On the Computational Complexity of Betti Numbers: Reductions from Matrix Rank*

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Abstract
We give evidence for the difficulty of computing Betti numbers of simplicial complexes over a finite field. We do this by reducing the rank computation for sparse matrices with m non-zero entries to computing Betti numbers of simplicial complexes consisting of at most a constant times m simplices. Together with the known reduction in the other direction, this implies that the two problems have the same computational complexity.

1 Introduction

The efficient computation of topological properties of a space is one of the main goals of computational topology [8,18]. Homology groups are important such properties that encode the connectivity of a space. For example, the zeroth homology group represents the connected components, and if the space is embedded in 3-dimensional Euclidean space, then the second homology group represents the voids. The computation of homology groups, and in particular of their ranks – the Betti numbers – is of practical importance. In most applications, homology groups are computed using field coefficients, in which case the groups are vector spaces and the Betti numbers are their dimensions. Computing these numbers and their more advanced persistent versions is central in applications of computational topology [9]. Consequently, hardness results and evidence for the difficulty of computing Betti numbers imply similar results for all of the applications.

For a general finite simplicial complex, the homology groups can be computed by reducing the boundary or incidence matrices using standard row- and column-operations. Similarly, the Betti numbers in the case of field coefficients can be computed by finding the ranks of these matrices. However, in some special cases, Betti numbers can be computed by other means and more efficiently. For instance, if the complex triangulates a 2-manifold, then the Betti numbers can be computed using the Euler relation. Another example is when the simplicial complex is embedded in 3-dimensional Euclidean space and the complement space is also triangulated; see [4]. In this situation, the Betti numbers can be computed in a time that is linear in the size of the complex. One of the motivations of this paper is to understand what can be said if the simplicial complex is embedded in the 4-dimensional Euclidean space.

We use the term *computational complexity* to refer to the run-time of an optimal algorithm solving a problem. It is commonly described as a function of the size of the problem instance. We reduce problems to each other using worst-case linear-time reductions on the common RAM model. However, our results are valid when the underlying model of computation allows these reductions to be done in linear time. For example, this is the case for randomized algorithms with expected run-time as computational complexity.

Here is an overview of our results, which concern themselves with computing the Betti numbers of finite 2-dimensional simplicial complexes, that is: the ranks of the homology groups defined for $\mathbb{F}_2$-coefficients. Recall that the Euler number of a complex is the alternating sum of simplex counts as well as the alternating sum of Betti numbers: $\chi = \beta_0 - \beta_1 + \beta_2$. For a 2-dimensional complex with m simplices, we can compute χ as well as β_0 in $O(m)$ time. The complexity of computing the first Betti number is therefore equal to that of the second Betti number. We have two main results:

I. The complexity of computing the Betti numbers of a 2-dimensional simplicial complex with m simplices is the same as that of computing the rank of an m-by-m 0-1 matrix with m 1s.

II. The complexity of computing the Betti numbers

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of a 2-dimensional simplicial complex with n vertices is at least that of computing the rank of an n-by-n 0-1 matrix.

Our first result says that computing the Betti numbers of a 2-dimensional complex has the same complexity as computing the rank of a sparse matrix. To prove this claim, we build a simplicial complex for a given matrix such that the second Betti number of the complex is the nullity of the matrix. The simplicial complex in this construction embeds in the 4-dimensional Euclidean space, which thus provides an answer to the question stated above. Our second result says that if we ignore the number of simplices and express run-time in terms of the number of vertices of the complex, then we cannot do better than for computing the rank of an n-by-n matrix. We remark that we can harvest a sparsification of 0-1 matrices as a side-product of our reductions; see Section 4.

The paper is organized as follows. Section 2 introduces background material and definitions, including a short review of the current best bounds for computing the rank of a matrix. Section 3 presents the reductions proving our two main results. Section 4 extends our results from $\mathbb{F}_2$ to other finite fields, and it relates them to matrix sparsification.

2 Background

In this section, we recall basic definitions and facts about simplicial and singular homology for general d-dimensional simplicial complexes. We also define vertical and horizontal homology classes. Unless explicitly stated, we will limit ourselves to coefficients in $\mathbb{F}_2$, the field of integer arithmetic modulo 2. All homology groups will be finite rank vector spaces over $\mathbb{F}_2$. They are determined up to isomorphism by their ranks.

