the other hand, \( r_{m+1} \) is no longer a vertex of \( R \), so all ears cut off \( R \) that are incident to \( r_{m+1} \) must be returned to \( R \)'s territory. When we move to \( qs_{i+1} \), some additional edges are removed from \( \mathcal{F} \) which, in effect, expands \( P \) and \( R \). The new vertices can be pushed on their respective stacks, one by one, so that the triangulation process can continue where it left off.

The only place where we waste time in this process (i.e., where time spent is not proportional to good ears found) is when ears cut off \( R \) are returned to \( R \). Since ears are returned for only one polygon, we can limit the waste by strictly alternating between cutting an ear of \( P \) and one of \( R \). This way, for each returned

![Fig. 5.1. The two rays define the current \( W \), and the dotted line segments indicate those ears removed from \( P \) and \( R \). If \( P \) is found to be noncompletable, the next candidate edge \( q_{i+1} \) lies in the updated \( W \) defined by \( \overrightarrow{rs_i} \) and the ray passing through \( R \).](image-url)
ear (except maybe the last) there is a permanently removed ear. Therefore, the total number of operations performed while inserting \( q_1, q_2, \ldots, q_t \) is linear in the number of edges in \( \mathcal{A} \) that intersect \( q_t \).

As in the proof of Theorem 5.1, a successful edge-insertion, complete with retriangulation, takes time linear in the number of old edges intersected by the new edge. We now prove that the old edges removed will never be reinserted in any later successful edge-insertion.

**Lemma 5.3.** Assume \( \mu \) satisfies condition (II), let \( \mathcal{A} \) be a triangulation of \( S \) with worst triangle \( pqr \), and let \( \mathcal{B} \) be obtained from \( \mathcal{A} \) by the successful insertion of edge \( q_2 \). Then no edge \( xy \) in \( \mathcal{A} \) that intersects \( q_2 \), can be an edge of any improvement of \( \mathcal{B} \).

**Proof.** Lemma 5.2 implies that every improvement of \( \mathcal{B} \) has an edge \( qw \) that lies inside the wedge \( W \) computed when \( q_t \) is inserted into \( \mathcal{A} \). Every edge \( xy \) in \( \mathcal{A} \) that intersects \( q_2 \) also intersects every other edge \( qt \) with \( t \in W \). In particular, \( xy \cap qw \neq \emptyset \) which implies that \( xy \) is neither in \( \mathcal{B} \) nor in any improvement of \( \mathcal{B} \).

**Theorem 5.4.** Let \( S \) be a set of \( n \) points in \( \mathbb{R}^2 \) and let \( \mu \) be a measure that satisfies (II) so that given a triangle its anchor can be computed in constant time.

1. A constrained or unconstrained triangulation of \( S \) that minimizes the maximum triangle measure can be constructed in time \( O(n^2 \log n) \) and storage \( O(n) \).
2. In the nondegenerate case (i.e., when \( \mu(xyz) \neq \mu(abc) \) unless \( xyz = abc \)) the (unique) triangulation that lexicographically minimizes the decreasing vector of triangle measures can be constructed in the same amount of time and storage.

**Proof.** As before, the algorithm uses the quad-edge data structure of [15] to store the triangulation. The bit array, however, is replaced by a priority queue that holds the triangles of \( \mathcal{A} \) ordered by measure. It admits inserting and deleting triangles and finding a triangle with maximum measure in logarithmic time [4]. Lemma 5.3 implies that only \( O(n^2) \) edges and triangles are manipulated in the main loop of the algorithm, which thus takes time \( O(n^2 \log n) \). Lemma 5.3 also implies a quadratic upper bound on the number of iterations of the repeat-loop, which implies that the total time needed to find worst triangles \( pqr \) is also \( O(n^2 \log n) \). This proves part (1), and part (2) follows from the same argument as in Theorem 5.1.

6. Maximizing the Minimum Height

The height \( h(xyz) \) of triangle \( xyz \) is the minimum distance from a vertex to the opposite edge. A maxmin height triangulation of \( S \) maximizes the smallest height of its triangles, over all triangulations of \( S \). Although the maxmin height,
the maxmin angle, and the minmax angle criteria all tend to avoid thin and elongated triangles, they do not necessarily define the same optima. Indeed, four-point examples can be constructed to show that the three criteria are pairwise different.

