The Morse Theory of Čech and Delaunay Filtrations*

Ulrich Bauer† Herbert Edelsbrunner†

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Abstract

Given a finite set of points in $\mathbb{R}^n$ and a positive radius, we study the Čech, Delaunay-Čech, alpha, and wrap complexes as examples of a generalized discrete Morse theory. We prove that the latter three complexes are simple-homotopy equivalent, and the same is true for their weighted versions. Our results have applications in topological data analysis and in the reconstruction of shapes from sampled data.

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†IST Austria (Institute of Science and Technology Austria). http://ulrich-bauer.org, http://ist.ac.at/~edels

1 Introduction

A fundamental task in topological data analysis is to turn the data into a topological space, or a filtration of spaces. Assuming the data consists of points in $\mathbb{R}^n$, we have several possible ways to construct a filtration using a non-negative scale parameter r. We can connect $p+1$ data points by the spanned p-simplex iff the closed balls of radius r centered at the points have a non-empty common intersection. This gives the *Čech complex* for radius r, which is known to have the same homotopy type as the union of balls [11, Chapter III]. We can intersect each ball with the Voronoi cell of its point and then take the nerve of these intersections. This gives the *alpha* or *Delaunay complex* for radius r, which embeds in $\mathbb{R}^n$ and also has the homotopy type of the union of balls [9]. Alternatively, we can select all simplices in the Čech complex that belong to the Delaunay triangulation. This gives the *Delaunay-Čech complex* for radius r, which also embeds in $\mathbb{R}^n$ but is known to collapse to the Delaunay complex only in $\mathbb{R}^2$ [6]. The results of this paper generalize this relation to arbitrary dimension.

Data analysis in low dimensions usually requires the reconstruction of the shape beyond the mere topological characterization. An example is the *wrap complex*, which is used commercially for this purpose [10]. We extend the original 3-dimensional notion to $\mathbb{R}^n$ and introduce a dependence on the radius parameter, showing that the wrap complex is simple-homotopy equivalent to the Delaunay and the Delaunay-Čech complexes. Among the four, the Delaunay complex has been defined for points with weights. We generalize the other three complexes to the weighted case in a way that preserve the homotopy and simple-homotopy equivalences between them.

The main technical ingredients to proving the relations between the complexes are generalized discrete Morse functions; see [14] for an introduction to discrete Morse theory and [15] for the generalization that allows for intervals larger than pairs in the discrete vector field. Our constructions are elementary, using the radii of smallest empty circumspheres and smallest enclosing spheres to define the generalized discrete Morse functions, which are then used to prove the collapsibility results.

Outline. Section 2 presents the background in combinatorial topology and discrete Morse theory. Section 3 introduces the radius functions and proves technical lemmas relating the corresponding gradients. Section 4 introduces the filtration of wrap complexes and proves the collapsing hierarchy in the unweighted case. Section 5 concludes the paper. Appendix A extends all results to the weighted case.

2 Background

All complexes in this paper are simplicial, either concrete geometric or abstract. After presenting the general background on simplicial complexes, we introduce three of the four types of complexes studied in this paper, and we give quick reviews of discrete Morse theory and its generalization.

Simplicial complexes. We recall that a *p-simplex* is the convex hull of $p+1$ affinely independent points in Euclidean space. The *faces* of the p-simplex are the simplices defined by subsets of the $p+1$ points. A *simplicial complex* is a collection of simplices that is closed under the face relation and has the property that the intersection of any two simplices in the complex is either empty or a common face. In this paper, we only consider finite simplicial complexes. The maximum number of affinely independent points in $\mathbb{R}^n$ is $n+1$. If we want simplices spanned by more than this number of points, we need to define them abstractly. Given a finite collection, an *abstract simplicial complex* over this collection is a system of subcollections that is closed under the subset relation. Each collection in this system is referred to as an *abstract simplex* whose *dimension* is its cardinality minus 1. For example, the collections of vertices of the simplices in a geometric simplicial complex form an abstract simplicial complex. Another example is the *nerve* of a collection of sets, which is the system of subcollections whose sets have a non-empty common intersection. We can *geometrically realize* an abstract simplicial complex by mapping the initial collection to points, but we may have to go to high dimensions to avoid improper intersections among the simplices. If the maximum dimension of any abstract simplex in the complex is m, then $2m + 1$ dimensions suffice for a geometric realization [11, Chapter III].

To apply standard concepts from point set topology, we associate with a (geometric) simplicial complex K in $\mathbb{R}^n$ its *underlying space*, $|K|$, which is the union of the simplices of K endowed with the subspace topology inherited from $\mathbb{R}^n$. Two topological spaces $\mathbb{X}$ and $\mathbb{Y}$ are *homeomorphic* if there is a continuous bijection $h\colon \mathbb{X} \to \mathbb{Y}$ whose inverse is also continuous. A weaker but often more useful notion is the following. The two spaces have the same *homotopy type* if there are continuous maps $f\colon \mathbb{X} \to \mathbb{Y}$ and $g\colon \mathbb{Y} \to \mathbb{X}$ such that $g \circ f$ is homotopic to the identity on $\mathbb{X}$ and $f \circ g$ is homotopic to the identity of $\mathbb{Y}$. Here, two maps are said to be *homotopic* if there is a continuous deformation from one map to the other. To apply this concept in the abstract setting, we first construct geometric realizations of all abstract simplicial complexes, which is done without mention. A useful tool in this context is the Nerve Theorem, which asserts that under certain conditions, the nerve of a collection of sets has the same homotopy type as the union of the sets [2, 19]. It applies, for example, if the collection is finite and the sets are convex.