Simplicial homology. Let K be a simplicial complex, and write n_i for its number of i-simplices. The *size* of K is the total number of simplices: $\sum_i n_i$. Assuming an ordering, we write $\Delta_{i,\ell}$ for the ℓ-th i-simplex, and we let

$$S_i(K) \quad = \quad \{\Delta_{i,\ell} \mid \ell = 1, 2, \ldots, n_i\}$$

be the set of i-simplices in K. Any subset of $S_i = S_i(K)$ is called an *i-chain* of K. It can be written as an n_i-vector of 0s and 1s: $c = (c(1), c(2), \ldots, c(n_i))$, where $c(\ell) = 1$ iff the simplex $\Delta_{i,\ell}$ belongs to c, for $1 \leq \ell \leq n_i$. Two i-chains, c and d, can be added by vector addition modulo 2:

$$(c+d)(\ell) \quad = \quad c(\ell) + d(\ell) \pmod 2.$$

This means that the sum is the chain that consists of all i-simplices in the symmetric difference of the two chains: $c + d = (c \cup d) - (c \cap d)$. With this notion of addition, the set of i-chains forms a group, denoted as $C_i = C_i(K)$. More specifically, C_i is the n_i-dimensional vector space over $\mathbb{F}_2$ with basis S_i.

The *boundary* of an i-simplex, denoted as $\partial_i(\Delta_{i,\ell})$, is the collection of its faces of dimension $i-1$. It is a chain in C_{i-1}. Since S_i is a basis for C_i, this definition can be extended to a unique homomorphism between vector spaces, $\partial_i : C_i \to C_{i-1}$ defined by

$$\partial_i(c) \quad = \quad \sum_{\Delta_{i,\ell} \in c} \partial_i(\Delta_{i,\ell}),$$

called the *i-th boundary homomorphism*. By definition, the zeroth boundary homomorphism, ∂_0, is the zero map. A chain with empty boundary is called a *cycle*. Hence, the i-cycles are the chains in the kernel of ∂_i, and we write $Z_i = Z_i(K) \subseteq C_i$ for this kernel. For example, if K is a triangulation of a 2-manifold (without boundary), then the chain c that includes all triangles in K is a 2-cycle. Indeed, every edge of K belongs to the boundary of exactly two triangles, which implies that the sum of the boundaries of all triangles is empty. A chain that is the boundary of another chain – necessarily of one higher dimension – is called a *boundary*. Hence, the i-boundaries are the chains in the image of ∂_{i+1}, and we write $B_i = B_i(K) \subseteq C_i$ for this image. Note that the Z_i and B_i are also vector spaces over $\mathbb{F}_2$, and that $B_i \subseteq Z_i$ because the boundary of a boundary is necessarily empty. We can therefore form the quotient, $H_i = H_i(K) = Z_i/B_i$, called the *i-th homology group* of K. This quotient is again a vector space over $\mathbb{F}_2$, and its dimension is called the *i-th Betti number* of K, denoted as $\beta_i = \beta_i(K)$. Since H_i is a quotient, its elements are not cycles but rather classes of *homologous* cycles. If such a class contains a cycle c, then we denote the class by $[c]$, and we call c a *representative* of the class. Note that the sum of any two representative cycles of the same class is a boundary.

Computing simplicial homology. We already have a basis for the vector space C_i, namely S_i or, equivalently, the vectors of length n_i with only a single 1 each. We need to find bases for Z_i and B_i. Let D_i be the matrix of the i-th boundary homomorphism, ∂_i, with respect to the bases S_i and S_{i-1}. The rows of D_i are indexed by $(i-1)$-simplices and the columns by i-simplices. The ℓ-th column of D_i corresponds to $\Delta_{i,\ell}$ and is the vector $\partial_i(\Delta_{i,\ell})$. It follows that we can write the boundary of a chain $c \in C_i$ in matrix notation as $\partial_i(c) = D_i c$. To find a basis for Z_i, we only need to find a basis for the null-space of D_i. We thus reduce the matrix D_i to diagonal form using the usual row- and column-operations. In this form, an initial segment of the diagonal consists of 1s and all other matrix entries are zero. Let R_i and R_{i-1} be the new bases with respect to which the i-th boundary matrix is diagonal. The vectors in R_i associated with zero columns form a basis of Z_i, and the vectors in R_{i-1} associated with non-zero rows form a basis of B_{i-1}. After reducing all boundary matrices, we have a basis for each Z_i and each B_i. If our interest is in the Betti numbers, we can count basis vectors and this way get the ranks of the D_i and the dimensions of the vector spaces:

$$\begin{aligned} \mathsf{dim}\, Z_i &= n_i - \mathsf{rank}\, D_i, \\ \mathsf{dim}\, B_i &= \mathsf{rank}\, D_{i+1}, \\ \mathsf{dim}\, H_i &= \mathsf{dim}\, Z_i - \mathsf{dim}\, B_i. \end{aligned}$$

However, to get bases for the homology groups, we need to do a bit more work. One possibility is to express the basis vectors of B_i in terms of the basis of Z_i, for $0 \leq i \leq d$, which can be done by matrix multiplication. Alternatively, we can rewrite D_i before reducing it such that its rows correspond to the computed bases of C_{i-1} after reducing D_{i-1}. The latter method avoids the need for matrix multiplication at the cost of doing the reduction on matrices that are possibly dense from the beginning. To summarize, assuming a constant dimension of K, bases for all homology groups can be computed in a constant number of matrix reductions, and all Betti numbers can be computed with a constant number of rank computations for sparse matrices.

