

THE MORSE THEORY OF ČECH AND DELAUNAY COMPLEXES

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ABSTRACT. Given a finite set of points in \mathbb{R}^n and a radius parameter, we study the Čech, Delaunay–Čech, Delaunay (or alpha), and Wrap complexes in the light of generalized discrete Morse theory. Establishing the Čech and Delaunay complexes as sublevel sets of generalized discrete Morse functions, we prove that the four complexes are simple-homotopy equivalent by a sequence of simplicial collapses, which are explicitly described by a single discrete gradient field.

1. INTRODUCTION

The burgeoning field of *topological data analysis* was born from the idea that results in algebraic topology can be fruitfully applied to timely challenges in data analysis [11, 20]. It rests on the time-tested method of modeling by abstraction, which in this setting means that we interpret the data as a finite sample of a topological space. Since we are given the data – but not the space – we construct a family of hypothetical spaces and gain insights into the data from general topological properties of these spaces and the relationships between them.

1.1. Results. Assuming the data consist of a set of points $X \subseteq \mathbb{R}^n$, we have several discrete geometric constructions at our disposal that use the Euclidean distance and a scale parameter, r , to convert the data into a filtration of spaces. If the balls of radius r centered at $p+1$ data points have a non-empty common intersections, then we may add the p -simplex they span to the space representation. This gives the *Čech complex* for radius r , denoted by $\check{C}ech_r(X)$, which is known to have the same homotopy type as the union of balls, $B_r(X)$ [24, Chapter III]. The construction of Čech complexes originates from the definitions of cohomology theories for general topological spaces due to Alexandroff and Čech [1, 13, 14]. Besides its foundational role in algebraic topology, it was introduced by Carlsson and de Silva [19] as a core construction in topological data analysis, and has received significant interest in this area as well as in stochastic geometry and topology [32].

Alternatively, we may first intersect each ball with the Voronoi domain of its center and then take the nerve. This gives the *alpha* or *Delaunay complex* for radius r , denoted by $Del_r(X)$, which embeds in \mathbb{R}^n , is a subcomplex of the Čech complex, and has the same homotopy type [21]. The Delaunay complex was first introduced under the name *alpha-shape* as a construction for associating a geometric shape to a finite set of points in the plane [25]. It has become a popular method both in computational geometry and topology; see the survey [23]. Not surprisingly,

2010 *Mathematics Subject Classification.* Primary 52C99; Secondary 51F99, 55U10, 57Q10.

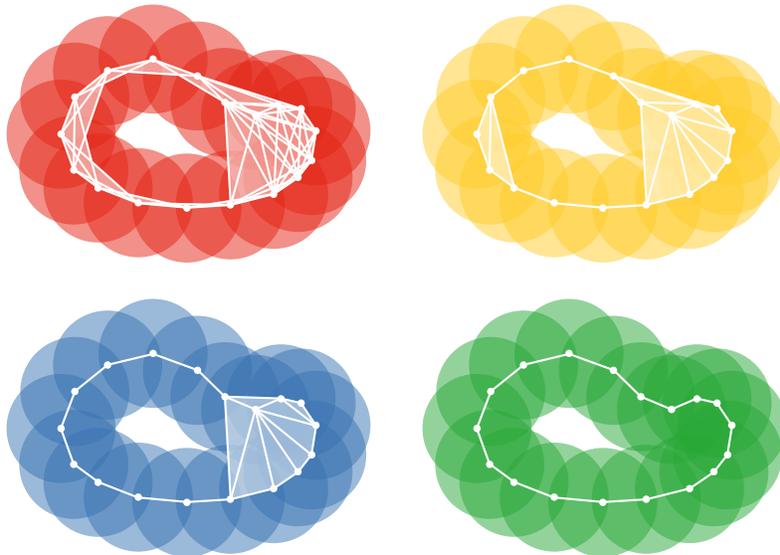


Figure 1: The four different geometric complexes appearing in the collapsing sequence of the main theorem, drawn on top of the union of balls $B_r(X)$, to which they are all homotopy-equivalent. *In sequence*: the high-dimensional Čech complex, $\check{C}ech_r(X)$, projected onto the plane, the Delaunay–Čech complex, $Del\check{C}ech_r(X)$, the Delaunay complex, $Del_r(X)$, and the Wrap complex, $Wrap_r(X)$.

the Čech complex collapses to the Delaunay complex for the same radius, but this was an open question prior to this paper.

As a third option, we may collect all simplices in the Čech complex that belong to the Delaunay triangulation. This gives the *Delaunay–Čech complex* for radius r , denoted by $Del\check{C}ech_r(X)$. This construction is a convenient alternative to Delaunay complexes [17, 34], requiring only the computation of the Delaunay triangulation and the smallest enclosing sphere of each Delaunay simplex.

As a fourth option, we may construct the *Wrap complex*, denoted by $Wrap_r(X)$, which is a subcomplex of the Delaunay triangulation [22]. Going beyond topological characterizations, it gives a geometric description of the data and has been successfully employed within commercial settings in software for surface reconstruction from point data, serving as the foundation of the software package Geomagic Wrap®. We extend the original 3-dimensional notion to \mathbb{R}^n and introduce a dependence on a radius parameter, with the original definition corresponding to radius ∞ . Formulating the Wrap complex within Forman’s discrete Morse theory [26], we answer an open question in [22]. Our main result is that the four complexes are related by simplicial collapses, and that this property extends to natural generalizations of the complexes to points with weights.

Theorem (Čech–Delaunay Collapsing Theorem). *Let X be a finite set of possibly weighted points in general position in \mathbb{R}^n . Then*

$$\check{C}ech_r(X) \searrow Del\check{C}ech_r(X) \searrow Del_r(X) \searrow Wrap_r(X)$$

for every $r \in \mathbb{R}$.

This establishes the *simple-homotopy* equivalence of the four different complexes, a particular type of homotopy equivalence that admits a purely combinatorial description in terms of elementary collapses and expansions and is algebraically characterized by the vanishing of Whitehead torsion [18].

Our proofs are based on the insight that the Delaunay and Čech complexes arise as sublevel sets of generalized discrete Morse functions with shared structural properties. We refer to [26] for an introduction to discrete Morse theory, and to [15, 27] for the generalization of the discrete gradient to allow for intervals larger than pairs. Our constructions are elementary, using the radii of smallest enclosing spheres and smallest empty circumspheres to define the generalized Morse functions. The geometric arguments are couched in the language of convex optimization, in which there is little difference between ordinary points and points with weights. Indeed, all our results generalize to the weighted setting, and thus relate to the theory of power diagrams and regular triangulations [28].

The common structure of Delaunay and Čech complexes leads naturally to a generalization of the two constructions, which we call the *selective Delaunay complex*, $\text{Del}_r(X, E)$, defined for a subset $E \subseteq X$ of *excluded* points. The construction of the selective Delaunay complex is based on smallest enclosing spheres of subsets $Q \subseteq X$ whose interiors are empty of the excluded points E . The Delaunay and Čech complexes arise as special cases with $E = X$ and $E = \emptyset$, respectively. The main collapsing theorem is derived from the following statement, which relates selective Delaunay complex with different excluded sets through simplicial collapses:

Theorem (Selective Delaunay Collapsing Theorem). *Let X be a finite set of possibly weighted points in general position in \mathbb{R}^n , and let $E \subseteq F \subseteq X$. Then*

$$\text{Del}_r(X, E) \searrow \text{Del}_r(X, E) \cap \text{Del}(X, F) \searrow \text{Del}_r(X, F)$$

for every $r \in \mathbb{R}$.

It is worth mentioning that the concept of selective Delaunay complexes enables the explicit construction of a sequence of maps in homology between Delaunay complexes $\text{Del}_r(X)$ and $\text{Del}_r(Y)$ that is equivalent to the maps in homology induced by the inclusions $B_r(X), B_r(Y) \hookrightarrow B_r(X \cup Y)$.

1.2. Related work. A simple-homotopy version of the Nerve Theorem appears in [4]. The author proves that a good cover of a simplicial complex by subcomplexes has a nerve that is simple-homotopy equivalent to the complex. Note however that this does not imply the stronger result that the Čech complex collapses simplicially to the Delaunay complex.