Proximity complexes. Let X be a finite set of points in $\mathbb{R}^n$. For $r \geq 0$, let $B_r(x) = \{y \in \mathbb{R}^n \mid \|x - y\| \leq r\}$ be the closed ball of radius r centered at $x \in X$. The *Čech complex* for radius r,

$$C_r(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} B_r(x) \neq \emptyset\},$$

is isomorphic to the nerve of the collection of closed balls. For $r \leq s$, we have $C_r(X) \subseteq C_s(X)$. For sufficiently large radius, the Čech complex is the full (abstract) simplex spanned by X, which we denote as $\Delta(X) = 2^X \setminus \{\emptyset\}$. Using the Euclidean metric in $\mathbb{R}^n$, we define the *Voronoi cell* of $x \in X$ as the set of points $a \in \mathbb{R}^n$ such that $\|x - a\| \leq \|y - a\|$ for all $y \in X$. Every Voronoi cell is a convex polyhedron, any two such polyhedra intersect at most along shared boundaries, and together the Voronoi cells cover $\mathbb{R}^n$. Letting $\mathrm{Vor}\,(x)$ be the Voronoi cell of $x \in X$, we write $\mathrm{Vor}\,(Q) = \bigcap_{x \in Q} \mathrm{Vor}\,(x)$ for the common intersection of Voronoi cells, for every $Q \subseteq X$. The *Delaunay triangulation*, denoted by $D(X)$, is isomorphic to the nerve of the collection of Voronoi cells: it consists of all simplices $Q \subseteq X$ with $\mathrm{Vor}\,(Q) \neq \emptyset$. A mild general position assumption is required to ensure that taking the convex hulls of the abstract simplices in $D(X)$ gives a simplicial complex in $\mathbb{R}^n$.

There are two natural ways to combine the two constructions. For the first, we intersect $B_r(x)$ with $\mathrm{Vor}\,(x)$ and we construct a complex isomorphic to the nerve of the collection of such sets, and for the second, we restrict the Čech complex to the Delaunay triangulation:

$$D_r(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} [B_r(x) \cap \mathrm{Vor}\,(x)] \neq \emptyset\},$$
$$DC_r(X) = \{Q \in D(X) \mid \bigcap_{x \in Q} B_r(x) \neq \emptyset\}.$$

In the literature, $D_r(X)$ is known as the *alpha complex*, for $\alpha = r$; see e.g. [12]. We prefer to call it the *Delaunay complex* for radius r, because this naming convention emphasizes the fact that the Delaunay complexes provide the canonical filtration of the Delaunay triangulation, as explained in Section 3. The second complex, $DC_r(X)$, is referred to as the *Delaunay-Čech complex* for radius r. For growing radius, both complexes give rise to a filtration of the Delaunay triangulation. While the two complexes are similar, they are not necessarily the same; see Figure 1. Instead of equality, we have $D_r(X) \subseteq DC_r(X)$, for all r. To see this, we just note that if the sets $B_r(x) \cap \mathrm{Vor}\,(x)$ have a non-empty common intersection, then the sets $B_r(x)$ have a non-empty common intersection and so do the sets $\mathrm{Vor}\,(x)$.

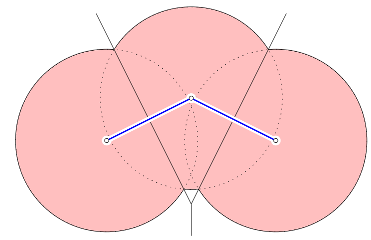

Figure 1: The Delaunay complex for the given radius has three vertices and two edges. In contrast, the Delaunay-Čech complex (not shown) has three vertices, three edges, and a triangle.

Discrete Morse theory. Let K be a simplicial complex, geometric or abstract. Note that the face relation defines a canonical partial order on K. We call a simplex P a *facet* of another simplex Q iff P is a face of Q and $\dim P = \dim Q - 1$, and we call (P, Q) a facet pair. The *Hasse diagram* of K is the directed graph whose nodes are the simplices and whose arcs are the facet pairs. It is the transitive reduction of the face relation. A *discrete vector field* is a partition V of the faces of K into singleton sets and facet pairs.

The Hasse diagram of K induces a digraph on V, denoted as $\mathcal{H}(V)$, obtained by contracting each facet pair in V. Specifically, the nodes of $\mathcal{H}(V)$ are the singletons and pairs in V, and there is an arc from μ to ν if there are simplices $P \in \mu$ and $Q \in \nu$ such that (P, Q) is a facet pair. The discrete vector field is *acyclic*, or a *discrete gradient field*, if $\mathcal{H}(V)$ is acyclic. In this case, the transitive closure of $\mathcal{H}(V)$ is a partial order on V, and we call $\mathcal{H}(V)$ the *Hasse diagram* of V. To prove acyclicity, we may construct a function $f\colon K \to \mathbb{R}$ that satisfies $f(P) = f(Q)$ whenever (P, Q) is a facet pair in V, and $f(P) < f(Q)$ whenever (P, Q) is a facet pair not in V. We call such an f a *discrete Morse function* that has V as its gradient. A simplex that does not belong to a pair in V is a *critical simplex*, and the corresponding value is a *critical value* of f.

To provide an intuition for the concept, we note that the pairs in a discrete gradient field correspond to elementary collapses (see e.g. [11, Chapter III]), except that the lower-dimensional simplex does not have to be free. An elementary collapse can be realized continuously by a deformation retraction. This implies that if we can transform a simplicial complex K to another simplicial complex K_0 using a sequence of such collapses, denoted by $K \searrow K_0$, then the two complexes have the same homotopy type. Indeed, the relation is slightly stronger, which is usually expressed by saying that K and K_0 are *simple-homotopy equivalent* [7]. A discrete Morse function can encode a simplicial collapse [13]:

Theorem 2.1 (Collapsing Theorem)**.** *If $f : K \to \mathbb{R}$ is a discrete Morse function and all critical simplices of f are contained in a subcomplex $K_t = f^{-1}(-\infty, t]$, then $K \searrow K_t$.*

Generalized discrete Morse theory. To generalize the concepts of discrete Morse theory, we recall that an *interval* of the poset K, with the partial order given by the face relation, is a subset of the form

$$[P, R] = \{Q \mid P \subseteq Q \subseteq R\}.$$

The interval is non-empty iff P is a face of Q. In this case, the interval contains both simplices, and we refer to P as the *lower bound* and to R as the *upper bound* of the interval. Next, consider a partition of K into intervals. Borrowing from the nomenclature of [15], we call this set of intervals a *generalized discrete vector field*, denoting it by V. Indeed, a discrete vector field is the special case in which all intervals are either singletons or pairs. The corresponding directed graph, denoted again as $\mathcal{H}(V)$, has the intervals as its nodes and an arc from μ to ν if there are simplices $P \in \mu$ and $Q \in \nu$ such that (P, Q) is a facet pair in K. The generalized discrete vector field is *acyclic* if $\mathcal{H}(V)$ is acyclic, and if it is, then we call V a *generalized discrete gradient field* and $\mathcal{H}(V)$ the *Hasse diagram* of V. As before, acyclicity of V is asserted by the existence of a function $f\colon K \to \mathbb{R}$ that satisfies $f(P) = f(Q)$ whenever P is a face of Q and both belong to the same interval, while $f(P) < f(Q)$ whenever P is a face of Q but they belong to different intervals. We call such an f a *generalized discrete Morse function* that has V as its (generalized) gradient. We call an interval of cardinality 1 *singular* and the simplex it contains *critical*. Correspondingly, the value of a critical simplex is a *critical value* of f.