Singular homology. Recall that a cycle in simplicial homology is a collection of simplices in a given triangulation of the space. We also need cycles that possibly cross over simplices in the triangulation and therefore introduce the formalism of singular homology. Formally, a *singular i-simplex* is a (continuous) map $\sigma : \Delta_i \to |K|$, where Δ_i is the standard i-simplex. We write $\bar{S}_i$ for the set of all such maps. A *singular i-chain*, $\bar{c}$, is a finite subset of $\bar{S}_i$. Equivalently, the chain is a function, $\bar{c} : \bar{S}_i \to \{0, 1\}$ with finite support. Thinking of such a function as an infinite vector, and using $\mathbb{F}_2$, we again define addition by setting

$$(\bar{c} + \bar{d})(\sigma) \quad = \quad \bar{c}(\sigma) + \bar{d}(\sigma) \pmod 2.$$

The *i-th singular chain group*, $\bar{C}_i$, is the set of singular i-chains together with addition, which is again a vector space over $\mathbb{F}_2$. The *boundary* of a singular i-simplex mapping Δ_i to $|K|$ is the sum of the restrictions of the map to the $(i-1)$-dimensional faces of Δ_i. Extending this definition to chains, we get the *boundary homomorphism*, $\bar{\partial}_i : \bar{C}_i \to \bar{C}_{i-1}$. With this, we define the *i-th singular cycle group*, $\bar{Z}_i$, as the kernel of $\bar{\partial}_i$, and the *i-th singular boundary group*, $\bar{B}_i$, as the image of $\bar{\partial}_{i+1}$. As before, we have $\bar{\partial}_i \circ \bar{\partial}_{i+1} = 0$, which implies that all singular boundaries are singular cycles. Finally, we define the *i-th singular homology group* by taking the quotient, $\bar{H}_i = \bar{Z}_i / \bar{B}_i$.

The simplices of K can also be thought of as members of $\bar{S}_i$. This inclusion induces homomorphisms $C_i \to \bar{C}_i$, which in turn define homomorphisms between the homology groups, $H_i \to \bar{H}_i$. A well-known result in algebraic topology asserts that these homomorphisms are isomorphisms; see e.g. [11, Theorem 2.27]. For a finite simplicial complex, the simplicial and singular homology groups are isomorphic and a basis for simplicial homology is also a basis for singular homology. The reason for introducing singular in addition to simplicial homology is that it simplifies our definitions and proofs. From this point on, we write S for $\bar{S}$, c for $\bar{c}$, etc.

Remark. Besides facilitating the comparison of spaces with each other, homology groups in low dimensions also have intuitive meanings. For example, the rank of the zeroth homology group is the number of connected components. For a graph, the rank of the first homology group is the number of independent cycles, which for a connected graph is the number of edges minus the number of vertices plus one. Note that for a tree, this number is zero.

Horizontal and vertical homology. Let K be a 2-dimensional simplicial complex, and $f : |K| \to \mathbb{R}$ a piecewise linear function on K. It is defined by assigning values to the vertices and then extending the map by linear interpolation on the simplices. We assume f is *generic*, by which we mean that it has distinct values on the vertices. We distinguish between homology classes that are represented by cycles carried by level sets of the functions, and classes that have no such cycles. This distinction has been introduced in [2] and studied in [5].

More formally, we call a homology class $\alpha \in H_i$ *horizontal* if it has a representative cycle, c, whose image under f is a finite set of values in $\mathbb{R}$. The horizontal classes form a subgroup of homology, called the *i-th horizontal homology group* of K with respect to f, denoted as $H_i^{\text{hor}} = H_i^{\text{hor}}(K, f)$. The *i-th vertical homology group* is $H_i^{\text{vcl}} = H_i / H_i^{\text{hor}}$. The ranks of the horizontal and vertical homology groups are called the *horizontal* and *vertical Betti numbers*, denoted as β_i^{hor} and β_i^{vcl}. Note that the i-th Betti number of K satisfies $\beta_i = \beta_i^{\text{hor}} + \beta_i^{\text{vcl}}$.