The *flow complex* introduced in [29] is conceptually similar to the Wrap complex. Being based on the gradient flow of the distance function to a point set, this construction is however less combinatorial. In general, the flow complex is not a subcomplex of the Delaunay triangulation, and its computation remains challenging [12]. It has been shown to have the same homotopy type as the Delaunay complex for the same radius [10]. Our results imply that it has the same homotopy type as the Delaunay–Čech complex and the Wrap complex, all for the same radius.

The structure of the generalized gradients has been described before for the special case of Čech filtrations in [2], and for Delaunay filtrations in [22, 36]. Our Morse-theoretic treatment of selective Delaunay complexes systematically unifies

and generalizes these results. The continuous Morse theory for distance functions to finite point sets in Euclidean space has also been investigated in [7, 33, 35]. By the homotopy-equivalence of unions of balls, Čech complexes, and Delaunay complexes, we obtain the same characterization of critical points and the same statements about the change of homotopy type of sublevel sets at critical values. Our main interest is in the additional structure provided by the interval partition of the discrete Morse function, and the explicit combinatorial description of homotopy equivalences induced by a discrete gradient.

This paper extends the collapse of the Delaunay–Čech to the Delaunay complex, which is the main result in an early version of the present paper [5] and has been known prior to that paper in \mathbb{R}^2 only [17]. To further include the Čech complex in the collapsing sequence, we had to substantially change the proof and unify the treatment of Čech and Delaunay functions in the framework of convex optimization.

The generalization of discrete Morse theory used in the present paper was suggested in [27]. The corresponding notion of collapses by intervals in the face poset has been considered before in [38], and collapses by even more general clusters of simplices have been considered in [30, 31]. Another popular construction of a geometric complex is the *Vietoris–Rips complex* [37]. The resulting filtrations however do not generally come from a generalized Morse function.

1.3. Outline. Section 2 presents background material in combinatorial topology and discrete Morse theory. Section 3 discusses the Čech and Delaunay complexes in the context of a larger family of proximity complexes in Euclidean space. Section 4 generalizes from points to points with weights, writing all conditions in the language of convex optimization. Section 5 proves the collapsing sequence. Section 6 describes consequences of the collapsing sequence. Section 7 concludes the paper.

2. BACKGROUND

All complexes in this paper are simplicial, with vertices from a finite set in \mathbb{R}^n . Since we do not restrict the dimension of our simplices, we will treat them as combinatorial rather than concrete geometric objects. Assuming the reader is familiar with abstract simplicial complexes, we give quick reviews of discrete Morse theory [26] and its generalization.

2.1. Discrete Morse theory. Given a finite set $X \subseteq \mathbb{R}^n$, we call a set $Q \subseteq X$ of $q + 1$ points a q -*simplex*. Its *dimension* is q , its *faces* are the subsets of Q , and its *facets* are the faces of dimension $q - 1$. A *simplicial complex* is a collection of simplices, K , that is closed under the face relation, and its *dimension* is the maximum dimension of any of its simplices. The face relation defines a canonical partial order on K , and the *Hasse diagram*, denoted as $\mathcal{H}(K)$, is the transitive reduction of this order. In other words, $\mathcal{H}(K)$ is the directed acyclic graph whose nodes are the simplices and whose arcs are the pairs (P, Q) in which P is a facet of Q . A *discrete vector field* is a partition V of K into singleton sets $\{C\}$ and pairs $\{P, Q\}$ corresponding to arcs (P, Q) in the Hasse diagram. Suppose now that there is a function $f: K \rightarrow \mathbb{R}$ that satisfies $f(P) \leq f(Q)$ whenever P is a face of Q , with equality holding in this case iff (P, Q) is a pair in V . Then f is called a *discrete Morse function* and V is its *discrete gradient*. Indeed, the existence of f attests for the acyclicity of the directed graph obtained from the Hasse diagram by contracting

the pairs in V . A simplex that does not belong to any pair in V is called a *critical simplex* and the corresponding value is a *critical value* of f .

To provide an intuition for the concept, we note that the pairs in a discrete gradient correspond to elementary collapses [24, Chapter III], except that the lower-dimensional simplex does not have to be free. An elementary collapse can be realized continuously by a deformation retraction. This implies that if we can transform a simplicial complex, K , to another, K' , using a sequence of such elementary collapses, then the two complexes have the same homotopy type. In fact, the implied relation is slightly stronger, which is usually expressed by saying that K and K' are *simple-homotopy equivalent* [18]. We say that K collapses onto K' and write $K \searrow K'$. A discrete gradient can encode a collapse [26]:

Theorem 2.1 (Gradient Collapsing Theorem). *Let K be a simplicial complex with a discrete gradient V , and let $K' \subseteq K$ be a subcomplex. If $K \setminus K'$ is a union of pairs in V , then $K \searrow K'$.*

We say that the collapse from K to K' is *induced by V* .

2.2. Generalized discrete Morse theory. To generalize discrete Morse theory we recall that an *interval* in the face relation of K is a subset of the form

$$(1) \quad [P, R] = \{Q \mid P \subseteq Q \subseteq R\}.$$

The interval is non-empty iff P is a face of R . In this case, the interval contains both simplices – which may be the same – and we refer to P as its *lower bound* and to R as its *upper bound*. Borrowing from the nomenclature of [27], we call a partition W of K into intervals a *generalized discrete vector field*. Indeed, a discrete vector field is the special case in which all intervals are either singletons or pairs. Suppose now that there is a function $f: K \rightarrow \mathbb{R}$ that satisfies $f(P) \leq f(Q)$ whenever P is a face of Q , with equality holding in this case iff P and Q belong to a common interval in W . Then f is called a *generalized discrete Morse function* and W is its *generalized discrete gradient*. If an interval contains only one simplex, then we call the interval *singular*, the simplex a *critical simplex*, and the value of the simplex a *critical value* of f .

It is easy to see that for every generalized discrete gradient, there is a discrete gradient that refines every non-singular interval $[P, R]$ into pairs: choose an arbitrary vertex $x \in R \setminus P$ and partition $[P, R]$ into pairs $\{Q \setminus \{x\}, Q \cup \{x\}\}$ for all $Q \in [P, R]$. We call this a *vertex refinement*. The generalized discrete gradient and its refinement have the same critical simplices, implying that the Gradient Collapsing Theorem 2.1 also applies to generalized discrete gradients. The refinement is in general not unique.

Theorem 2.2 (Generalized Gradient Collapsing Theorem). *Let K be a simplicial complex with a generalized discrete gradient V , and let $K' \subseteq K$ be a subcomplex. If $K \setminus K'$ is a union of non-singular intervals in V , then $K \searrow K'$.*

3. PROXIMITY COMPLEXES

We introduce Čech and Delaunay complexes as members of the larger family of selective Delaunay complexes. After writing these complexes as sublevel sets of real-valued functions, we introduce the Delaunay–Čech complexes as subcomplexes of the Delaunay triangulation.

3.1. Čech complexes. Write $d(x, y)$ for the Euclidean distance between $x, y \in \mathbb{R}^n$. For $r \geq 0$, let $B_r(x) = \{y \in \mathbb{R}^n \mid d(x, y) \leq r\}$ be the closed ball of radius r centered at $x \in X$. The *Čech complex* of a finite set $X \subseteq \mathbb{R}^n$ for radius $r \geq 0$,

$$(2) \quad \check{\text{Cech}}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{x \in Q} B_r(x) \neq \emptyset \right\},$$

is isomorphic to the nerve of the collection of closed balls. Therefore, by the Nerve Theorem [8], it is homotopy equivalent to the union of the balls, $B_r(X) = \bigcup_{x \in X} B_r(x)$. For sufficiently large radius, the Čech complex is the full (abstract) simplex spanned by X , which we denote as $\Delta(X) = 2^X \setminus \{\emptyset\}$. For $r \leq t$, we have $\check{\text{Cech}}_r(X) \subseteq \check{\text{Cech}}_t(X)$, so the Čech complexes form a *filtration* of $\Delta(X)$.