It is easy to see that for every generalized discrete gradient field, there is a discrete gradient field that refines the non-singular intervals into pairs. Both have the same critical simplices, implying that the Collapsing Theorem (2.1) also applies to generalized Morse functions. The refinement is in general not unique.

3 Radius Functions

The proximity complexes introduced in Section 2 have equivalent dual definitions, which are more convenient for our purposes. We begin by introducing these dual definitions, together with a precise statement of the general position assumption we use throughout this paper.

Čech and Delaunay functions. A *circumsphere* of a finite set $Q \subseteq \mathbb{R}^n$ is an $(n-1)$-sphere that passes through all points of Q. A sufficient condition for the existence of a circumsphere is that Q is affinely independent. For $p < n$, the circumsphere is not unique, but there is only one *smallest circumsphere*, namely the one whose center lies in the affine hull of Q. We formulate conditions under which the smallest circumspheres do not contain more than the necessary points.

Definition 3.1 (General position)**.** A finite set $X \subseteq \mathbb{R}^n$ is in *general position* if for every subset Q of at most $n+1$ points in X,

(i) Q is affinely independent, and

(ii) the smallest circumsphere of Q does not pass through any points of $X \setminus Q$.

From now on, let X be a finite set of points in general position in $\mathbb{R}^n$. A circumsphere of a subset Q is *empty* if the open ball bounded by the sphere contains no point of X. It is not difficult to see that the Delaunay triangulation, as defined in Section 2, consists of all simplices $Q \subseteq X$ that have empty circumspheres; see also [8]. By Condition (i) of our general position assumption, all sets Q that have empty circumspheres are affinely independent, which suffices for $D(X)$ to be geometrically realizable in $\mathbb{R}^n$.

To make the step from the Delaunay triangulation to the Delaunay complexes for a radius, we note that among the empty circumspheres of a set $Q \in D(X)$, there is a unique one with smallest radius. We denote this *smallest empty circumsphere* by $S_D(Q)$. In addition, we introduce the *Delaunay function*, $f_D \colon D(X) \to \mathbb{R}$, defined by mapping Q to the squared radius of $S_D(Q)$. We will see shortly that f_D can be used to characterize Delaunay complexes.

Before we get there, we note that an *enclosing sphere* of $Q \subseteq \mathbb{R}^n$ is an $(n-1)$-sphere S such that the closed ball $\operatorname{conv} S$ contains all points of Q. Among all enclosing spheres, there is a unique one with smallest radius. We denote this *smallest enclosing sphere* by $S_C(Q)$, and we introduce the *Čech function*, $f_C \colon \Delta(X) \to \mathbb{R}$, defined by mapping Q to the squared radius of $S_C(Q)$. With these two functions, it is easy to say which simplices belong to the complexes.

Lemma 3.2 (Radius Functions Lemma)**.** *Let X be a finite set of points in general position in $\mathbb{R}^n$. Then*

$$C_r(X) = f_C^{-1}[0, r^2], \tag{1}$$

$$D_r(X) = f_D^{-1}[0, r^2], \tag{2}$$

for every $r \geq 0$.

Proof. A subset Q of X belongs to the Čech complex for radius r iff the closed balls with radius r centered at the points in Q have a non-empty common intersection. The points in this intersection are precisely the centers of all enclosing spheres of radius r. This implies (1).

The subset Q belongs to the Delaunay complex for radius r iff the Voronoi cells of the points in Q have a common intersection that includes points at a distance at most r from the points in Q. These points are precisely the centers of all empty circumspheres of radius at most r. This implies (2). □

It is easy to see that $f_C(Q) \leq f_D(Q)$ for every $Q \in D(X)$. This gives an alternative proof for the fact that $D_r(X) \subseteq DC_r(X) \subseteq C_r(X)$ for every $r \geq 0$.

Čech intervals. We will prove collapsibility using structural properties of the two radius functions. We begin with the easier case of the Čech function. Given a simplex $Q \subseteq X$, recall that $S_C(Q)$ is the smallest enclosing sphere of Q. We find two simplices bounding an interval that consists of the simplices with the same smallest enclosing sphere as Q:

$$L_C(Q) = Q \cap S_C(Q);$$
$$U_C(Q) = X \cap \operatorname{conv} S_C(Q).$$

Note that $L_C(Q)$ is the unique simplex such that $S_C(Q)$ is the smallest circumsphere of $L_C(Q)$. Write V_C for the collection of such intervals $[L_C(Q), U_C(Q)]$, over all $Q \subseteq X$. We show that V_C is a generalized discrete gradient field.

Lemma 3.3 (Čech Function Lemma)**.** *Let X be a finite set of points in general position in $\mathbb{R}^n$. Then $f_C\colon \Delta(X) \to \mathbb{R}$ is a generalized discrete Morse function with gradient V_C.*

Proof. Let $Q \subseteq X$ and write $L = L_C(Q)$, $U = U_C(Q)$, and $S = S_C(Q)$. Clearly $L \subseteq Q \subseteq U$. Furthermore, every other simplex Q' with $L \subseteq Q' \subseteq U$ defines the same smallest enclosing sphere and therefore also the same simplices $L_C(Q') = L$ and $U_C(Q') = U$. Hence, the interval $[L, U]$ consists exactly of the simplices Q' with the same smallest enclosing sphere $S_C(Q') = S$.

Note that V_C is a generalized discrete vector field: its intervals partition $\Delta(X)$ since $S_C(Q') = S_C(Q)$ whenever $L_C(Q) \subseteq Q' \subseteq U_C(Q)$. It remains to show that V_C is a generalized gradient. But this is clear because V_C is defined by f_C, which is non-decreasing along increasing chains of the face relation, and it stays constant only within the intervals of V_C. □

We recall that our general position assumption implies that a simplex $Q \subseteq X$ has a smallest circumsphere iff $\dim Q \leq n$, and that the smallest circumspheres of different simplices are different. We thus have $\sum_{p=0}^{n} \binom{\dim X+1}{p+1}$ smallest circumspheres and the same number of intervals. Very few of them are singular, namely only the ones for which $L_C(Q) = U_C(Q)$. In other words, the critical simplices are the $Q \subseteq X$ such that the smallest enclosing sphere of Q is also an empty circumsphere. All these simplices also belong to the Delaunay triangulation.