The edge-flipping strategy [17], [18] applied to the maxmin height criterion does not always succeed in computing an optimal triangulation. Consider a regular pentagon $abcde$ and the circle through the five points. Perturb $a$ slightly to a point outside the circle and $c$ and $d$ slightly to points inside the circle so that $h(c, db) < h(d, ec) < h(b, ca) = h(e, ad) < h(a, be)$, where we write $h(x, yz)$ for the minimum distance between a point $x$ and a line through points $y$ and $z$. See Fig. 6.1. The maxmin-height triangulation uses diagonals $ac$ and $ad$. If the current triangulation uses $be$ and $ce$, however, no edge-flip can result in a better triangulation.

We now show that $-\eta$ satisfies condition (II), when we define the vertices of $xyz$ with maximum angle to be anchors. It follows that maxmin height triangulations can be constructed by the $O(n^2 \log n)$-time implementation of the edge-insertion paradigm.

**Lemma 6.1.** Let $xyz$ be a triangle of a triangulation $\mathcal{A}$ of $S$ and let $\eta(xyz) = h(y, zx)$. Then $\eta(\mathcal{F}) < \eta(xyz)$ for any triangulation $\mathcal{F}$ of $S$ that neither contains $xyz$ nor breaks $xyz$ at $y$.

**Proof.** The height $\eta(xyz) = h(y, zx)$ is the distance between $y$ and a point $x \in zx$. Assume that $xyz$ is not in $\mathcal{F}$ and that $\mathcal{F}$ does not break $xyz$ at $y$.

Therefore, there exists a triangle $uvw$ in $\mathcal{F}$ so that either $u = x$ and $uv \cap yz \neq \emptyset$ (rename vertices if necessary), or $uv$ intersects both $yx$ and $yz$. In both cases, $\eta(uvw) \leq h(y, uv) < \eta(xyz)$ because $uv \cap yz \neq \emptyset$. \hfill $\Box$

It should be clear that Lemma 6.1 also holds for constrained triangulations of $S$. Theorem 5.4 then implies that a maxmin height triangulation, and in the nondegenerate case a triangulation lexicographically maximizing the increasing vector of heights, can be computed in time $O(n^2 \log n)$ and storage $O(n)$. 
7. Minimizing the Maximum Eccentricity

Consider a triangle $xyz$ and let $(c_1, \rho_1)$ be its circumcircle, with center $c_1$ and radius $\rho_1$. The eccentricity of $xyz$, $e(xyz)$, is the infimum over all distances between $c_1$ and points of $xyz$. Clearly, $e(xyz) = 0$ iff $c_1$ lies in the closure of $xyz$. Note that eccentricity is related to the size of the maximum angle, $\alpha(xyz)$, only with large triangles counting more. Specifically, unless $e(xyz) = e(abc) = 0$,

$$\alpha(xyz) < \alpha(abc) \quad \text{iff} \quad \frac{e(xyz)}{\rho_1} < \frac{e(abc)}{\rho_2},$$

where $\rho_2$ is the radius of the circumcircle of $abc$. The triangulation of the pentagon in Fig. 6.1 can be used to show that edge-flipping does not always succeed in minimizing the maximum eccentricity.

Eccentricity is our first example of a measure satisfying condition (I), but not (II). Consider Fig. 7.1. In this figure, vertices $u$ and $v$ lie very close to $yx$ and $yz$, respectively, so that the circumcircle of $uvy$ is significantly smaller than the one of $xyz$, and $e(uvy) < e(xyz)$. In fact, $e(xyz)$ exceeds the eccentricity of every triangle of the minmax-eccentricity triangulation $T$, even though $T$ does not break $xyz$ at any of its vertices. We now show that $e$ satisfies the weaker condition (I). It turns out that $y$ is an anchor of $xyz$ only if a largest angle in $xyz$ is at $y$.

**Lemma 7.1.** Let $xyz$ be a triangle of a triangulation $T$ of $S$, such that $e(xyz) > 0$, and let $y$ be a vertex with maximum angle in $xyz$. Then $\max\{e(T), e(S)\} > e(xyz)$ for every triangulation $T$ of $S$ that neither contains $xyz$ nor breaks $xyz$ at $y$.