3.2. Delaunay complexes. Let $X \subseteq \mathbb{R}^n$ be again finite and $x \in X$. The *Voronoi domain* of x with respect to X , and the *Voronoi ball* of x with respect to X for a radius $r \geq 0$ are

$$(3) \quad \text{Vor}(x, X) = \{y \in \mathbb{R}^n \mid d(y, x) \leq d(y, p) \text{ for all } p \in X\},$$

$$(4) \quad \text{Vor}_r(x, X) = B_r(x) \cap \text{Vor}(x, X).$$

The *Delaunay complex* of X for radius $r \geq 0$,

$$(5) \quad \text{Del}_r(X) = \left\{ Q \subseteq X \mid \bigcap_{x \in Q} \text{Vor}_r(x, X) \neq \emptyset \right\},$$

often referred to as alpha complex, is isomorphic to the nerve of the collection of Voronoi balls. For sufficiently large r , this is the *Delaunay triangulation* of X , which we denote as $\text{Del}(X) = \text{Del}_\infty(X)$. For $r \leq t$, we have $\text{Del}_r(X) \subseteq \text{Del}_t(X)$, so the Delaunay complexes form a filtration of the Delaunay triangulation. If we assume that the points are in general position, then the Delaunay triangulation is geometrically realized as a simplicial complex in \mathbb{R}^n . While the assumption of general position is not required in the abstract setting, we will need it in the construction of generalized discrete Morse functions.

3.3. Delaunay–Čech complexes. The *Delaunay–Čech complex* for radius $r \geq 0$ is the restriction of the Čech complex to the Delaunay triangulation. It contains all simplices in the Delaunay triangulation such that the balls of radius r centered at the vertices have a non-empty common intersection:

$$(6) \quad \text{Del}\check{\text{Cech}}_r(X) = \left\{ Q \in \text{Del}(X) \mid \bigcap_{x \in Q} B_r(x) \neq \emptyset \right\}.$$

Similar to the Delaunay complex, we have $\text{Del}\check{\text{Cech}}_r(X) \subseteq \text{Del}\check{\text{Cech}}_t(X) \subseteq \text{Del}(X)$ whenever $r \leq t$, so the Delaunay–Čech complexes also form a filtration of the Delaunay triangulation. While the Delaunay and the Delaunay–Čech complexes are similar, they are not necessarily the same, as illustrated in Figure 1. Instead of equality, we have $\text{Del}_r(X) \subseteq \text{Del}\check{\text{Cech}}_r(X)$ for all r . To see this, we just note that if the Voronoi balls have a non-empty common intersection, then the balls have a non-empty common intersection and so do the Voronoi domains.

3.4. Selective Delaunay complexes. The proof of the main result in this paper makes essential use of a family of complexes that contain the Delaunay and the Čech complexes as extremal cases. To introduce this family, let $X \subseteq \mathbb{R}^n$ be a finite set, $E \subseteq X$ a subset, and $r \geq 0$ a real number. Note that for a point $x \in X$ we can also consider the *Voronoi ball* of x with respect to a subset $E \subseteq X$ not necessarily containing x . Specifically, $\text{Vor}_r(x, E)$ is the set of points $y \in \mathbb{R}^n$ whose distance to x is bounded from above by r and by the distances to the points in E . The *selective Delaunay complex* for $E \subseteq X$ and r contains all simplices over X whose vertices have Voronoi balls for the subset E with non-empty common intersection:

$$(7) \quad \text{Del}_r(X, E) = \left\{ Q \subseteq X \mid \bigcap_{x \in Q} \text{Vor}_r(x, E) \neq \emptyset \right\}.$$

It is isomorphic to the nerve of the set of these Voronoi balls; see Figure 2. While

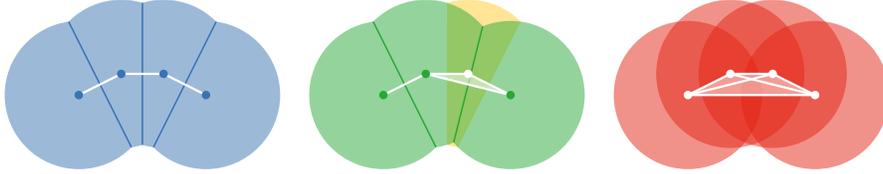


Figure 2: The Voronoi balls of a set X of four points in \mathbb{R}^2 . In the *middle*, three of the balls belong to the subset $E \subseteq X$ that impose constraints. The corresponding selective Delaunay complex, $\text{Del}_r(X, E)$, has four edges and one triangle. It properly contains the Delaunay complex, $\text{Del}_r(X)$, shown on the *left*, and it is properly contained in the Čech complex, $\check{\text{Cech}}_r(X)$, shown on the *right*.

individual Voronoi balls depend on E , their union does not. To see this, we note that $\text{Vor}_r(x, F) \subseteq \text{Vor}_r(x, E)$ whenever $E \subseteq F$ for all $x \in X$. We get the largest Voronoi balls for $E = \emptyset$, in which case each domain is a ball of radius r . We get the smallest Voronoi balls for $E = X$, in which case the Voronoi balls form a convex decomposition of the union of balls. Since the union does not depend on E , the Nerve Theorem [8] implies that for a given point set X and radius r , all selective Delaunay complexes have the same homotopy type. Note also that $\text{Del}_r(X, F) \subseteq \text{Del}_t(X, E)$ whenever $r \leq t$ and $E \subseteq F$. For $E = \emptyset$, the selective Delaunay complex is the Čech complex: $\check{\text{Cech}}_r(X) = \text{Del}_r(X, \emptyset)$. For $E = X$, the selective Delaunay complex is the Delaunay complex: $\text{Del}_r(X) = \text{Del}_r(X, X)$. In analogy to the Delaunay triangulation, we define $\text{Del}(X, E) = \text{Del}_\infty(X, E)$.

3.5. Radius functions. There is an equivalent, dual definition of selective Delaunay complexes, which is natural from the point of view of discrete Morse theory and will reveal important structural properties. To state this definition, consider two point sets $Q, E \subseteq X \subseteq \mathbb{R}^n$. We say an $(n-1)$ -dimensional sphere S in \mathbb{R}^n *includes* $Q \subseteq X$ if all points of Q lie on or inside S , and it *excludes* $E \subseteq X$ if all points of E lie on or outside S . If Q and E share points, then they necessarily lie on S . The set of such spheres may be empty, but if it is not, then we define $S(Q, E)$ to be the smallest such sphere, referring to it as the *Delaunay sphere* of Q with respect to E ,

and we write $s(Q, E)$ for its squared radius. The *radius function* for E maps each simplex to the squared radius of the Delaunay sphere:

$$(8) \quad s_E: \text{Del}(X, E) \rightarrow \mathbb{R}$$

defined by $s_E(Q) = s(Q, E)$. Considering the special case $E = X$ of Delaunay complexes, we call s_X the *Delaunay radius function* of X , and $S(Q, X)$ the *Delaunay sphere* of Q , which is commonly referred to as the *smallest empty circumsphere* of Q . Similarly, for the special case $E = \emptyset$ of Čech complexes, we call s_\emptyset the *Čech radius function* of X , and $S(Q, \emptyset)$ the *Čech sphere* of Q , which is commonly referred to as the *smallest enclosing sphere* of Q .

It is not difficult to see that Q belongs to the selective Delaunay complex for radius r iff its value under the radius function exists and does not exceed r^2 . For completeness, we present the formal claim with proof.

Lemma 3.1 (Radius Function Lemma). *Let $X \subseteq \mathbb{R}^n$ be finite, $E \subseteq X$, and $r \geq 0$. A simplex $Q \in \text{Del}(X, E)$ belongs to $\text{Del}_r(X, E)$ iff $s_E(Q) \leq r^2$.*

Proof. Suppose $Q \in \text{Del}_r(X, E)$, consider the Voronoi balls of its vertices with respect to E , and let y be a common point of these balls. Let ρ be the maximum distance between y and any point in $Q \cap E$. By construction, the sphere with center y and radius ρ includes Q and excludes E . We have $s_E(Q) \leq \rho^2 \leq r^2$ since $s_E(Q)$ is the squared radius of the smallest such sphere.

Conversely, if $s_E(Q) \leq r^2$, then the center of the smallest sphere that includes Q and excludes E belongs to the Voronoi ball of every point in Q with respect to E , which implies $Q \in \text{Del}_r(X, E)$. \square

We observed earlier that $\text{Del}_r(X, F) \subseteq \text{Del}_r(X, E)$ whenever $E \subseteq F$. Correspondingly, we have $s_E(Q) \leq s_F(Q)$ whenever both are defined, which is clear because E generates fewer constraints than F and therefore allows for a radius that is smaller than or equal to the smallest radius we get for F .