Delaunay intervals. Given a simplex $Q \in D(X)$, we now find two simplices that enclose between them the simplices with the same smallest empty circumsphere. To motivate the definition, we first investigate the intersections of simplices having a common smallest empty circumsphere:

Lemma 3.4 (Delaunay Face Lemma)**.** *Let X be a finite set of points in general position in $\mathbb{R}^n$. If $Q, Q' \subseteq X$ have the same smallest empty circumsphere, then this is also the smallest empty circumsphere of $Q \cap Q'$.*

Proof. Setting $P = Q \cap Q'$, we note that $\operatorname{Vor}(P)$ contains both $\operatorname{Vor}(Q)$ and $\operatorname{Vor}(Q')$ as faces. Indeed, $\operatorname{Vor}(P)$ is the smallest common intersection of Voronoi cells with this property. The center of the common smallest empty circumsphere satisfies $z \in \operatorname{Vor}(Q) \cap \operatorname{Vor}(Q')$ and therefore $z \in \operatorname{Vor}(P)$. Let now x be a point in P. Among the points in $\operatorname{Vor}(Q)$, z minimizes the distance to every point in Q and therefore also to x. Equivalently, the $(n-1)$-plane H normal to $x - z$ that passes through z *weakly separates* x from $\operatorname{Vor}(Q)$, i.e.,

z is contained in one closed half-space bounded by H, and $\mathrm{Vor}(Q)$ is contained in the other; see [3, Section 4.2.3]. Similarly, H weakly separates x from $\mathrm{Vor}(Q')$. Since $\mathrm{Vor}(P)$ is the smallest common intersection of Voronoi cells that contains $\mathrm{Vor}(Q)$ and $\mathrm{Vor}(Q')$ as faces, this implies that H also weakly separates z from $\mathrm{Vor}(P)$, and hence, z minimizes the distance to x among all points in $\mathrm{Vor}(P)$. We conclude that P has the same smallest empty circumsphere as Q and Q'. □

Writing $S_D(Q)$ for the smallest empty circumsphere of Q, the lower bound, $L_D(Q)$, is the intersection of all faces of Q whose smallest empty circumsphere is $S_D(Q)$, and the upper bound, $U_D(Q)$, consists of all points on $S_D(Q)$:

$$L_D(Q) = \bigcap_{S_D(Q')=S_D(Q)} Q';$$
$$U_D(Q) = X \cap S_D(Q').$$

Note that $U_D(Q)$ is the unique simplex such that $S_D(Q)$ is the smallest circumsphere of $U_D(Q)$. Write V_D for the collection of intervals $[L_D(Q), U_D(Q)]$, over all simplices $Q \in D(X)$. Using the Delaunay Face Lemma (3.4), we show that V_D is a generalized discrete gradient field.

Lemma 3.5 (Delaunay Function Lemma)**.** *Let X be a finite set of points in general position in $\mathbb{R}^n$. Then $f_D\colon D(X) \to \mathbb{R}$ is a generalized discrete Morse function with gradient V_D.*

Proof. Given $Q \in D(X)$, write $P = L_D(Q)$, $R = U_D(Q)$, and $S = S_D(Q)$. We first show that $S_D(Q') = S$ iff $P \subseteq Q' \subseteq R$. By definition, R contains all points of X on S, so $S_D(Q') = S$ implies $Q' \subseteq R$. Hence, it remains to show that S is the smallest empty circumsphere of $Q' \subseteq R$ iff $P \subseteq Q'$. If $P \nsubseteq Q'$ then, by definition of P, the simplex Q' has an empty circumsphere that is smaller than S. On the other hand, if $P \subseteq Q'$, then Q' does not have an empty circumsphere smaller than S, since any empty circumsphere of Q' is also an empty circumsphere of $P \subseteq Q'$, and S is the smallest empty circumsphere of P according to the Delaunay Face Lemma (3.4).

The intervals of V_D are of the form $[L_D(Q), U_D(Q)]$, over all $Q \in D(X)$. These intervals partition the Delaunay triangulation. Finally, f_D is non-decreasing along increasing chains of the face relation, and stays constant only within the constructed intervals, so V_D is the gradient of f_D. □

We have $L_D(Q) = U_D(Q)$ iff the center of $S_D(Q)$ is contained in $\mathrm{conv}\, U_D(Q)$, and if it is, then our general position assumption implies that it is an interior point of the simplex. This case is noteworthy since it implies that every singular interval of f_D is also a singular interval of f_C. Conversely, every singular interval of f_C is a singular interval of f_D. In other words, f_C and f_D have the same critical simplices. We summarize:

Lemma 3.6 (Critical Value Lemma)**.** *A simplex Q is a critical simplex of V_D iff it is a critical simplex of V_C iff $f_D(Q) = f_C(Q)$.*

Radon's theorem. We prepare the analysis of the relation between Čech and Delaunay intervals by discussing a variant of Radon's theorem. In the original form, it asserts that any $p+1$ points in $\mathbb{R}^{p-1}$ can be partitioned into two non-empty sets whose convex hulls have a non-empty common intersection [20]. We are interested in a version of this theorem in which the $p+1$ points are the vertices of a p-simplex. Writing $R = \{x_0, x_1, \ldots, x_p\}$, we express the center of the smallest circumsphere as an affine combination:

$$z = \zeta_0 x_0 + \zeta_1 x_1 + \ldots + \zeta_p x_p.$$

The coefficients satisfy $\zeta_0 + \zeta_1 + \ldots + \zeta_p = 1$, but they are not necessarily all non-negative because z is not necessarily contained in $\mathrm{conv}\, R$. The p-simplex has $p+1$ *facets*, namely $R_i = R \setminus \{x_i\}$ for $0 \le i \le p$. We call R_i a *front facet* if $\zeta_i < 0$ and a *back facet* if $\zeta_i > 0$. Assuming general position, z does not lie in the $(p-1)$-plane of any facet, so every facet is either front or back. It is convenient to re-index such that R_0 to R_{j-1} are front facets and R_j to R_p are back facets. We call

$$F = \{x_j, x_{j+1}, \ldots, x_p\},$$
$$B = \{x_0, x_1, \ldots, x_{j-1}\}.$$

the *smallest front face* and the *smallest back face* of R, noting that F is either R or the common intersection of all front facets, and B is the possibly empty common intersection of all back facets. Observe that each front facet has the same smallest empty circumsphere as R, while every back facet has a smaller such sphere. If $R = U_D(Q)$ is an upper bound of a Delaunay interval, the Delaunay Face Lemma (3.4) asserts that $F = L_D(R)$ is the smallest face that shares the smallest empty circumsphere with R. The new version of Radon's theorem is an assertion about the position of the circumcenter relative to these two faces.