Reeb graphs. The distinction between first horizontal and first vertical homology classes is significant because there are fast algorithms for computing the latter but no such algorithms for computing the former. To explain this, we now introduce a map from a simplicial complex, K, and a function, $f : |K| \to \mathbb{R}$, to a graph that sends horizontal classes to zero. We call two points $x, y \in |K|$ *equivalent*, denoted as $x \sim y$, if $f(x) = f(y)$ and the two points belong to the same component of $f^{-1}(t)$, where $t = f(x) = f(y)$. The *Reeb graph* of K and f is the quotient space obtained by identifying equivalent points, denoted as $K/\sim$. Intuitively, the Reeb graph has a point for each connected component of each level set, and these points are connected like the corresponding components of the level sets. The map from K to $K/\sim$ is continuous and induces a homomorphism from the first homology group of K to the first homology group of $K/\sim$. It is clear that this homomorphism sends horizontal classes to zero. It is also true that it preserves the 1-dimensional vertical classes. More specifically, the first vertical homology group of the complex and the function is isomorphic to the first homology group of the Reeb graph: $H_1^{\text{vcl}}(K, f) \cong H_1(K/\sim)$; see [5, Theorem 3.2].

The Reeb graph of a generic function on a simplicial complex of size m can be computed in $O(m \log m)$ time; see [6, 10, 13]. Writing n' for the number of nodes, m' for the number of arcs, and ℓ' for the number of connected components of the Reeb graph, we have $\beta_1(K/\sim) = m' - n' + \ell'$. It follows that we can compute the rank of the first vertical homology group of K and f in $O(m \log m)$ time.

Complexity of rank computation. To get the first homology of K, we still need to compute the first horizontal homology of K and f. The main point of this paper is to show that this is more difficult than computing the vertical homology.

It is known that the rank of an arbitrary matrix can be computed in matrix multiplication time [1]. The best asymptotic run-time for multiplying two matrices is a major open problem in algebraic complexity theory. Let ω be a number such that a worst-case optimal algorithm that multiplies two n-by-n matrices runs in $O(n^{\omega+\varepsilon})$ time, for each $\varepsilon > 0$. The number ω is called the *exponent* of matrix multiplication. The currently best upper bound is $\omega < 2.3727$; see [3, 15]. However, it is not known whether the sparsity of matrices can help in computing the rank. While there exists a theoretical algorithm for multiplying two n-by-n matrices each with $O(n)$ non-zero entries in $O(n^{2+\varepsilon})$ time, for every $\varepsilon > 0$ [17], it is not known whether this helps in rank computation or Gaussian elimination for sparse matrices.

It is worth mentioning that there is a randomized algorithm that computes the rank of a matrix in time roughly proportional to n^2. Specifically, Wiedemann's Monte Carlo algorithm computes the rank of an n-by-n matrix with m non-zero entries in $O(n^{2+\varepsilon} + nm)$ time [14]. Moreover, there exists a Las Vegas algorithm whose expected run-time for matrices with $O(n)$ non-zero entries is $O(n^{2.28})$; see [7] but also [16].

3 Main Results

In this section, we state and prove our main results. They consist of reductions from computing the rank of a matrix to computing the Betti numbers of a complex. For simplicity, we consider only square matrices, while the generalization to rectangular matrices is straightforward. Any n-by-n matrix, M, determines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$. The kernel of this map is the *null-space*, and the dimension of the kernel is the *nullity* of the matrix, denoted as $\mathsf{null}\, M$. Since the rank of the matrix is the dimension of the image of this map, we have $\mathsf{null}\, M = n - \mathsf{rank}\, M$. The nullity is therefore the maximum number of independent solutions to the equations $Mx = 0$.

Statements. We begin by formally stating our first theorem. It assumes a representation of the input matrix that gives amortized constant time access to its non-zero entries. It also assumes that the matrix contains no zero rows or columns, which implies that $n \leq m$.

THEOREM 3.1. *Let M be an n-by-n 0-1 matrix with m non-zero entries. In time $O(m)$, it is possible to build a 2-dimensional simplicial complex, K, of size $O(m)$ and a piecewise linear function $f : |K| \to \mathbb{R}$, such that $H_1^{\text{hor}}(K)$ is isomorphic to the null-space of M and $H_2(K)$ is isomorphic to the null-space of the transpose, M^t.*

Theorem 3.1 implies

$$(3.1) \qquad b(m) = \Omega(r(n, m)),$$

where $b(m)$ is the complexity of computing the Betti numbers of a 2-dimensional simplicial complex of size m, and $r(n, m)$ is the computational complexity of computing the rank of an n-by-n 0-1 matrix with m non-zero entries. The constructed complex, $K = K(M)$, embeds in $\mathbb{R}^4$. Equation (3.1) implies the result labeled I in the introduction. Recall that for a complex of size m the Betti numbers can be computed by a constant number of rank computations for matrices which are at most m-by-m and have $O(m)$ non-zero entries. Therefore, $b(m) = O(r(m, m))$. The theorem shows $b(m) = \Omega(r(m, m))$ which gives $b(m) = \Theta(r(m, m))$.

The second construction focuses on the number of entries of the matrix, ignoring the difference between sparse and dense. Again, we assume access in constant amortized time to the non-zero entries of the input matrix.