4. CONVEX OPTIMIZATION

Assuming the points are in general position, we prove that the radius functions defined in Section 3 are generalized discrete Morse functions. For this purpose, we translate the geometric constructions into the language of convex optimization. In this setting, there is little difference between points and weighted points, so we generalize all results to weighted points.

4.1. Weighted points. Using the Radius Function Lemma 3.1, we can determine whether or not a simplex Q belongs to $\text{Del}_r(X, E)$ by solving a convex optimization problem: $S(Q, E)$ is the sphere with center z and radius $r \leq 0$ that

$$(9) \quad \underset{r, z}{\text{minimizes}} \quad r^2$$

$$(10) \quad \text{subject to } d(z, q)^2 \leq r^2, \quad \forall q \in Q,$$

$$(11) \quad d(z, e)^2 \geq r^2, \quad \forall e \in E.$$

We generalize the setting to allow for points $x \in \mathbb{R}^n$ with weight $w_x \in \mathbb{R}$, a concept well known for Voronoi diagrams and Delaunay triangulations, whose weighted versions are sometimes referred to as *power diagrams* [3] and *regular triangulations* [28]. To explain, we substitute $s = r^2$, allowing s to be negative as well. Appealing to geometric intuition, we speak of a sphere S with squared radius s and center

z nonetheless. We say S *includes* a point x with weight w_x if $d(z, x)^2 \leq s + w_x$, and S *excludes* x if $d(z, x)^2 \geq s + w_x$. Similarly, x *lies on* S if it is simultaneously included and excluded. With this extension of the relations, we can read everything we said about spheres and points as statements about spheres and weighted points. To obtain an intuitive geometric interpretation of the weight, we consider the sphere S_x with center x and positive squared radius w_x . The weighted point x then lies on S iff the two spheres S and S_x intersect orthogonally. Similarly, S includes x iff the two spheres are orthogonal or closer to each other than orthogonal, and S excludes x iff the two spheres are orthogonal or further from each other than orthogonal.

With the new notation, we rewrite the convex optimization problem so it applies to the more general, weighted setting:

$$(12) \quad \underset{s, z}{\text{minimize}} \quad s$$

$$(13) \quad \text{subject to } d(z, q)^2 \leq s + w_q, \quad \forall q \in Q,$$

$$(14) \quad d(z, e)^2 \geq s + w_e, \quad \forall e \in E.$$

This effectively generalizes the notion of selective Delaunay complexes from equal sized balls to sets in which balls can have different sizes. Such more general data occurs in a number of applications, including the modeling of biomolecules.

4.2. Karush–Kuhn–Tucker conditions. In the next step, we reformulate the optimization problem so that the objective function is convex and the constraints are affine. Before we get there, consider a general optimization problem:

$$(15) \quad \underset{y}{\text{minimize}} \quad f(y)$$

$$(16) \quad \text{subject to } g_j(y) \leq 0, \quad \forall j \in J,$$

$$(17) \quad g_k(y) = 0, \quad \forall k \in K,$$

$$(18) \quad g_l(y) \geq 0, \quad \forall l \in L,$$

in which J, K, L are pairwise disjoint index sets. Assuming f is convex and the g_i are affine, the *Karush–Kuhn–Tucker conditions* say that a feasible point y is an optimal solution iff there exist coefficients $\lambda_i \in \mathbb{R}$ for all $i \in I = J \cup K \cup L$ such that

$$\text{(stationarity)} \quad \nabla f(y) + \sum_{i \in I} \lambda_i \nabla g_i(y) = 0,$$

$$\text{(complementary slackness)} \quad \lambda_i g_i(y) = 0, \quad \forall i \in I,$$

$$\text{(dual feasibility)} \quad \lambda_j \geq 0, \quad \forall j \in J,$$

$$\lambda_l \leq 0, \quad \forall l \in L;$$

see [9]. In particular, necessity is provided by the linearity of the constraints [9, p. 226], while sufficiency is provided by the convexity of the objective function and the inequality constraints [9, p. 244]. To get our problem into this form, we introduce $a = \|z\|^2 - s$, write $y = (z, a)$, set $K = Q \cap E$, $J = Q \setminus E$, $L = E \setminus Q$, and define

$$(19) \quad f(y) = s = \|z\|^2 - a,$$

$$(20) \quad g_x(y) = \|z - x\|^2 - s - w_x = -2\langle z, x \rangle + a + \|x\|^2, \quad \forall x \in Q \cup E.$$

Noting that a point in $K = Q \cap E$ gives rise to two inequalities that combine to an equality constraint, we see that the thus defined optimization problem is equivalent to the original optimization problem (12), (13), (14). The gradients are

$$(21) \quad \nabla f(y) = (2z, -1),$$

$$(22) \quad \nabla g_x(y) = (-2x, 1).$$

The stationarity condition is equivalent to writing z as an affine combination of the points in $Q \cup E$, with non-negative coefficients for the points in $Q \setminus E$ and non-positive coefficients for the points in $E \setminus Q$. The complementary slackness condition translates into vanishing coefficients whenever the inequality is strict; that is: whenever the point does not lie on the sphere. We can now specialize the general conditions.

Theorem 4.1 (Special KKT Conditions). *Let S be a sphere that includes $Q \subseteq X$ and excludes $E \subseteq X$. Then S is the smallest such sphere iff its center is an affine combination of the points $x \in Q \cup E$,*

$$z = \sum \lambda_x x \quad \text{with} \quad 1 = \sum \lambda_x,$$

such that

- (I) $\lambda_x = 0$ whenever x does not lie on S ,
- (II) $\lambda_x \geq 0$ whenever $x \in Q \setminus E$, and
- (III) $\lambda_x \leq 0$ whenever $x \in E \setminus Q$.

4.3. Combinatorial formulation. We are almost ready to prove the important technical result that the radius functions are generalized discrete Morse functions provided the points are in general position. We begin by formalizing the latter condition. A *circumsphere* of a set $P \subseteq \mathbb{R}^n$ is an $(n-1)$ -sphere such that all points of P lie on the sphere. A sufficient condition for the existence of a circumsphere is that P be affinely independent. For sets of n or fewer points, the circumsphere is not unique, but by the Special KKT Conditions 4.1, there is only one *smallest circumsphere*, namely the one whose center lies in the affine hull of the points. We formulate conditions under which only the minimum number of points lie on any smallest circumsphere.

Definition 4.2 (General Position Assumption). A finite set X in \mathbb{R}^n is in *general position* if for every $P \subseteq X$ of at most $n+1$ points

- (a) P is affinely independent, and
- (b) no point of $X \setminus P$ lies on the smallest circumsphere of P .

The formulation applies to points as well as to points with weights. We therefore assume the more general case in which X is a finite set of weighted points in general position in \mathbb{R}^n . Let S be an $(n-1)$ -sphere, write $\text{Incl } S, \text{Excl } S \subseteq X$ for the subsets of included and excluded points, and set $\text{On } S = \text{Incl } S \cap \text{Excl } S$. Now assume that S is the smallest circumsphere of some set P , that is: the center z of S lies in the affine hull of P , and $P = \text{On } S$ by general position. We have

$$z = \sum_{x \in \text{On } S} \rho_x x \quad \text{with} \quad 1 = \sum_{x \in \text{On } S} \rho_x.$$

By general position, the affine combination is unique, and $\rho_x \neq 0$ for all $x \in \text{On } S$. We call

$$(23) \quad \text{Front } S = \{x \in \text{On } S \mid \rho_x > 0\},$$

$$(24) \quad \text{Back } S = \{x \in \text{On } S \mid \rho_x < 0\}$$

the *front face* and the *back face* of $\text{On } S$, respectively. We have $\text{Back } S = \emptyset$ iff the circumcenter z is contained in the convex hull of $\text{On } S$. Using these definitions, we now give a combinatorial version of the Karush–Kahn–Tucker conditions.

Theorem 4.3 (Combinatorial KKT Conditions). *Let X be a finite set of weighted points in general position in \mathbb{R}^n . Let $Q, E \subseteq X$ for which there exists a sphere S with $Q \subseteq \text{Incl } S$ and $E \subseteq \text{Excl } S$. It is the smallest such sphere, $S = S(Q, E)$, iff*

- (i) S is the smallest circumsphere of $\text{On } S$,
- (ii) $\text{Front } S \subseteq Q$, and
- (iii) $\text{Back } S \subseteq E$.