**Proof.** Assume that $T$ neither contains $xyz$ nor breaks it at $y$. Therefore, $T$ must contain a triangle $uvy$ so that $u = x$ and $uv \cap yz \neq \emptyset$ (renaming vertices if necessary), or $uv$ intersects $yx$ and $yz$, as in Fig. 7.2. Let $(c_1, \rho_1)$ be the circumcircle of $xyz$. If neither $u$ nor $v$ are enclosed by this circle, then $e(xyz) < e(uvy) \leq e(T)$. Otherwise, assume that $v$ is enclosed by $(c_1, \rho_1)$ and consider the line segment $c_1v$.

![Fig. 7.1.](image)

$T$ is the triangulation with diagonals $uw$, $vw$, and $wu$, and $T$ the one with diagonals $xy$, $yz$, and $zx$. Then $e(T) < e(xyz) \leq e(S)$, but $T$ does not break $xyz$ at any of its vertices, in contradiction to condition (II).
Fig. 7.2. The triangle $xyz$ in $\mathcal{A}$ is neither contained in $\mathcal{F}$ nor is it broken at $y$ by $\mathcal{F}$. Therefore, $\mathcal{F}$ contains a triangle $wyz$ that intersects $xyz$ as shown. There must be a triangle in $\mathcal{A}$ with eccentricity greater than $\varepsilon(xyz)$ intersecting $c_1,v$.

It intersects a sequence of edges of $\mathcal{A}$, ordered from $c_1$ to $v$. For an edge $ab$ in this sequence let $abc$ be the supporting triangle so that $c$ and $c_1$ lie on different sides of $ab$. Assume that $ab$ is the first edge in the sequence so that $(c_1, p_1)$ encloses $c$ but not $a$ and not $b$. Then $\varepsilon(\mathcal{A}) \geq \varepsilon(abc) > \varepsilon(xyz)$.

Theorem 5.1 thus implies that a minmax-eccentricity triangulation of $n$ points can be constructed in time $O(n^2)$ and storage $O(n^2)$. In the nondegenerate case the same time and storage suffice to construct a triangulation lexicographically minimizing the decreasing vector of eccentricities.

8. Minimizing the Maximum Slope

Consider a function $f: \mathbb{R}^3 \to \mathbb{R}$ defining a surface $x_3 = f(x_1, x_2)$ in $\mathbb{R}^3$. The gradient of $f$ is the vector $\nabla f = (\partial f / \partial x_1, \partial f / \partial x_2)$, each component of which is itself a function from $\mathbb{R}^2$ to $\mathbb{R}$. Define $\nabla^2 f = (\partial f / \partial x_1)^2 + (\partial f / \partial x_2)^2$, and call $\sqrt{\nabla^2 f}$ at a point $(x_1, x_2)$ the slope at this point.

Let $S$ be a point set in $\mathbb{R}^2$ and let $\hat{S}$ be the corresponding set in $\mathbb{R}^3$ where each point of $S$ has a third coordinate called elevation. For a point $x$ of $S$, we write $\hat{x}$ for the “lifted” point, that is, the corresponding point in $\hat{S}$. Analogous to the definitions in $\mathbb{R}^2$, $\hat{x}y$ denotes the relatively open line segment with endpoints $\hat{x}$ and $\hat{y}$, and $\hat{x}y\hat{z}$ denotes the relatively open triangle with corners $\hat{x}, \hat{y}, \hat{z}$. We can think of $\hat{x}y\hat{z}$ as a partial function $f$ on $\mathbb{R}^2$, defined within $xyz$. At each point in $xyz$, the gradient is well defined and the same as for any other point in $xyz$. We can therefore set $\varepsilon(xyz)$ equal to the slope at any point of $xyz$, and call it the slope of $xyz$. For a triangulation $\mathcal{A}$ of $S$ define $\varepsilon(\mathcal{A}) = \max\{\varepsilon(xyz)|xyz$ a triangle of $\mathcal{A}\}$, as usual. A minmax-slope triangulation of $S$ minimizes the maximum $\varepsilon$ of any triangle.

Triangulations are commonly used to compute surfaces interpolating point set data with elevations. Rippa [23] recently proved that, regardless of elevations, the Delaunay triangulation minimizes the integral (over the convex hull of $S$) of $\nabla^2 f$ among all triangulations of $S$. See [7] for other interesting optimization criteria.

The five-point example of Fig. 6.1 again shows that the edge-flipping strategy does not in general compute a minmax-slope triangulation. Just imagine that points $a, b, c, d, e$ are not perturbed and thus form a regular pentagon. Let the
elevations of \( a, b, c, d, e \) be 5, 11, 0, 10, 0, in this sequence. The optimal triangulation is defined by the diagonals \( ac \) and \( ad \), and the current triangulation (with diagonals \( be \) and \( ce \) as shown) cannot be improved by a single edge-flip.