THEOREM 3.2. *Let M be an n-by-n 0-1 matrix with m non-zero entries. In time $O(m)$, we can build a 2-dimensional simplicial complex, L, with $O(n)$ vertices and a piecewise linear function $g : |L| \to \mathbb{R}$, such that $H_1^{\text{hor}}(K)$ is isomorphic to the null-space of M and $H_2(K)$ is isomorphic to the null-space of M^t.*

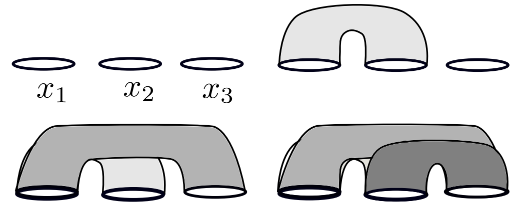
Theorem 3.2 implies our second main result, labeled II in the Introduction:

$$B(n) = \Omega(r(n)), \tag{3.2}$$

where $B(n)$ is the complexity of computing the Betti numbers of a 2-dimensional simplicial complex with n vertices, and $r(n)$ is the complexity of computing the rank of an n-by-n 0-1 matrix. The complex, $L = L(M)$, does not necessarily embed in $\mathbb{R}^4$.

The first reduction. We interpret M as the matrix of a system of linear equations. The nullity is the dimension of the space of solutions to the equations $\sum_{\ell=1}^{n} M(k, \ell)x_\ell = 0$, for $1 \leq k \leq n$.

Construction. We start by constructing a cycle made out of a constant number of edges for each column. We refer to the cycle corresponding to column ℓ by x_ℓ. Placing these cycles in a 2-dimensional plane Π in $\mathbb{R}^4$, we set the function values of their vertices to 0. For each row, we add a sphere with as many holes as there are non-zero entries, gluing the boundaries of the holes to the cycles corresponding to the non-zero entries; see Figure 1. Letting p be the number of holes, we call this surface a *p-cap*, since it is obtained by removing p disks from a sphere. It generalizes a cap, which is a sphere with a single disk removed. It



$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Figure 1: Starting with three cycles, the simplicial complex that corresponds to the 3-by-3 matrix is constructed by adding one cap at a time.

is easy to construct a triangulation of the p-cap that consists of $O(p)$ vertices, edges, and triangles and embeds in a 3-dimensional plane containing Π. The function values of the vertices in the triangulation that are glued to the initial cycles are 0, and those of all other vertices are chosen to be strictly larger than 0 and smaller than 1. As suggested in Figure 1, we can think of the function values as the heights of the vertices above Π. Choosing a pencil of 3-dimensional planes, all passing through Π, we get n caps that are pairwise disjoint except for possibly shared cycles in Π. It follows that the k-th cap is a 2-chain that introduces the relation $\sum_{\ell=1}^{n} M(k, \ell)[x_\ell] = 0$ on the classes of cycles, for $1 \leq k \leq n$. After adding the n caps, we obtain a simplicial complex, which we call $K = K(M)$.

Analysis. At the beginning, after adding the cycles and before adding any caps, every x_ℓ represents a 1-dimensional horizontal homology class. The class remains horizontal throughout the construction, but it can of course become null-homologous. Indeed, the effect of a cap is to render the corresponding sum of cycles to be null-homologous. We show that the cap does not affect the first horizontal homology group in any other way.

LEMMA 3.1. *For every $1 \leq k \leq n$, the addition of the k-th cap does not create any new horizontal homology class, and it kills only one class, namely $\sum_{\ell=1}^{n} M(k, \ell)[x_\ell]$.*

PROOF. Let p be the number of non-zero entries in the k-th row of M, and recall that the corresponding p-cap is a sphere with p holes. To show that the addition of the p-cap does not add any new horizontal classes, we construct the p-cap from the p circles that bound its holes as follows. Connect the p circles with $p-1$ arcs whose interior points have function values strictly larger than 0. Because each arc covers an interval of function values that has a non-empty interior, these arcs do not change the horizontal homology. We can form a closed curve that traverses each circle once and each arc twice, once in each direction. The p-cap can now be completed by adding a disk whose boundary is glued to the closed curve. Finally, we note that the boundary cycle of the added disk is homologous to $\sum_{\ell=1}^{n} M(k,\ell)x_\ell$. This is the only relation implied by the disk, which finishes the proof. □

Lemma 3.1 implies that the first horizontal homology group of K is generated by $[x_1], [x_2], \ldots, [x_n]$ subject to $Mx = 0$, where $x = ([x_1], [x_2], \ldots, [x_n])$. The number of independent generators is therefore $n - \mathsf{rank}\, M = \mathsf{null}\, M$. The cap added for row k creates a new 2-cycle iff its boundary can be written as a linear combination of the boundaries of the preceding caps. It follows that the second Betti number is equal to the the number of rows minus the rank of M, which is the nullity of matrix M^t.