Proof. We first show that the Special KKT Conditions 4.1 imply the Combinatorial KKT Conditions 4.3. The center of S lies in the affine hull of $P = \{x \in Q \cup E \mid \lambda_x \neq 0\}$, and Condition (I) implies that $P \subseteq \text{On } S$ which in turn implies (i). By general position, we have $\text{On } S = P$, and we can apply the definition of front and back face, letting $\rho_x = \lambda_x$ for all $x \in \text{On } S$. Condition (II) now says that $x \notin E$ implies $\rho_x \geq 0$, or, equivalently, that $\rho_x < 0$ implies $x \in E$. Hence, $\text{Back } S \subseteq E$. Similarly, Condition (III) yields $\text{Front } S \subseteq Q$.

We next show that Conditions (i) to (iii) imply (I) to (III). First note that (ii) and (iii) imply $\text{On } S \subseteq Q \cup E$. We define $\lambda_x = \rho_x$ for all $x \in \text{On } S$, and $\lambda_x = 0$ for $x \in (Q \cup E) \setminus \text{On } S$. Now (I) is satisfied by construction, and the inclusion $\text{Back } S \subseteq E$ implies (II), while the inclusion $\text{Front } S \subseteq Q$ implies (III). \square

4.4. Partition into intervals. Fix $E \subseteq X$ and recall that s_E maps each simplex $Q \in \text{Del}(X, E)$ to the squared radius of $S = S(Q, E)$. This implies $s_E(P) = s_E(Q)$ for all $P \in [\text{Front } S, \text{Incl } S]$. To prove that s_E is a generalized discrete Morse function, it remains to show that $s_E(P) < s_E(Q)$ whenever $P \subseteq Q$ do not belong to the same interval. But this is clear from the General Position Assumption 4.2.

Theorem 4.4 (Selective Delaunay Morse Function Theorem). *Let X be a finite set of weighted points in general position in \mathbb{R}^n , and $E \subseteq X$. Then the radius function, $s_E: \text{Del}(X, E) \rightarrow \mathbb{R}$, is a generalized discrete Morse function whose discrete gradient consists of the intervals $[\text{Front } S, \text{Incl } S]$ over all Delaunay spheres $S = S(Q, E)$ with $Q \in \text{Del}(X, E)$.*

Setting $E = \emptyset$, we have $\text{Front } S = \text{On } S$ because $\text{Back } S = \emptyset$ by the Combinatorial KKT Conditions 4.3. Symmetrically, for $E = X$, we have $\text{Incl } S = \text{On } S$. This implies the following two special cases of the above theorem.

Corollary 4.5 (Čech Morse Function Corollary). *The Čech radius function of a finite set of weighted points in general position is a generalized discrete Morse function. Its gradient consists of the intervals $[\text{On } S, \text{Incl } S]$ over all Čech spheres S of X .*

Corollary 4.6 (Delaunay Morse Function Corollary). *The Delaunay radius function of a finite set of weighted points in general position is a generalized discrete Morse function. Its gradient consists of the intervals $[\text{Front } S, \text{On } S]$ over all Delaunay spheres S of X .*

Another straightforward consequence of the Combinatorial KKT Conditions 4.3 is the invariance of the critical simplices. To state this theorem, we call $Q \in \text{Del}(X)$ a *centered Delaunay simplex* if the center of $S = S(Q, X)$ is contained in the convex hull of Q . Equivalently, we have $\text{Front } S = \text{On } S = \text{Incl } S$. Note that in this case $S = S(Q, E)$ for all sets $E \subseteq X$.

Corollary 4.7 (Critical Simplex Corollary). *Let X be a finite set of weighted points in general position in \mathbb{R}^n . Independent of E , a subset $Q \subseteq X$ is a critical simplex of s_E iff $s(Q, \emptyset) = s(Q, E) = s(Q, X)$ iff Q is a centered Delaunay simplex.*

4.5. Wrap complex. It is now easy to define the Wrap complex using the gradient V_X of the Delaunay radius function $s_X: \text{Del}(X) \rightarrow \mathbb{R}$, which partitions the Delaunay triangulation into intervals of the form $[\text{Front } S, \text{On } S]$. Let \mathcal{G} be the directed graph whose nodes are the intervals in V_X , with an arc from μ to ν if there are simplices $P \in \mu$ and $Q \in \nu$ with $P \subseteq Q$. It defines a partial order on V_X . The *lower set* of a subset $A \subseteq V_X$, denoted by $\downarrow A$, is the collection of intervals from which A can be reached along directed paths in \mathcal{G} . The lower set of a singular interval is akin to the stable manifold of a critical point in smooth Morse theory, except that the lower sets of the critical simplices do not necessarily form a partition. Indeed, the lower sets can overlap, and some of the simplices may not belong to the lower set of any critical simplex. The latter can be considered to belong to the lower set of the ‘outside’, but it will not be necessary to formalize this intuition. The *Wrap complex* for $r \geq 0$ consists of all simplices in the lower set of the singular intervals with a Delaunay sphere of radius at most r :

$$(25) \quad \text{Sing}_r(X) = \{[Q, Q] \in V_X \mid s_X(Q) \leq r^2\},$$

$$(26) \quad \text{Wrap}_r(X) = \bigcup \downarrow \text{Sing}_r(X).$$

The original definition of the Wrap complex [22, Section 6] corresponds to $\text{Wrap}_\infty(X)$, which we simply denote as $\text{Wrap}(X)$. In the terminology of [22], a *confident simplex* is the upper bound of a non-singular Delaunay interval, while all other simplices in the interval are *equivocal*. The critical Delaunay simplices are called *centered*, in accordance with the Critical Simplex Corollary 4.7. Clearly, $\text{Wrap}_r(X) \subseteq \text{Wrap}_t(X)$ whenever $r \leq t$. Moreover, from the construction as a union of lower sets we immediately have $\text{Wrap}_r(X) \subseteq \text{Del}_r(X)$ for every $r \in \mathbb{R}$.

5. SIMPLE-HOMOTOPY EQUIVALENCE

In this section, we prove that the various complexes considered in this paper are simple-homotopy equivalent. Throughout, we write $Q - x = Q \setminus \{x\}$ and $Q + x = Q \cup \{x\}$, noting that one of these two simplices is equal to Q . The proof strategy is based on the construction of two discrete gradients. The first one is defined on the full simplex on X and induces the simplicial collapse $\check{\text{Cech}}_r(X) \searrow \text{Del}\check{\text{Cech}}_r(X)$ by removing all non-Delaunay simplices. The second discrete gradient is defined on the Delaunay triangulation $\text{Del}(X)$ and induces the collapse $\text{Del}\check{\text{Cech}}_r(X) \searrow \text{Del}_r(X)$. While sketched here for the collapse of the Čech to the Delaunay complex, the construction more generally establishes a collapse of selective Delaunay complexes $\text{Del}_r(X, E) \searrow \text{Del}_r(X, F)$ for $E \subseteq F \subseteq X$.

The discrete gradients are constructed by assigning to each collapsed simplex $Q \in \text{Del}_r(X, F) \setminus \text{Del}_r(X, E)$ a point $x \in F \setminus E$ that turns the sphere $S(Q, E)$ infeasible for the excluded set F . As a consequence, the sphere $S(Q, F)$ will either

have a larger radius or not exist at all. The choice of the vertex x will be such that both simplices $Q - x$ and $Q + x$ will be assigned the same vertex x . The resulting collections of pairs $\{Q - x, Q + x\}$ are then verified to be the pairs of a discrete gradient that induces the desired collapses.

5.1. Auxiliary lemmas. Before we begin, we present several simple lemmas, which will be useful in our proofs. First, we characterize a special class of discrete gradients that are constructed using a distinguished vertex.