As in the case of eccentricity, we can show that \( \sigma \) does not satisfy the strong condition (II). Figure 8.1 gives a six-point example in which an improvement \( T \) of \( \mathcal{D} \), does not break a triangle \( xyz \) with \( \sigma(T) < \sigma(xyz) \).

Observe that the direction of steepest descent at any point on a triangle \( xyz \) is given by \( \Delta = -\nabla f \) at that point. We call the vertex \( y \) a peak of \( xyz \) unless the line \( y + \lambda \Delta, \lambda \in \mathbb{R} \), intersects the closure of \( xyz \) only at \( y \). In other words, a peak is a vertex first hit when sweeping with a line perpendicular to the direction of steepest descent. In the nondegenerate case \( xyz \) has only one peak, but if \( \Delta \) is parallel to an edge, then there are two peaks. Call the intersection of the closure of \( \hat{x}\hat{y}\hat{z} \) with the plane parallel to the \( x_3 \)-axis through \( y + \lambda \Delta \) the descent line \( \ell(xyz) \) of \( xyz \), assuming \( y \) is an anchor of \( xyz \).

The remainder of this section shows that \( \sigma \) does satisfy the weak condition (I). In fact, each peak of a worst triangle is an anchor. For technical reasons it is necessary to assume that no four points of \( \hat{S} \) are coplanar. Indeed, the strict inequality in Lemma 8.1 is incorrect without this assumption. (This general position assumption, however, does not diminish the generality of our algorithm, because a simulated perturbation of the points can be used to enforce general position [9]).

**Lemma 8.1.** Let \( xyz \) be a triangle of a triangulation \( \mathcal{D} \) of \( S \), and let the intersection of line \( y + \lambda \Delta \) with the closure of \( xyz \) be strictly larger than point \( y \), and let \( y \) be a peak of \( xyz \). Then \( \max\{\sigma(\mathcal{D}), \sigma(T)\} > \sigma(xyz) \) for every triangulation \( T \) of \( S \) that neither contains \( xyz \) nor breaks \( xyz \) at \( y \).

**Proof.** The slope of \( xyz \), \( \sigma(xyz) \), is also the slope of the descent line \( \ell_1 = \ell(xyz) \). Assume without loss of generality that \( \ell_1 \) descends from \( \hat{y} \) down to where it meets the closure of \( \hat{x}\hat{z} \). (If it ascends, we use the same argument only with the \( x_3 \)-axis reversed.) Assume also that \( T \) neither contains \( xyz \) nor breaks it at \( y \). It follows that \( T \) contains an edge \( uv \) so that either \( u = x \) and \( uv \cap yz \neq \emptyset \) (rename vertices if necessary) or \( uv \) intersects both \( xy \) and \( yz \). If \( \sigma(uvw) > \sigma(xyz) \), then \( \sigma(T) > \sigma(xyz) \) and there is nothing to prove.
Otherwise, the edge \( \hat{u}\hat{v} \) must pass above \( \ell_1 \) in \( \mathbb{R}^3 \). By this we mean that there is a line parallel to the \( x_3 \)-axis that meets \( \hat{u}\hat{v} \) and \( \ell_1 \) and the elevation of its intersection with \( \hat{u}\hat{v} \) exceeds the elevation of its intersection with \( \ell_1 \), as in Fig. 8.2. Then at least one of \( \hat{u} \) and \( \hat{v} \) must lie above the plane \( h_1 \) through points \( \hat{x}, \hat{y}, \hat{z} \); say \( \hat{v} \) lies above \( h_1 \). Consider the triangle \( yz \hat{v} \), and note that it is not necessarily a triangle of \( \mathcal{A} \) or \( \mathcal{T} \), nor even an empty triangle of \( S \). We have \( \sigma(yz \hat{v}) > \sigma(xy) \) because the \( x_3 \)-parallel projection of \( \ell_1 \) onto the plane \( h_2 \) through \( \hat{y}, \hat{v}, \hat{z} \) is steeper than \( \ell_1 \) but not steeper than \( \ell_2 = \ell(yz) \). We distinguish three cases depending on which vertex is the peak of \( yz \hat{v} \), that is, through which one a line of steepest descent of \( yz \hat{v} \) passes.