Complexity. We recall that M provides constant amortized time access to its non-zero entries, implying that the above construction can be done in $O(m)$ time. This completes the proof of Theorem 3.1.

REMARKS. (1) As explained above, the complex embeds in $\mathbb{R}^4$. In the $O(m)$ time needed for its construction, we can also triangulate the entire space, keeping K as a subcomplex of the triangulation. In sharp contrast to the 3-dimensional situation, this implies that even the availability of such a triangulation does not make it easy to compute the Betti numbers.

(2) The particular structure of K makes it possible to construct the first vertical Betti number of K in linear time, without running the Reeb graph algorithm. Indeed, the Reeb graph is homotopy equivalent to the bipartite graph whose nodes are the rows and the columns of M, with an arc from a row to a column iff they intersect in a non-zero entry of M. We have $2n$ nodes and m arcs, and we can compute ℓ, the number of connected components, in $O(m)$ time. The number of independent loops in the Reeb graph is $m-2n+\ell$, which is also the first vertical Betti number of K. We note that Equation (3.1) now follows from Theorem 3.1 in two ways: first by combining the above calculation of vertical Betti number with $\beta_1^{\text{hor}} = \mathsf{null}\, M$ and $\beta_1 = \beta_1^{\text{hor}} + \beta_1^{\text{vcl}}$, and second by using $\beta_2 = \mathsf{null}\, M^t$.

(3) The assumption of access in constant amortized time to the non-zero entries of the matrix is not essential. Without it, we can construct the complex K in $O(n^2+m)$ time, which implies a slightly weaker claim that suffices for our purposes.

The second reduction. Similar to Theorem 3.1, we prove Theorem 3.2 by interpreting the matrix M as a system of linear equations from which we construct a simplicial complex. The main difference is that we now allow ourselves only $O(n)$ vertices, which limits the possibilities. We still manage to construct a simplicial complex, $L = L(M)$, and a piecewise linear function, $g : |L| \to \mathbb{R}$, such that the first horizontal homology group of L and g is isomorphic to the null-space of M. However, L will not necessarily embed in $\mathbb{R}^4$.

Construction. We start by creating n square cycles, denoted as y_ℓ, one for each column of M. We assign the same function value, g_ℓ, to all four vertices of y_ℓ, making sure that different square cycles receive different function values. For each row k of M, we introduce the relation $\sum_{\ell=1}^{n} M(k,\ell)[y_\ell] = 0$ by adding some edges and triangles to the complex. We cannot afford adding a cap, as in the proof of Theorem 3.1, because this would require an additional number of vertices proportional to the number of non-zero entries in row k. Instead, we connect the square cycles by pairs of triangles, as illustrated in Figure 2. In particular, if the non-zero entries in row k belong to columns $\ell_1 < \ell_2 \ldots < \ell_p$, then we connect the right edge of y_{ℓ_j} with the left edge of $y_{\ell_{j+1}}$, for $j = 1, 2, \ldots, p-1$. We add the connecting pair of triangles unless the two square cycles are already so connected.

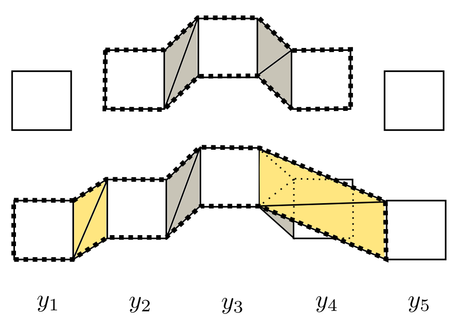


Figure 2: Top: a step in the construction connecting the middle three square cycles and representing $[y_2+y_3+y_4] = 0$ by adding the cone over the dashed cycle. Bottom: another step doing the same for $[y_1 + y_2 + y_3 + y_5] = 0$.

Finally, we consider the cycle, c_k, that goes around the sequence of connected squares (the ones corresponding to non-zero entries in row k) and added triangles, and we add a cone over c_k. Assuming row k has p non-zero entries, c_k has $4p$ vertices and $4p$ edges. The cone over c_k thus consists of $4p$ edges and $4p$ triangles, and it adds only one new vertex to the complex. We choose the function value of this vertex different from the function values of all previous vertices.

Analysis. We argue that L has the desired homology groups. Indeed, adding a pair of connecting triangles does not alter the first horizontal homology group. To see this, we add the three edges and the two triangles in sequence. The first edge does not affect the first horizontal homology group for the simple reason that it connects vertices with different function values, say $g_{\ell_j} < g_{\ell_{j+1}}$. The function values of the points on the edge thus cover the interval $[g_{\ell_j}, g_{\ell_{j+1}}]$, which has non-empty interior. We now add the other two edges and the two triangles using two anti-collapses, which preserve the homotopy type of the complex, and therefore also its homology groups.