Lemma 5.1 (Vertex Gradient Lemma). *Let K be a simplicial complex, V a discrete vector field on K , and x a vertex of K . If every pair in V can be written as $\{Q - x, Q + x\}$ for some simplex Q , then V is a discrete gradient.*

Proof. Consider the function $f: K \rightarrow \mathbb{R}$ defined by taking average dimensions:

$$(27) \quad f(Q) = \begin{cases} \frac{1}{2}(\dim(Q - x) + \dim(Q + x)) & \text{if } \{Q - x, Q + x\} \in V, \\ \dim(Q) & \text{otherwise.} \end{cases}$$

Clearly $f(Q - x) = f(Q + x)$ whenever $\{Q - x, Q + x\} \in V$. Suppose now that $P \subseteq Q$ with $\dim P = \dim Q - 1$. If P is critical or Q is critical, then $f(P) < f(Q)$ is easy to see. Assume therefore that $\{P - x, P + x\}$ and $\{Q - x, Q + x\}$ are different pairs in V . Then $f(P) < f(Q)$ unless $P = P - x$ and $Q = Q + x$, but the latter case would imply $Q = P + x$, which contradicts the disjointness of the pairs. \square

We will also make use of the following lemma, which allows us to extend a discrete gradient on a subcomplex by a discrete gradient on its complement. A proof can be found in [31, Lemma 4.3].

Lemma 5.2 (Gradient Composition Lemma). *Let $K \subseteq L$ be simplicial complexes with discrete gradients V of K and W of L . If every pair in W is disjoint from K , then the pairs in $V \cup W$ define a discrete gradient on L .*

The following lemma will be useful to obtain a common refinement of two generalized gradients by taking the sum of the two corresponding generalized Morse functions. We omit the proof, which is straightforward.

Lemma 5.3 (Sum Refinement Lemma). *Let $f: K \rightarrow \mathbb{R}$ and $g: L \rightarrow \mathbb{R}$ be generalized discrete Morse functions with gradients V and W . Then $f + g: K \cap L \rightarrow \mathbb{R}$ is a generalized Morse function with gradient $\{I \cap J \mid I \in V, J \in W, I \cap J \neq \emptyset\}$.*

In order to analyze the discrete gradients of radius functions, we note that for $Q, E \subseteq X$ and $S = S(Q, E)$, removing a point $x \in \text{Incl } S$ from Q affects the smallest sphere only if $x \in \text{Front } S$. Likewise, removing a point $y \in \text{Excl } S$ from E affects the smallest sphere only if $y \in \text{Back } S$.

Lemma 5.4 (Same Sphere Lemma). *Let Q be a simplex in $\text{Del}(X, E)$ and $S = S(Q, E)$ the smallest sphere that includes Q and excludes E . Then*

- (i) $S = S(Q - x, E) = S(Q + x, E)$ iff $x \in \text{Incl } S \setminus \text{Front } S$,
- (ii) $S = S(Q, E - y) = S(Q, E + y)$ iff $y \in \text{Excl } S \setminus \text{Back } S$.

Proof. We show (i), omitting the proof of (ii), which is analogous. By the Combinatorial KKT Conditions 4.3, we have $\text{Front } S \subseteq Q \subseteq \text{Incl } S$. Now $x \in \text{Incl } S \setminus \text{Front } S$ is equivalent to $\text{Front } S \subseteq Q - x$ and $Q + x \subseteq \text{Incl } S$, which implies that $Q - x$ and $Q + x$ belong to the same interval $[\text{Front } S, \text{Incl } S]$. The claim follows. \square

5.2. Pairing lemmas. Next, we prove two key technical lemmas that will facilitate the construction of discrete gradients proving our collapsibility results. Let $E \subseteq F \subseteq X$ and consider a simplex Q whose spheres $S(Q, E)$ and $S(Q, F)$ both exist but are different. We show that there is a point in $F \setminus E$ such that adding the point to Q or removing it from Q affects neither of the spheres.

Lemma 5.5 (First Simplex Pairing Lemma). *Let $E \subseteq F \subseteq X$ and $Q \in \text{Del}(X, F)$ with $S(Q, E) \neq S(Q, F)$. Then there exists a point $x \in F \setminus E$ such that*

- (i) $S(Q - x, E) = S(Q + x, E)$,
- (ii) $S(Q - x, F) = S(Q + x, F)$.

Proof. To construct the point in question, we write $S = S(Q, E)$ and $T = S(Q, F)$. By the Combinatorial KKT Conditions 4.3, we have $T = S(Q, \text{Back } T)$. By assumption we have $S \neq T$, and because $E \subseteq F$, the sphere S is smaller than T . It can therefore not exclude all points of $\text{Back } T$, and we let x be any point in $\text{Back } T \setminus \text{Excl } S$. Clearly $x \in F \setminus E$. Finally, we apply the first claim in the Same Sphere Lemma 5.4 twice to get the claimed relations. First we use $x \notin \text{Excl } S$ to get $x \in \text{Incl } S \setminus \text{Front } S$ so applying the lemma gives $S(Q - x, E) = S(Q + x, E)$ as claimed in (i). Second we use $x \notin \text{Front } T$ to get $x \in \text{Incl } T \setminus \text{Front } T$ so applying the lemma gives $S(Q - x, F) = S(Q + x, F)$ as claimed in (ii). \square

We note that the set $\text{Back } T \setminus \text{Excl } S$ and the point x in this set selected for Q in the above proof work for both $Q - x$ and for $Q + x$. In other words, substituting $Q - x$ or $Q + x$ for Q in the First Simplex Pairing Lemma 5.5 does not affect the claimed relations, and we can consistently select the same point x for both $Q - x$ and $Q + x$. The following lemma makes this observation precise.

Lemma 5.6 (First Consistent Pairing Lemma). *Assuming E is a proper subset of F , there is a map*

$$\varphi: \{Q \in \text{Del}(X, F) \mid S(Q, E) \neq S(Q, F)\} \rightarrow F \setminus E$$

such that $x = \varphi(Q)$ satisfies the properties of the First Simplex Pairing Lemma 5.5, and $x = \varphi(Q - x) = \varphi(Q + x)$.

Proof. To define the map φ , we let x_1, x_2, \dots, x_m be an arbitrary but fixed ordering of the vertices in X . Let $Q \in \text{Del}(X, F)$ be a simplex, and let $S = S(Q, E)$ and $T = S(Q, F)$. Now consider the vertex $x = x_i \in \text{Back } T \setminus \text{Excl } S$ with the smallest index in the chosen ordering. This vertex x satisfies the properties of the First Simplex Pairing Lemma 5.5, as shown in its proof. The choice of this vertex depends only on the two spheres $S = S(Q, E)$ and $T = S(Q, F)$, and by the First Simplex Pairing Lemma 5.5, using either $Q - x$ or $Q + x$ in place of Q yield this same pair of spheres. Defining $\varphi(Q) = x$, we conclude $x = \varphi(Q - x) = \varphi(Q + x)$. \square

Consider next a simplex that belongs to the selective Delaunay complex for E but not for F . We show that there exists a point x that has properties similar to those established in the First Simplex Pairing Lemma 5.5.

Lemma 5.7 (Second Simplex Pairing Lemma). *Let $E \subseteq F \subseteq X$ and let Q be a simplex in $\text{Del}(X, E)$ but not in $\text{Del}(X, F)$. Then there exists a point $x \in F \setminus E$ such that*

- (i) $S(Q - x, E) = S(Q + x, E)$,
- (ii) both $Q - x$ and $Q + x$ are not in $\text{Del}(X, F)$.

Proof. To construct the point in question, we write $F_Q = F \cap \text{Excl } S(Q, E)$ and note that $S = S(Q, E) = S(Q, F_Q)$ by the Combinatorial KKT Conditions 4.3. In particular, $Q \in \text{Del}(X, F_Q)$. Let $A \subseteq F$ be a subset of the points F and $x \notin A$ satisfying $F_Q \subseteq A \subseteq A + x \subseteq F$ such that Q belongs to $\text{Del}(X, A)$ but not to $\text{Del}(X, A + x)$. It is clear that such A and x exist. Since $x \notin \text{Excl } S$, we have $x \in \text{Incl } S \setminus \text{On } S \subseteq \text{Incl } S \setminus \text{Front } S$. Applying the Same Sphere Lemma 5.4, we get $S = S(Q - x, E) = S(Q + x, E)$, as claimed in (i). Similarly, $Q \in \text{Del}(X, A) \setminus \text{Del}(X, A + x)$ implies that $x \notin \text{Excl } S(Q, A)$, and we also get $S(Q - x, A) = S(Q + x, A)$, which will be useful shortly. In particular, note that $Q - x, Q + x$ both belong to $\text{Del}(X, A)$.