Lemma 3.7 (Front-Back Separation Lemma)**.** *Let R be a set of $p+1$ points in general position such that the center z of the smallest circumsphere lies outside* conv R*. Any $(p-1)$-plane that weakly separates the smallest front face from the smallest back face also weakly separates z from the smallest back face.*

Proof. Assume without loss of generality that the points of R lie in $\mathbb{R}^p$ and the center of the smallest circumsphere lies at the origin. Writing this center as the affine combination of the $p+1$ vertices of R, we get $0 = \zeta_0 x_0 + \zeta_1 x_1 + \ldots + \zeta_p x_p$. By assumption of general position, the origin does not belong to the $(p-1)$-plane defined by any facet, which implies that all coefficients are non-zero. We re-index such that ζ_0 to ζ_{j-1} are negative and ζ_j to ζ_p are positive, define

$$y'_F = \zeta_j x_j + \zeta_{j+1} x_{j+1} + \ldots + \zeta_p x_p,$$
$$y'_B = \zeta_0 x_0 + \zeta_1 x_1 + \ldots + \zeta_{j-1} x_{j-1},$$

and observe that $y'_F = -y'_B$. Scaling the two points to $y_F = y'_F/(\zeta_j + \zeta_{j+1} + \ldots + \zeta_p)$ and $y_B = y'_B/(\zeta_0 + \zeta_1 + \ldots + \zeta_{j-1})$, we note that they are convex combinations of F and B, and that the two points are still collinear with the circumcenter at the origin; see Figure 2. By construction, the order of the points along the common

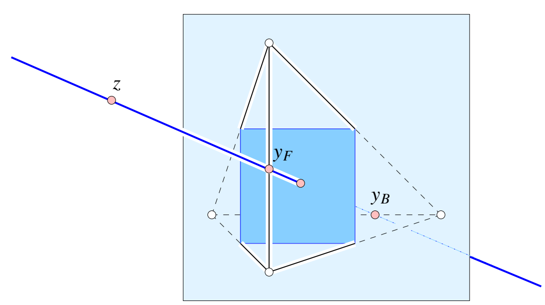

Figure 2: The line that passes through the circumcenter and intersects the smallest front as well as the smallest back face, and a plane that separates these two faces.

line is such that y_F lies between z and y_B. Any $(p-1)$-plane that weakly separates F from B intersects this line in a point between y_F and y_B. It then follows that this $(p-1)$-plane also weakly separates z from y_B, as claimed. □

To elucidate the connection to the Radon Theorem, we note that the central projections of F and of B from the circumcenter to a suitable $(p-1)$-plane give the Radon partition of the projection of R.

Interval intersection. By the Critical Value Lemma (3.6), the Čech and the Delaunay functions have the same critical simplices, which form singular intervals. All other intervals are non-singular. While Delaunay

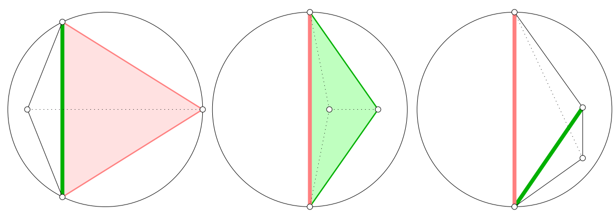

Figure 3: Three examples of a Delaunay tetrahedron within its smallest enclosing sphere, viewed from the center of the circumsphere. The circle is the intersection of the two spheres. From left to right: the lower bound of the Čech interval is a (pink) triangle, edge, edge, and the lower bound of the Delaunay interval is a (green) edge, triangle, edge.

intervals tend to be smaller than Čech intervals, there is no particular relation between them. Consider for example $L = L_C(Q)$ and $P = L_D(Q)$, for some $Q \in D(X)$. As illustrated in Figure 3, it is possible that P is a proper subset of L, that L is a proper subset of P, and that L and P are incomparable. Nevertheless, we have one important relationship, namely that the intersection of two non-singular intervals is non-singular.

Lemma 3.8 (Excluded Singularity Lemma)**.** *Let X be a finite set of points in general position in $\mathbb{R}^n$. The intersection of a non-singular Čech interval and a non-singular Delaunay interval is a possibly empty non-singular interval.*

Proof. Writing $[L, U]$ for the Čech interval and $[P, R]$ for the Delaunay interval, we assume that their intersection is non-empty. This common intersection is necessarily an interval, with lower bound $L \cup P$ and upper bound $U \cap R$. Setting $Q = L \cup P$, we aim to show that there exists a point $x \in U \cap R$ that does not belong to Q. It then follows that $Q \cup \{x\}$ is a second simplex in the intersection, implying that the intersection is not singular, as desired.

To establish the existence of such a point x, we set $S_0 = S_C(Q)$ and $S_1 = S_D(Q)$. Let H be the $(n-1)$-plane that contains $S_0 \cap S_1$, which is an $(n-2)$-sphere. We orient H such that S_1 encloses S_0 in the open half-space *in front* of H, and S_0 encloses S_1 in the open half-space *behind* H; see Figure 4. The center z_0

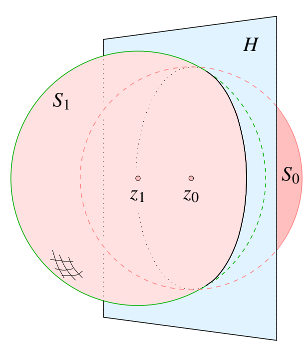

Figure 4: The plane contains the center of the smallest enclosing sphere and the circle in which this sphere intersects the smallest empty circumsphere.

of S_0 is interior to conv L, and L is a face of R contained in H. Because R is non-critical, by assumption, the center z_1 of S_1 lies outside conv R and in front of H. All points of R lie on S_1, and the points of $Q \subseteq R$ lie on or behind H. By definition of the Delaunay lower bound, $P = F$ is the smallest front face of R. Recall that $B = R \setminus F$ is the smallest back face of R. If $Q = R$, then $Q = F \cup L$ and $R = F \cup B$ imply $B \subseteq L$. Hence, B lies on H, $F \subseteq Q$ lies on or behind H, and H weakly separates F from B. By the Front-Back Separation Lemma (3.7), F lies on or in front of H, implying that all points of R lie on H and therefore on $S_0 \cap S_1$. In particular, S_0 is a circumsphere of R. But S_0 has a smaller radius than S_1, contradicting the fact that S_1 is the smallest circumsphere of R. We conclude that Q does not contain all points of R.