The cone over c_k is single disk, and adding it to the complex does not affect the first homology other than by introducing the relation $[c_k] = 0$. We argue that c_k is homologous to $d_k = \sum_{\ell=1}^{n} M(k,\ell) y_\ell$. In other words, $c_k + d_k$ is a boundary. But this is clear because $c_k + d_k$ is the boundary of the sum of triangles connecting square cycles of non-zero entries in row k.

In summary, the first horizontal homology group of the final complex L is generated by the $[y_\ell]$, for $\ell = 1, 2, \ldots, n$, subject to the relation $My = 0$, where $y = ([y_1], [y_2], \ldots, [y_n])$. The first horizontal Betti number is therefore $\beta_1^{\text{hor}}(L) = n - \mathsf{rank}\, M = \mathsf{null}\, M$. In the case in which c_k is null-homologous before the cone is added, the addition of the cone creates a new 2-cycle. Hence, we also have $\beta_2(L) = \mathsf{null}\, M^t$, as required.

Complexity. The number of vertices in the complex $L = L(M)$ is $5n$, namely 4 for each column and 1 for each row. The number of edges is at most $4n + 7m$, and the number of triangles is less than $6m$. All these simplices can be constructed in $O(m)$ time, assuming again a representation that permits access to the non-zero entries of M in constant amortized time. This completes the proof of Theorem 3.2.

Remarks (2) and (3) given after the proof of Theorem 3.1 also apply here.

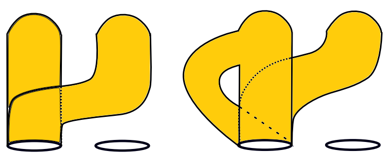
4 Extensions

In this section, we consider three extensions of our results: from simplicial to more general complexes, from $\mathbb{F}_2$ to more general finite fields, and from sparse matrices to matrices that have at most three non-zero entries per row and per column.

More general complexes. Given a matrix, M, with integer entries, there is a standard construction of a 2-dimensional CW complex whose second boundary matrix is M; see for example [11, Corollary 1.28]. To construct this complex, we start with a wedge of n oriented circles pinned together at a common point, which we denote as ω. We order the circles and write z_ℓ for the ℓ-th circle in this ordering. Each circle corresponds to a column of the matrix. Consider any loop that starts at ω and ends at ω. We can write this loop as $\sum_\ell a_\ell z_\ell$, in which a_ℓ is the number of times the path traverses z_ℓ (using the sign to distinguish traversals with or against the orientation). To complete the construction, it suffices to attach a disk by gluing its boundary to the loop corresponding to the row, for each row. It is not difficult to see that the first homology group of the final complex is generated by the circles z_ℓ subject to the relations given by the equations $Mz = 0$, in which $z = ([z_1], [z_2], \ldots, [z_n])$ is a column vector. In particular, the $\mathbb{F}_q$-Betti numbers of the complex give the nullity and hence the rank of the matrix M over $\mathbb{F}_q$, for q prime.

The above construction is related to computing the homology of a simplicial complex by simplification that merges simplices into larger and more complicated cells. The number of cells is reduced but at the cost of making the boundary matrix denser, albeit smaller in size. Eventually, we compute the rank of a smaller but denser matrix. This approach to homology computation is justified as long as the complexity of computing the rank of a sparse matrix is not known to be less than that of computing the rank of a dense matrix.

Finite fields. Our two theorems and the implied complexity bounds for computing Betti numbers extend from $\mathbb{F}_2$ to general finite fields. Assume we are given a matrix, M, with elements in $\mathbb{F}_q$; that is: integer numbers modulo q, with q a prime number. Let $m_k = \sum_\ell M(k,\ell)$ be the sum of entries in row k, where we take the sum in $\mathbb{Z}$ and not modulo q. We construct the simplicial complex, $K = K(M)$, as before but with an m_k-cap added for the k-th row such that the number of legs that are attached to the cycle x_ℓ is $M(k,\ell)$. The same proof then shows that after attaching the m_k-caps, we will have $\sum_\ell M(k,\ell)[x_\ell] = 0$ in homology with $\mathbb{F}_q$ coefficients. It follows that the first horizontal $\mathbb{F}_q$-Betti number of K is the nullity of M, and similarly for the second Betti number of K. In other words, we have a statement for $\mathbb{F}_q$ like that given in Theorem 3.1 for $\mathbb{F}_2$. It follows that the complexity of computing the Betti numbers over $\mathbb{F}_q$ is at least that of computing the rank of an n-by-n matrix with entries in $\mathbb{F}_q$ whose sum of entries is m.