**Case 1:** \( v \) is a peak of \( yz \hat{v} \). Then \( \ell_2 \) connects \( \hat{v} \) with a point on the closure of \( \hat{y}\hat{z} \). Consider the intersection of \( \mathcal{A} \) with a plane parallel to the \( x_3 \)-axis through \( \ell_2 \). This intersection includes a polygonal chain that connects \( \hat{v} \) with that same point on the closure of \( \hat{y}\hat{z} \) (since \( yz \) is an edge in \( \mathcal{A} \)). One of the segments in the chain must have slope at least the average slope of the chain; hence one of the triangles \( abc \) in \( \mathcal{A} \) has \( \sigma(abc) \geq \sigma(yz \hat{v}) > \sigma(xy) \), and therefore \( \sigma(\mathcal{A}) > \sigma(xy) \).

**Case 2:** \( z \) is a peak of \( yz \hat{v} \). Then \( \ell_2 \) connects \( \hat{z} \) with a point on the closure of \( \hat{y}\hat{v} \). Then we use the same argument as in Case 1, only applied to \( \mathcal{T} \). Since \( yv \) is an edge in \( \mathcal{T} \) at least one of the triangles \( abc \) in \( \mathcal{T} \) that intersect the projection of \( \ell_2 \) has \( \sigma(abc) \geq \sigma(yv) > \sigma(xy) \), and therefore \( \sigma(\mathcal{T}) > \sigma(xy) \).

**Case 3:** \( y \) is a peak of \( yz \hat{v} \). In this case \( \ell_2 \) connects \( \hat{y} \) with a point \( \hat{w} \) on the closure of \( \hat{y}\hat{z} \). Furthermore, it is impossible that \( \ell_2 \) descends from \( \hat{y} \) to \( \hat{w} \) because \( \hat{w} \) lies above \( h_1 \), which contradicts \( \sigma(yz \hat{v}) > \sigma(xy) \). Thus, it must be that \( \ell_2 \) descends from \( \hat{w} \) down to \( \hat{y} \). Then \( \sigma(uv) > \sigma(yz \hat{v}) \) because \( \hat{u}\hat{v} \) passes above \( \ell_2 \). However, \( \sigma(yz \hat{v}) > \sigma(xy) \), so we have shown \( \sigma(\mathcal{T}) > \sigma(xy) \).

Note that Lemma 8.1 also holds for constrained triangulations of \( S \). We can therefore apply Theorem 5.1 and obtain an \( O(n^3) \)-time and \( O(n^2) \)-storage algorithm for constructing a minmax slope triangulation, and in the nondegenerate case for constructing a triangulation lexicographically minimizing the decreasing vector of slopes.

**Remark.** It would be interesting to find other optimality criteria for point sets with elevations, that are amenable to edge-insertion. However, we know that
several natural measures, e.g., $\mu(\text{xyz})$ equal to the maximum angle on the lifted triangle $\hat{x}\hat{y}\hat{z}$, do not satisfy either (I) or (II). A six-point counterexample can be formed with the vertices of a regular hexagon. There are two triangulations of the hexagon with an equilateral triangle in the middle; no single edge-insertion tranforms one into the other. By appropriately setting elevations, these two triangulations can be made local optima.

9. Conclusion

The main result of this paper is the formulation of the edge-insertion paradigm as a general method to compute optimal triangulations, and the identification of two classes of criteria for which the paradigm indeed finds the optimum. The paradigm is an abstraction of the algorithm introduced in [10] for computing minmax angle triangulations.

The algorithms of this paper have been implemented by Waupootisch [11]. (The programs are currently available through anonymous ftp from the directory “/SGI/MinMaxer” at the site “ftp.ncsa.uiuc.edu.”) The experience shows that the $O(n^2 \log n)$-time algorithm is fairly practical, also for large point sets. This is because its running time for most data sets is significantly less than the pessimistic worst-case prediction. This phenomenon, on the other hand, was not observed for the $O(n^3)$-time algorithm.

Though usually simple to verify, conditions (I) and (II) are somewhat restrictive. It would be interesting to find conditions weaker than (I) even though the price to pay may be implementations of the paradigm that take more than cubic time. Listings of optimality criteria can be found in [1], [2], [19], and [24]. Furthermore, implementations for criteria satisfying (I) and (II) that run in time $o(n^3)$ and $o(n^2 \log n)$ are sought.

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