We aim at proving that at least one of the points in $R \setminus Q$ lies behind H. If so, then this point x is enclosed by S_0 and therefore $x \in U \cap R$, as desired. To derive a contradiction, we assume that all points of $R \setminus Q$ lie in front of H. All points of L lie on H, so all points of B lie on or in front of H. Hence, H weakly separates F from B. But both $R \setminus Q \subseteq B$ and z_1 are in front of H, so H does not weakly separate z_1 from B, which contradicts the Front-Back Separation Lemma (3.7). □

4 Collapsing Hierarchy

In this section, we prove our main result, which is the collapsibility of the Delaunay-Čech to the Delaunay complex, and of the Delaunay complex to the wrap complex. We begin with a sufficient condition for collapsibility.

Compatible restrictions. We prove our collapsibility results using a structural insight into gradients. Let V be a generalized discrete gradient field on a simplicial complex K, let $K_0 \subseteq K$ be a subcomplex, and let V_0 be the restriction of V to K_0. We say that V *restricts compatibly* to K_0 if V_0 is again a generalized discrete gradient field, and a singular interval belongs to V iff it belongs to V_0.

Lemma 4.1 (Compatible Restriction Lemma)**.** *Let $K_0 \subseteq K$ be a pair of simplicial complexes. If there exists a generalized gradient V on K that restricts compatibly to K_0, then $K \searrow K_0$.*

Proof. We prove the claim by refining V to a (non-generalized) discrete gradient field W on K that restricts compatibly to K_0. Let $[L, U]$ be a non-singular interval in V. Its intersection with K_0 is either empty or a non-singular interval $[L, T]$ in K_0. In the first case, we choose an arbitrary vertex $x \in U \setminus L$ and decompose $[L, U]$ into pairs $[Q, Q \cup \{x\}]$ for all $x \notin Q \in [L, U]$. The second case is similar, except we make sure to choose $x \in T \setminus L$. Clearly, W is a discrete vector field on K. We note that $Q \in K_0$ iff $R \in K_0$ for every pair $[Q, R] \in W$. Since V restricts compatibly to K_0, this implies that W restricts compatibly to K_0. To prove that W is acyclic, we construct a discrete Morse function, $g\colon K \to \mathbb{R}$, with gradient W from a generalized discrete Morse function, $f\colon K \to \mathbb{R}$, with generalized gradient V. Specifically, for each singular $[Q, Q] \in W$, we set $g(Q) = f(Q)$, and for each pair $[Q, R] \in W$ contained in $[L, U] \in V$, we set

$$g(Q) = g(R) = \begin{cases} f(L) + \varepsilon \dim R & \text{if } Q, R \in K_0, \\ f(L) + \varepsilon \dim R + \lambda & \text{if } Q, R \notin K_0, \end{cases}$$

where $\varepsilon > 0$ is less than the smallest absolute difference between different values of f divided by the maximum dimension of any simplex, and λ is greater than the largest difference between values of f. It is straightforward to verify that g is a discrete Morse function with gradient W. Finally, to prove that K collapses to K_0, we let $m = \max_{Q \in K} f(Q)$ and $m_0 = \max_{Q \in K_0} f(Q)$ so that $K = g^{-1}(-\infty, m]$ and $K_0 = g^{-1}(-\infty, m_0]$. Moreover, since all critical simplices of V belong to K_0, no critical value of g is larger than m. Now the Collapsing Theorem (2.1) yields $K \searrow K_0$. ◻

Wrap complex. Recall that the Delaunay function defines a generalized discrete gradient field, represented by the Hasse diagram, $\mathcal{H}(V_D)$, whose nodes are the Delaunay intervals. Every critical simplex forms a singular interval $[Q, Q]$ and therefore its own node in $\mathcal{H}(V_D)$. The *lower set* of a critical simplex $Q \in D(X)$, denoted by $\downarrow Q$, is the collection of simplices contained in intervals from which this node can be reached along directed paths. This concept is akin to the stable manifold of a critical point in smooth Morse theory, except that the lower sets of the critical simplices do not necessarily form a partition. Indeed, the lower sets can overlap, and some of the simplices may not belong to any lower set. The latter can be considered to belong to the lower set of the "outside", but it will not be necessary to formalize this intuition. The *wrap complex* for $r \geq 0$ is the union of the lower sets of all critical simplices Q with smallest empty circumsphere of radius at most r:

$$W_r(X) = \bigcup_{\substack{[Q,Q] \in V_D \\ f_D(Q) \leq r^2}} \downarrow Q.$$

Clearly, $W_r(X) \subseteq W_s(X)$ whenever $r \leq s$. Alternatively, we can define the wrap complexes as sublevel sets of another function, namely of $f_W\colon D(X) \to \overline{\mathbb{R}}$ defined by mapping $P \in D(X)$ to the minimum $f_D(Q)$ of any critical simplex Q for which $P \in \downarrow Q$, and to ∞ if P does not belong to the lower set of any critical simplex. Note that $f_D(P) \leq f_W(P)$ for every P, which implies $W_r(X) \subseteq D_r(X)$ for every $r \geq 0$.

Collapses. We are ready to state and prove the main result of this paper.

Theorem 4.2 (Main Theorem)**.** *Let X be a finite set of points in general position in $\mathbb{R}^n$. Then*

$$DC_r(X) \searrow D_r(X) \searrow W_r(X),$$

for every $r \geq 0$.