To see (ii), we note that $Q \notin \text{Del}(X, A + x)$ by assumption and therefore also $Q + x \notin \text{Del}(X, A + x)$. Since $A + x \subseteq F$, this implies the second relation in (ii). To derive a contradiction, we assume $Q - x \in \text{Del}(X, F)$ or, by implication, $Q - x \in \text{Del}(X, A + x)$. Using $S(Q - x, A) = S(Q + x, A)$, we get $x \notin \text{Excl } S(Q - x, A)$. Hence, x lies inside the sphere $S(Q - x, A)$ and we have $S(Q - x, A + x) \neq S(Q - x, A)$. On the other hand, since $Q + x \notin \text{Del}(X, A + x)$, we know that $x \notin \text{Incl } S(Q - x, A + x)$. Applying the second claim in the Same Sphere Lemma 5.4, we get $S(Q - x, A + x) = S(Q - x, A)$, a contradiction to the above. We thus conclude that $Q - x, Q + x$ both do not belong to $\text{Del}(X, A + x)$ and therefore also not to $\text{Del}(X, F)$. \square

Similar to above, the set A and the point $x \notin A$ selected for Q in the above proof work for both $Q - x$ and for $Q + x$, and we can consistently select the same point x for both $Q - x$ and $Q + x$.

Lemma 5.8 (Second Consistent Pairing Lemma). *Assuming E is a proper subset of F , there is a map*

$$\psi: \text{Del}(X, E) \setminus \text{Del}(X, F) \rightarrow F \setminus E$$

such that $x = \psi(Q)$ satisfies the properties of the Second Simplex Pairing Lemma 5.7, and $x = \psi(Q - x) = \psi(Q + x)$.

Proof. To define ψ , we let x_1, x_2, \dots, x_m be an arbitrary but fixed ordering of the vertices in X . Let $Q \in \text{Del}(X, E)$ be a simplex considered in the Second Simplex Pairing Lemma 5.7 and recall that $F_Q = F \cap \text{Excl } S(Q, E)$. There is a unique index j such that $A = F_Q \cup (F \cap \{x_1, x_2, \dots, x_{j-1}\})$ and $x = x_j$ satisfy the criteria spelled out in the proof of the Second Simplex Pairing Lemma 5.7; concretely, we have $Q \in \text{Del}(X, A) \setminus \text{Del}(X, A + x)$ and $x \in F \setminus E$. We use this choice of vertex to define $\psi(Q) = x_j$.

Note that the choice of F_Q depends only on $S(Q, E)$, and since $S(Q - x, E) = S(Q + x, E)$, we get the same A and x for $Q - x$ as well as for $Q + x$. We also have $Q - x, Q + x \in \text{Del}(X, A) \setminus \text{Del}(X, A + x)$, as pointed out in the proof of the Second Simplex Pairing Lemma 5.7, so we conclude that $x = \psi(Q - x) = \psi(Q + x)$, as claimed. \square

5.3. Collapsing. We are now ready to prove two collapsibility results for selective Delaunay complexes. They will imply the main results of this paper.

Theorem 5.9 (Selective Delaunay Collapsing Theorem). *Let X be a finite set of possibly weighted points in general position in \mathbb{R}^n , and let $E \subseteq F \subseteq X$. Then*

$$(28) \quad \text{Del}_r(X, E) \searrow \text{Del}_r(X, E) \cap \text{Del}(X, F) \searrow \text{Del}_r(X, F)$$

for every $r \in \mathbb{R}$.

Proof. We show that both collapses are induced by discrete gradients constructed with the help of the two Simplex Pairing Lemmas. We first prove the second collapse, $\text{Del}_r(X, E) \cap \text{Del}(X, F) \searrow \text{Del}_r(X, F)$. Let V_E and V_F be the generalized discrete gradients of the radius functions $s_E: \text{Del}(X, E) \rightarrow \mathbb{R}$ and $s_F: \text{Del}(X, F)$. By the Sum Refinement Lemma 5.3, the function $s_E + s_F: \text{Del}(X, F) \rightarrow \mathbb{R}$ is a generalized discrete Morse function, and its generalized discrete gradient is

$$(29) \quad W = \{I \cap J \mid I \in V_E, J \in V_F, I \cap J \neq \emptyset\}.$$

For any simplex Q that belongs to $\text{Del}_r(X, E) \cap \text{Del}(X, F)$ but not to $\text{Del}_r(X, F)$, the sphere $S(Q, E)$ has radius at most r but $S(Q, F)$ has radius larger than r . The set of such simplices is partitioned by a subset of the intervals in W . Since in particular $S(Q, E) \neq S(Q, F)$, the First Simplex Pairing Lemma 5.5 implies that this partition contains no singular intervals. By the Generalized Gradient Collapsing Theorem 2.2, W induces the collapse $\text{Del}_r(X, E) \cap \text{Del}(X, F) \searrow \text{Del}_r(X, F)$. Note that the pairs

$$(30) \quad P_0 = \{\{Q - x, Q + x\} \mid Q \subseteq X, x \in X, \varphi(Q) = x\}$$

defined using the map φ from the First Simplex Pairing Lemma 5.5 yield a vertex refinement of the generalized gradient W .

We next prove the first collapse, $\text{Del}_r(X, E) \searrow \text{Del}_r(X, E) \cap \text{Del}(X, F)$, using the pairs obtained from the Second Simplex Pairing Lemma 5.7 to partition the complement, $\text{Del}(X, E) \setminus \text{Del}(X, F)$. Revisiting the construction of ψ , we fix a total order x_1, x_2, \dots, x_m on X and define

$$(31) \quad K_0 = \text{Del}(X, E),$$

$$(32) \quad K_i = K_{i-1} \setminus \{Q \subseteq X \mid \psi(Q) = x_i\}.$$

We thus get a filtration $K_m \subseteq K_{m-1} \subseteq \dots \subseteq K_0$ that starts with $\text{Del}(X, F)$ and ends with $\text{Del}(X, E)$. By the Vertex Gradient Lemma 5.1, the pairs

$$(33) \quad P_i = \{\{Q - x_i, Q + x_i\} \mid Q \subseteq X, \psi(Q) = x_i\}, \quad i = 1, \dots, m,$$

give rise to a discrete gradient, V_i , and since V_i partitions $K_i \setminus K_{i+1}$ into pairs, it induces a collapse $K_i \searrow K_{i+1}$ by the Gradient Collapsing Theorem 2.1. The union of all such sets of pairs,

$$(34) \quad \bigcup_{i=1}^m P_i = \{\{Q - x, Q + x\} \mid Q \subseteq X, x \in X, v(Q) = x\},$$

forms a partition of $\text{Del}(X, E) \setminus \text{Del}(X, F)$ and, by applying the Gradient Composition Lemma 5.2 inductively, yields a gradient on $\text{Del}(X, E)$ inducing the collapse $\text{Del}(X, E) \searrow \text{Del}(X, F)$. \square

We remark that the pairs of the First Simplex Pairing Lemma 5.5 and the Second Simplex Pairing Lemma 5.7 can be combined according to the Gradient Composition Lemma 5.2. The result is a single discrete gradient V , with the pairs $\bigcup_{i=0}^m P_i$, which induce both collapses, for any choice of r simultaneously.

5.4. The collapsing sequence. We are now ready to state and prove the collapsing sequence, according to which the Čech, the Čech–Delaunay, the Delaunay, and the Wrap complexes – all for the same parameter $r \in \mathbb{R}$ – are simple-homotopy equivalent. The relation between the first three complexes follows directly from the

Selective Delaunay Collapsing Theorem 5.9, and it is not difficult to expand the relation to include the Wrap complex.

Theorem 5.10 (Čech–Delaunay Collapsing Theorem). *Let X be a finite set of possibly weighted points in general position in \mathbb{R}^n . Then*

$$(35) \quad \check{C}ech_r(X) \searrow \text{Del}\check{C}ech_r(X) \searrow \text{Del}_r(X) \searrow \text{Wrap}_r(X)$$

for every $r \in \mathbb{R}$.

