Revisiting Alexander Duality with Tessellations and Mosaics

Ranita Biswas
IST Austria (Institute of Science and Technology Austria), Klosterneuburg, Austria
ranita.biswas@ist.ac.at

Sebastiano Cultrera di Montesano
IST Austria (Institute of Science and Technology Austria), Klosterneuburg, Austria
sebastiano.cultrera@ist.ac.at

Herbert Edelsbrunner
IST Austria (Institute of Science and Technology Austria), Klosterneuburg, Austria
herbert.edelsbrunner@ist.ac.at

Morteza Saghafian
Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran
morteza.saghafian65@student.sharif.edu

Abstract
The Voronoi tessellation in $\mathbb{R}^d$ is defined by locally minimizing the power distance to given weighted points. Symmetrically, the Delaunay mosaic can be defined by locally maximizing the negative power distance to other such points. We prove that the average of the two piecewise quadratic functions is piecewise linear, and that all three functions have the same critical points and values. The proof relies on tools from discrete and continuous mathematics, which we develop in parallel and combine to present a geometric view of Alexander duality.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases Voronoi tessellations, Delaunay mosaics, PL functions, radius functions, discrete Morse theory, Alexander duality.

Funding This project has received funding from the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme, grant no. 788183, from the Wittgenstein Prize, Austrian Science Fund (FWF), grant no. Z 342-N31, and from the DFG Collaborative Research Center TRR 109, ‘Discretization in Geometry and Dynamics’, Austrian Science Fund (FWF), grant no. I 02979-N35.

Lines 476

1 Introduction

The starting point for the work reported in this paper is the role of the general position assumption in the construction of Delaunay mosaics, and more specifically of their radius functions. Without general position assumption, the mosaics are not simplicial and the radius functions are not discrete Morse. How do we relax the theory to allow for non-generic data? Related to this question is the symmetry between Voronoi tessellations and Delaunay mosaics that appears when we introduce weights, and non-generic data is essential to realize this symmetry. In this paper, we weave the two strands of inquiry into a discrete geometric approach to Alexander duality in algebraic topology. Along the way, we prove new results on Voronoi tessellations and Delaunay mosaics and related continuous and discrete functions.

The Voronoi tessellation and the dual Delaunay mosaic are classic topics in discrete geometry and go back at least to the seminal papers by Voronoi [19] and by Delaunay [3]. The radius function on the Delaunay mosaic was first introduced in [7], along with its sublevel
sets, which are the alpha shapes of the given points. Three-dimensional alpha shapes have found ample applications in shape modeling \cite{9, 12, 14} and in the analysis of biomolecules [8]. The connection to discrete Morse theory, as introduced by Forman [10] and generalized by Freij [11], was exploited for the purpose of surface reconstruction in [5]; see also [18]. We formulate the extension of discrete Morse theory needed to encompass radius functions on non-generic Delaunay mosaics and thus facilitate their application when non-generic position is essential, such as in crystallography.

Non-general position of points with weights is also essential when we interpret a Voronoi tessellation as a Delaunay mosaic and vice versa. By this we do not mean to take the tessellation to its dual mosaic but rather to construct a different set of weighted points whose Delaunay mosaic is essentially identical to the Voronoi tessellation of the first set. Viewing the tessellation and the mosaic as projections of the boundary complexes of convex polytopes, this construction follows by observing that the polar of a convex polyhedron is still a convex polyhedron. Notwithstanding, we get new insights into a much studied subject by looking into the details of this symmetry. We mention three such results, the first of which is combinatorial.

Let \( \mu \neq \nu \) be cells of a Voronoi tessellation, and write \( \mu^*, \nu^* \) for the corresponding cells in the dual Delaunay mosaic. Then \( \text{int} \, \mu \cap \nu^* \neq \emptyset \) implies \( \text{int} \, \nu \cap \mu^* = \emptyset \).

The second result is about the piecewise quadratic functions, \( \text{vor}, \text{del} : \mathbb{R}^d \to \mathbb{R} \), whose pieces define the Voronoi tessellation and the dual Delaunay mosaic, respectively. Choosing opposite signs, the average defined by \( \text{sd}(x) = \frac{1}{2} [\text{vor}(x) + \text{del}(x)] \) is piecewise linear. We use the above combinatorial lemma to prove the following result.

Extending concepts from smooth Morse theory to piecewise quadratic and piecewise linear functions, we show that \( \text{vor}, \text{del}, \text{sd} : \mathbb{R}^d \to \mathbb{R} \) have the same critical points and the same critical values.

Discretizing the two piecewise quadratic functions, we get the radius functions on the Voronoi tessellation and Delaunay mosaic, \( \text{vor} : \text{Vor}(X) \to \mathbb{R} \) and \( \text{del} : \text{Del}(X) \to \mathbb{R} \). For generic collections of weighted points, they are discrete Morse but not so for non-generic collections.