Proof. We use the Compatible Restriction Lemma (4.1) for both collapsibility results. Consider first the case in which $K = DC_r(X)$ and $K_0 = D_r(X)$. The generalized discrete gradient field is obtained by intersecting the Delaunay intervals with the Čech intervals:

$$V = \{[L_C(Q), U_C(Q)] \cap [L_D(Q), U_D(Q)] \mid Q \in D(X)\}.$$

To see that V is indeed acyclic, we consider $f \colon D(X) \to \mathbb{R}$ defined by $f(Q) = \frac{1}{2}(f_C(Q) + f_D(Q))$. It is not difficult to see that f has the described partition V of the Delaunay triangulation as its gradient. Each interval of V is either disjoint of K or contained in K, which implies that the restriction of V to K is the gradient of the restriction of f to K. The Critical Value Lemma (3.6) implies that all critical simplices of K are contained in K_0. Together with the Excluded Singularity Lemma (3.8), this implies that the restriction of V to K restricts compatibly to K_0. Thus, by the Compatible Restriction Lemma (4.1), we have $DC_r(X) \searrow D_r(X)$.

Consider second the case in which $K = D_r(X)$ and $K_0 = W_r(X)$. Here we use the Delaunay intervals V_D. Each Delaunay interval is either disjoint of K or contained in K, and similar for K_0. Moreover, by construction, a critical simplex belongs to K iff it belongs to K_0. Hence, the generalized discrete gradient field compatibly restricts to K_0, and the Compatible Restriction Lemma (4.1) implies $D_r(X) \searrow W_r(X)$. □

5 Discussion

The two main contributions to the state of the art are new insights into the relationship between the Čech and the Delaunay complex – two constructions heavily used in topological data analysis – and a generalization of the wrap algorithm – a methodology for surface reconstruction used also in industry. In particular, we show that the restriction of the Čech complex for radius r to the Delaunay triangulation collapses to the Delaunay or alpha complex of the same radius. Furthermore, this Delaunay complex collapses to the wrap complex for the same radius. We leave the question whether or not the Čech complex itself collapses to the Delaunay-Čech complex as an open question. The wrap algorithm is generalized in three ways:

- it depends on a parameter that selects the critical simplices contributing to the construction;
- it allows for weights that can be used to tune the importance of points;
- it works in Euclidean space of any fixed dimension.

The *flow complex* introduced in [17] is conceptually similar to the wrap complex. Being based on the gradient flow of the distance function to a point set, this construction is however less combinatorial. It has been shown to have the same homotopy type as the Delaunay complex for the same radius [4]. Our results imply that it has the same homotopy type as the Delaunay-Čech complex and the wrap complex, all for the same radius. In general, the flow complex is not a subcomplex of the Delaunay triangulation, and its computation remains challenging [5]. An algorithm for computing the adjacency structure of the stable flow regions developed in [18] uses ideas closely related to the concepts developed in the present paper. The Morse theoretic view on Delaunay filtrations may shed additional light on the connection between the two constructions.

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A Weighted Points

The weighted case arises when we use balls of different size in the construction of a complex. Encoding the difference by giving weights to the points, we use a real-valued parameter to grow and shrink all balls simultaneously. This concept is well known for Voronoi diagrams and Delaunay triangulations, whose weighted versions are sometimes referred to as *power diagrams* [1] and *regular triangulations* [16].

Weighted complexes. It is not difficult to extend the four types of complexes considered in this paper to the weighted case. Let X be a finite set of points in $\mathbb{R}^n$, let $w\colon X \to \mathbb{R}$ be a *weight function*, and call $w(x)$ the *weight* of $x \in X$. The *weighted squared distance* of a point $a \in \mathbb{R}^n$ from x is the squared Euclidean distance minus the weight: $\pi_x(a) = \|x - a\|^2 - w(x)$. With this notion, we generalize the closed ball of radius r centered at x to $B_r^w(x) = \{a \in \mathbb{R}^n \mid \pi_x(a) \leq r^2\}$. Its center is x and its squared radius is $w(x) + r^2$, which can also be negative, in which case the ball is the empty set. Indeed, we allow the parameter to be negative, denoting it by $r^2 \in \mathbb{R}$ to remind us of its connection to the squared radius in the unweighted case. Technically, we let r take on values in $\sqrt{\mathbb{R}}$, which we define as the non-negative real numbers together with the non-negative multiples of the imaginary unit. Note that $B_r^w(x) = B_r(x)$ if $w(x) = 0$. The *weighted Čech complex* for $r \in \sqrt{\mathbb{R}}$ is

$$C_r^w(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} B_r^w(x) \neq \emptyset\},$$

which is isomorphic to the nerve of the collection of balls $B_r^w(x)$, $x \in X$. To define the other complexes, we still need the notion of the *weighted Voronoi cell* of $x \in X$, denoted by $\mathrm{Vor}^w(x)$, which is the set of points $a \in \mathbb{R}^n$ such that $\pi_x(a) \leq \pi_y(a)$ for all $y \in X$. Similarly to the unweighted case, the weighted Voronoi cells are convex polyhedra, any two polyhedra intersect at most along shared boundaries, and together the weighted Voronoi cells cover $\mathbb{R}^n$. There are also differences: the weighted Voronoi cell of x does not necessarily contain x, and it is possible that $\mathrm{Vor}^w(x)$ be empty. The *weighted Delaunay triangulation*, denoted by $D^w(X)$, is isomorphic to the nerve of the collection of weighted Voronoi cells. As in the unweighed case, we need a mild general position assumption for this to give a simplicial complex in $\mathbb{R}^n$. A precise statement of that assumption is given shortly. We can now define the *weighted Delaunay complex* and the *weighted Delaunay-Čech complex* for $r \in \sqrt{\mathbb{R}}$:

$$D_r^w(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} [B_r^w(x) \cap \mathrm{Vor}^w(x)] \neq \emptyset\},$$
$$DC_r^w(X) = \{Q \in D^w(X) \mid \bigcap_{x \in Q} B_r^w(x) \neq \emptyset\}.$$

Clearly, $D_r^w(X) \subseteq DC_r^w(X) \subseteq C_r^w(X)$, for every $r \in \sqrt{\mathbb{R}}$, just like in the unweighted case.