$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}$$

Figure 3: Left: the complex constructed for the matrix on the right assuming the entries are in $\mathbb{F}_2$. Middle: the complex constructed for the same matrix but assuming the entries are in $\mathbb{F}_3$.

We illustrate the construction in Figure 3, which shows two complexes for the same matrix, the complex on the left for the 2-element field, and the complex in the middle for the 3-element field. For $\mathbb{F}_2$, we have $-1 = 1$ and a complex that consists of a 2-sphere and a circle. Its first horizontal and its second Betti numbers equal the nullity of the matrix, as claimed. For $\mathbb{F}_3$, we have $-1 = 2$ and a complex that consists of a Klein bottle with an additional disk attached and an isolated cycle. As before, the first horizontal and the second Betti numbers equal the nullity of the matrix.

Matrix sparsification. Let A be an n-by-n matrix with m non-zero entries. Following [16], we call an N-by-N matrix B with $N = O(m)$ and $O(m)$ non-zero entries a *sparsification* of A if we can compute the rank of A from that of B. If the reduction of A to B and the computation of the rank of A from the rank of B can both be done efficiently, say in $O(m)$ time or almost $O(m)$ time, then we say the rank computation problem has been reduced to a *sparse problem*. In [16], the sparse matrix has the property that each row and each column contains at most three non-zero entries. Here we show that the first construction in Section 3 can be used to obtain a similar sparsification of a 0-1 matrix.

We view our construction of a 2-dimensional simplicial complex given in the proof of Theorem 3.1 as a process that from M generates two matrices, namely the boundary matrices of K, such that rank of M can be computed from the ranks of these matrices. The first boundary matrix, D_1, relates edges with vertices and thus has two non-zero entries per column. The second boundary matrix, D_2, relates triangles with edges and has three non-zero entries per column. The rows of D_2 correspond to edges of the complex. Most of these edges belong to exactly two triangles. However, if an edge belongs to x_ℓ, the number of incident triangles equals the number of rows with non-zero entries in position ℓ, which can be more than two.

If we apply the construction again, to D_1 and to D_2, we obtain four matrices, D_{11}, D_{12}, D_{21} and D_{22}. Consider $K(D_1)$. In D_1, each column has exactly two non-zero entries, which implies that $K(D_1)$ is a 2-manifold without boundary of size $O(m)$. Hence its Betti numbers can be determined in $O(m)$ time using its Euler characteristic. This corresponds to the fact that the rank of D_1 can be computed easily since each column has only two non-zeros.

On the other hand, $K(D_2)$ is a complex which may not be a manifold. Therefore, the difficulty is in finding the Betti numbers of this complex. This we can do by computing the rank of the boundary matrices. The rank of D_{21} can again be computed efficiently. So we turn to D_{22}, whose columns and rows have at most three non-zero entries each. After computing the rank of D_{22} and an additional $O(m)$ work, we obtain the rank of the original matrix, M. Therefore, we have a sparsification of a 0-1 matrix that is a special case of Theorem 1.1 in [16]:

Theorem 4.1. *Let A be an n-by-n 0-1 matrix with m non-zero entries. Another 0-1 matrix B of size $O(m)$-by-$O(m)$ and at most three non-zero entries in each row and each column can be constructed in $O(m)$ time such that the rank of A can be computed from the rank of B in $O(m)$ time.*

Remark. Recall that the Betti numbers of a triangulated 2-manifold are easy to compute. Every edge in such a manifold belong to two triangles, and the same is true for most edges of the complex $K(D_2)$, except for the edges that belong to the cycles x_ℓ, which belong to three triangles each. While this may be a small deviation from being a manifold, computing the Betti numbers is already difficult, namely equivalent to computing the rank of a sparse matrix.

5 Discussion

In this paper, we explain how the computation of the rank of a matrix with entries from a finite field can be reduced to computing the Betti numbers of a simplicial complex. The first vertical Betti number for any generic piecewise linear function has a fast algorithm via the Reeb graph, but the computation of the first horizontal Betti number is at least as difficult as computing the rank. This splitting of homology might have additional applications.

The relations with matrix multiplication is of special interest. Since complexes provide geometric interpretations of algebraic problems in matrix multiplication, we hope that such reductions can give new insights into the exponent of matrix multiplication.

We close this paper with a question about the size of a 2-dimensional simplicial complex with n vertices that is embedded in $\mathbb{R}^4$; see [12, Section 5.1]. It is conjectured that such a complex has at most some constant times n^2 triangles, but currently no upper bound better than a constant times n^3 is known. A positive answer would imply that the complexity of computing the Betti numbers of simplicial complexes embedded in $\mathbb{R}^4$ is indeed lower than for simplicial complexes with the same number of vertices that are not necessarily embedded in $\mathbb{R}^4$.

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