Proof. Setting $E = \emptyset$ and $F = X$ in the Selective Delaunay Collapsing Theorem 5.9, we get the first two relations. To see the third, recall that $\text{Wrap}_r(X) \subseteq \text{Del}_r(X)$ for every $r \in \mathbb{R}$. To show that the latter complex collapses onto the former complex, we consider the gradient V_X of the Delaunay radius function $s_X : \text{Del}(X) \rightarrow \mathbb{R}$. Each interval in V_X is either disjoint of $\text{Del}_r(X)$ or a subset of $\text{Del}_r(X)$, and similarly for $\text{Wrap}_r(X)$. Moreover, by construction all critical simplices of $\text{Del}_r(X)$ are also contained in $\text{Wrap}_r(X)$. It follows that $\text{Del}_r(X) \setminus \text{Wrap}_r(X)$ is the disjoint union of non-singular intervals in V_X , and the Generalized Gradient Collapsing Theorem 2.2 implies $\text{Del}_r(X) \searrow \text{Wrap}_r(X)$. \square

Independent of r , all collapses are induced by the same discrete gradient constructed in the Selective Delaunay Collapsing Theorem 5.9.

6. CONSEQUENCES

Next, we discuss two structural results implied by our collapsing sequence. The first concerns the persistent homology of the filtrations obtained by letting $s = r^2$ range over \mathbb{R} , while the second uses selective Delaunay complexes to compare Delaunay complexes of different point sets.

6.1. Naturality and persistence. Regarding a filtration as a diagram of topological spaces connected by inclusions, a *natural transformation* from a filtration $(K_t)_{t \in \mathbb{R}}$ to another filtration $(L_t)_{t \in \mathbb{R}}$ is a family of continuous maps $K_t \rightarrow L_t$ such that the diagram

$$\begin{array}{ccc} K_r & \longrightarrow & L_r \\ \downarrow & & \downarrow \\ K_t & \longrightarrow & L_t \end{array}$$

commutes for all $r \leq t$. The *persistent homology* of $(K_t)_{t \in \mathbb{R}}$ is the diagram of homology groups $H_*(K_t)$ connected by the homomorphisms induced by the inclusions $K_r \hookrightarrow K_t$ for $r \leq t$. Since homology is a functor, it sends a natural transformation of filtrations to a natural transformation of their persistent homology. By the Čech–Delaunay Collapsing Theorem 5.10, the diagram

$$\begin{array}{ccccccc} \text{Wrap}_r(X) & \hookrightarrow & \text{Del}_r(X) & \hookrightarrow & \text{Del}\check{C}ech_r(X) & \hookrightarrow & \check{C}ech_r(X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Wrap}_t(X) & \hookrightarrow & \text{Del}_t(X) & \hookrightarrow & \text{Del}\check{C}ech_t(X) & \hookrightarrow & \check{C}ech_t(X) \end{array}$$

commutes for all $r \leq t$. The horizontal inclusion maps in this diagram correspond to the collapses of the Čech–Delaunay Collapsing Theorem 5.10. This means that

for any two of the four filtrations, the inclusion maps constitute a natural transformation, which is a simple-homotopy equivalence at each filtration index. As a consequence, we have the following implication on the persistent homology of the filtrations.

Corollary 6.1 (Persistence Isomorphism Corollary). *The Čech, Delaunay–Čech, Delaunay, and Wrap filtrations have isomorphic persistent homology.*

It should be clear that this corollary extends to the filtrations of complexes considered in the Selective Delaunay Collapsing Theorem 5.9.

6.2. Connecting Delaunay complexes for different point sets. We now describe an application of the Selective Delaunay Collapsing Theorem 5.9 that highlights the selective Delaunay complexes as interesting objects in their own right.

Assume we are given two finite point sets X and Y in \mathbb{R}^n , and denote the corresponding unions of balls of radius r by $B_r(X)$ and $B_r(Y)$. We think of X as a geometric approximation of Y , or of both as different approximations of some compact space. We are interested in the homomorphisms in homology induced by the inclusions $B_r(X), B_r(Y) \hookrightarrow B_r(X \cup Y)$. Ideally, the induced homomorphisms are isomorphisms for an appropriate choice of r . However, this cannot always be achieved and no such choice of r may exist, even if the Hausdorff distance $\delta = d_H(X, Y)$ is small. Nevertheless, the induced homomorphisms constitute natural transformations that can be thought of as approximate isomorphisms, up to a shift by δ in the index r of the diagrams. Known as δ -interleavings [6, 16], they translate the geometric closeness of X and Y into a structural similarity of their persistent homology. The persistent homology is described uniquely up to isomorphism by a collection of intervals, called the *persistence barcode*, and indeed the mentioned homomorphisms further induce a matching between the persistence barcodes of $(B_r(X))_{r \in \mathbb{R}}$ and $(B_r(Y))_{r \in \mathbb{R}}$ that makes this similarity explicit [6]. The relevance of interleavings motivates the interest in the above inclusion maps and their homology.

To construct the homology of these inclusion maps combinatorially, we observe that the maps $B_r(X), B_r(Y) \hookrightarrow B_r(X \cup Y)$ can be described up to isomorphisms on the level of Čech complexes, as both $\check{C}ech_r(X)$ and $\check{C}ech_r(Y)$ are subcomplexes of $\check{C}ech_r(X \cup Y)$; see the diagram below. The situation is different for Delaunay complexes because there is no canonical simplicial map $Del_r(X) \rightarrow Del_r(X \cup Y)$ corresponding to the inclusion $X \hookrightarrow X \cup Y$. To cope, we use selective Delaunay complexes and construct a zigzag of inclusions connecting the two Delaunay complexes $Del_r(X)$ and $Del_r(Y)$; see the last two rows of the following diagram.

$$\begin{array}{ccccc}
 \check{C}ech_r(X) & \hookrightarrow & \check{C}ech_r(X \cup Y) & \longleftarrow & \check{C}ech_r(Y) \\
 \uparrow & & \nearrow \simeq & & \nwarrow \simeq \\
 \simeq \uparrow & & Del_r(X \cup Y, X) & & Del_r(X \cup Y, Y) \\
 \uparrow & \searrow & \nwarrow \simeq & \nearrow \simeq & \uparrow \\
 Del_r(X) & & Del_r(X \cup Y) & & Del_r(Y)
 \end{array}$$

First note that $Del_r(X) = Del_r(X, X)$ is a subcomplex of $Del_r(X \cup Y, X)$, because adding Y does not impose any constraints on how the points in X connect

to each other. Moreover, $\text{Del}_r(X \cup Y) = \text{Del}_r(X \cup Y, X \cup Y)$ is a subcomplex of $\text{Del}_r(X \cup Y, X)$, because the former has a larger set of excluded points than the latter. By the Selective Delaunay Collapsing Theorem 5.9, $\text{Del}_r(X \cup Y, X)$ collapses to $\text{Del}_r(X \cup Y)$. Similar relations hold if we swap X and Y .

From this diagram, one can see that the inclusions $\check{\text{Cech}}_r(X) \hookrightarrow \check{\text{Cech}}_r(X \cup Y)$ are naturally homotopy-equivalent to the inclusions $\text{Del}_r(X) \hookrightarrow \text{Del}_r(X \cup Y, X)$. In particular, the natural transformation $H_*(\check{\text{Cech}}_r(X) \hookrightarrow \check{\text{Cech}}_r(X \cup Y))$ is isomorphic to $H_*(\text{Del}_r(X) \hookrightarrow \text{Del}_r(X \cup Y, X))$. Furthermore, the discrete gradient from the Selective Delaunay Collapsing Theorem 5.9 inducing the collapse $\text{Del}_r(X \cup Y, X) \searrow \text{Del}_r(X \cup Y)$ can be used to construct the induced isomorphism $H_*(\text{Del}_r(X \cup Y, X)) \rightarrow H_*(\text{Del}_r(X \cup Y))$ on the level of cycles, see [26, Sections 7 and 8].

7. DISCUSSION

The main result of this paper is the construction of collapses from the Čech to the Delaunay–Čech to the Delaunay and finally to the Wrap complex of a finite set of possibly weighted points in general position in \mathbb{R}^n . This is achieved by finding a common refinement of the generalized discrete gradients of the Čech and Delaunay radius functions into a generalized discrete gradient with the same set of critical simplices. While the collapse of the Delaunay–Čech to the Delaunay complex is induced by a canonical generalized gradient, the construction of the collapse from the Čech to the Delaunay–Čech complex required the choice of a total order. Is there an alternative proof that does not rely on such a choice? The hope is that a natural such construction would reveal some of the constraints on the collapse imposed by the geometry of the data.

We remark that the proof in this paper makes essential use of the general position assumption. Can the generalized discrete Morse theory be further generalized so that this assumption is no longer necessary?

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