Weighted radius functions. The weighted complexes have equivalent dual definitions based on generalizations of circumspheres and enclosing spheres. Note that we can write the sphere with center $a \in \mathbb{R}^n$ and radius ϱ as the zero set of the weighted squared distance function from the point a with weight ϱ^2:

$$S_\varrho(a) = \{b \in \mathbb{R}^n \mid \|b - a\|^2 - \varrho^2 = 0\}.$$

We say a point $x \in \mathbb{R}^n$ with weight $w(x) \in \mathbb{R}$ is *orthogonal* to $S_\varrho(a)$ if $\|x - a\|^2 - \varrho^2 - w(x) = 0$. Similarly, we say x is *closer than orthogonal* if the left hand side of the equation is negative, and *further than orthogonal* if the left hand side is positive. Generalizing the notion of circumsphere, we call $S_\varrho(a)$ an *orthosphere* of Q if every weighted point in Q is orthogonal to the sphere. Given x with weight $w(x)$ and $a \in \mathbb{R}^n$, we can find a unique weight ϱ^2 such that x and a are orthogonal. In other words, a single weighted point has an

n-dimensional family of orthogonal spheres. Generically, $p+1$ weighted points have an $(n-p)$-dimensional family of orthogonal spheres. One of the two requirements in the definition of general position is that there be no families of dimension higher than $n - p$.

Definition A.1 (Weighted general position)**.** A finite set $X \subseteq \mathbb{R}^n$ with weight function $w\colon X \to \mathbb{R}$ is in *general position* if for every subset Q of at most $n+1$ points in X,

(i) the family of orthospheres of Q has dimension $n - p$, and

(ii) the smallest orthosphere of Q is not orthogonal to any weighted point in $X \setminus Q$.

Using the squared radius as the measure of size, we generalizing the smallest empty circumsphere: the *Delaunay sphere* of $Q \in D^w(X)$ is the smallest $(n-1)$-sphere, $S^w_D(Q)$, that is orthogonal to every weighted point in Q and orthogonal or further than orthogonal to every weighted point in $X \setminus Q$. The corresponding *weighted Delaunay function*, $f^w_D\colon D^w(X) \to \mathbb{R}$, is defined by mapping Q to the squared radius of $S^w_D(Q)$. Recalling that the squared radius is really the weight of the center of the sphere, we note that this squared radius may be negative. Generalizing the smallest enclosing sphere, we define the *Čech sphere* of $Q \subseteq X$ as the smallest $(n-1)$-sphere, $S^w_C(Q)$, that is orthogonal or closer than orthogonal to every weighted point in Q. The corresponding *weighted Čech function*, $f^w_C\colon \Delta(X) \to \mathbb{R}$, is defined by mapping Q to the squared radius of $S^w_C(Q)$. As for the weighted Delaunay function, the value of a simplex under the weighted Čech function may be negative. With these two functions, it is easy to characterize which simplices belong to the weighted Čech and weighted Delaunay complexes.

Lemma A.2 (Weighted Radius Functions Lemma)**.** *Let X be a finite set of points with weight function $w\colon X \to \mathbb{R}$ in general position in $\mathbb{R}^n$. Then*

$$C^w_r(X) = (f^w_C)^{-1}(-\infty, r^2],$$
$$D^w_r(X) = (f^w_D)^{-1}(-\infty, r^2],$$

for every $r \in \sqrt{\mathbb{R}}$.

Intervals. It is not difficult to establish that the two weighted radius functions are indeed generalized discrete Morse functions. We begin with the description of the lower and upper bounds of the intervals, which are straightforward generalizations of these bounds in the unweighted case. For V^w_C, the lower bound of the interval that contains $Q \subseteq X$ is the simplex $L = L^w_C(Q) \subseteq Q$ of weighted points orthogonal to the Čech sphere of Q, and the upper bound is the simplex $U = U^w_C(Q) \subseteq X$ of weighted points orthogonal or closer than orthogonal to this Čech sphere. For V^w_D, the upper bound of the interval that contains $Q \in D^w(X)$ is the simplex $R = U^w_D(Q)$ of weighted points orthogonal to the Delaunay sphere of Q. A facet R_i of R is a *front facet* if it has the same Delaunay sphere as R, and a *back facet*, otherwise. As in the unweighted case, the lower bound is the common intersection of the front facets, $P = L^w_D(Q)$. These lower and upper bounds define the intervals of the gradients of the two weighted radius functions.

Lemma A.3 (Weighted Čech and Delaunay Functions Lemma)**.** *Let X be a finite set of points with weight function $w\colon X \to \mathbb{R}$ in general position in $\mathbb{R}^n$. Then $f^w_C\colon \Delta(X) \to \mathbb{R}$ and $f^w_D\colon D^w(X) \to \mathbb{R}$ are generalized discrete Morse functions with gradients V^w_C and V^w_D, respectively.*

A crucial step toward proving the collapsibility of the weighted Delaunay-Čech complex to the weighted Delaunay complex is the comparison of the two gradient fields. The situation is again analogous to the unweighted case. A simplex $Q \subseteq X$ is critical for f^w_C iff every weighted point in $X \setminus Q$ is further than orthogonal to the Čech sphere of Q. But then the Čech sphere is equal to the Delaunay sphere, and Q is critical for f^w_D. Conversely, if Q is critical for f^w_D then it is critical for f^w_C. Besides having the same critical simplices, we need that the non-singular intervals intersect in non-singular intervals.

Lemma A.4 (Weighted Excluded Singularity Lemma)**.** *Let X be a finite set of points with weight function $w\colon X \to \mathbb{R}$ in general position in $\mathbb{R}^n$. The intersection of a non-singular Čech interval and a non-singular Delaunay interval is a possibly empty non-singular interval.*

The proof is omitted because it is almost verbatim the same as in the unweighted case; see Section 3.

Collapsibility in the weighted case. Before generalizing the Main Theorem (4.2) from the unweighted to the weighted case, we still need to introduce the *weighted wrap complex* for $r \in \sqrt{\mathbb{R}}$, which is the union of the lower sets of all critical simplices Q whose value under the weighted Delaunay function is at most r^2:

$$W_r^w(X) = \bigcup_{\substack{[Q,Q]\in V_D^w \\ f_D^w(Q)\leq r^2}} \downarrow Q.$$

Similar to the unweighted case, the lower sets do not necessarily partition the triangulation, and they do not even necessarily cover it.

Theorem A.5 (Weighted Main Theorem)**.** *Let X be a finite set of points with weight function $w\colon X \to \mathbb{R}$ in general position in $\mathbb{R}^n$. Then*

$$DC_r^w(X) \searrow D_r^w(X) \searrow W_r^w(X),$$

for every $r \in \sqrt{\mathbb{R}}$.

The proof is omitted because it is almost verbatim the same as in the unweighted case presented in Section 4.