Shedding light on the relation between the sub- and superlevel sets of these discrete functions, we prove that the underlying spaces of \( \text{del}^{-1}(-\infty, t) \) and \( \text{vor}^{-1}(t, \infty) \) are disjoint for all non-critical values \( t \).

The channel between the two underlying spaces is free of critical points, the level set of the piecewise linear function, \( \text{sd}^{-1}(t) \), splits it into two halves, and each half deformation retracts to the respective underlying space. Keeping track of the homology of the complementary subcomplexes, we get the basic relation of Alexander duality.

**Outline.** Section 2 presents background in discrete geometry. Section 3 studies the piecewise quadratic functions that define the Voronoi tessellation and Delaunay mosaic as well as their average, which is piecewise linear. Section 4 considers the corresponding discrete functions and their sub- and superlevel sets. Section 5 introduces a framework for monotonic functions and proves Alexander duality for complementary subcomplexes of the Voronoi tessellation and the Delaunay mosaic. Section 6 concludes the paper. Appendix A introduces Voronoi tessellations, Delaunay mosaics, and discrete Morse theory. Appendix B contains all proofs and extra illustrations.

### 2 Background

The central geometric constructs in this paper are the Voronoi tessellation and the dual Delaunay mosaic, which we introduce in Appendix A.1 for points with real weights in
Euclidean space. In addition, we review the standard polarity transform and its relation to the tessellation and the mosaic.

2.1 Polarity

We are interested in the most elementary version of polarity, which relates a point \( u = (u_1, u_2, \ldots, u_{d+1}) \) in \( \mathbb{R}^{d+1} \) with the hyperplane of points \( x \in \mathbb{R}^{d+1} \) that satisfy \( x_{d+1} = u_1 x_1 + \ldots + u_d x_d - u_{d+1} \). We denote this hyperplane by \( u^* \), we call \( u^* \) the polar hyperplane of \( u \), and we call \( u = (u^*)^* \) the polar point of \( u^* \). Importantly, the transform preserves incidences, that is: \( u \in v^* \) iff \( v \in u^* \) for any two points \( u, v \in \mathbb{R}^{d+1} \). The transform also preserves sidedness, which we introduce by saying that \( u \) lies below, on, above \( v^* \) if \( u_{d+1} \) is less than, equal to, greater than \( v_1 u_1 + \ldots + v_d u_d - v_{d+1} \). Specifically, \( u \) is above \( v^* \) iff \( v \) is above \( u^* \), and together with the preservation of incidences, this implies \( u \) is below \( v^* \) iff \( v \) is below \( u^* \).

To express the relation between the Voronoi tessellation and the Delaunay mosaic in terms of the polarity transform, we map every weighted point in \( \mathbb{R}^d \times \mathbb{R} \) to an unweighted point and its polar hyperplane in \( \mathbb{R}^{d+1} \). Given \( a \in \mathbb{R}^d \times \mathbb{R} \), we represent the two by a constant map and an affine map, \( \hat{a}, \hat{a}: \mathbb{R}^d \to \mathbb{R} \):

\[
\hat{a}(x) = \frac{1}{2} \| pt(a) \|^2 - \frac{1}{2} wt(a), \tag{1}
\]

\[
\hat{a}(x) = (pt(a), x) - \hat{a}(pt(a)), \tag{2}
\]

so that \( (pt(a), \hat{a}(pt(a))) \) is the point and \( \text{img} \, \hat{a} = \hat{a}(\mathbb{R}^d) \) is the hyperplane. It is not difficult to verify that these maps preserve incidences and sidedness as required:

\[
\text{sgn}[\hat{a}(pt(a)) - \hat{b}(pt(a))] = \text{sgn}[\hat{b}(pt(b)) - \hat{a}(pt(b))], \tag{3}
\]

in which \( \text{sgn}[r] \) is \(-1, 0, 1\) if \( r \) is negative, zero, positive. Returning to the connection with the weighted points, we introduce the paraboloid map, \( \varpi: \mathbb{R}^d \to \mathbb{R} \), defined by \( \varpi(x) = \frac{1}{2} \| x \|^2 \).

The zero-set of \( \hat{a} - \varpi \) consists of the points \( x \in \mathbb{R}^d \) for which

\[
\varpi(x) - \hat{a}(x) = -\frac{1}{2} \| x - pt(a) \|^2 + \frac{1}{2} wt(a) = -\frac{1}{2} \pi_a(x) \tag{4}
\]

vanishes. In words, the zero-set of \( \hat{a} - \varpi \) is also the zero-set of \( \pi_a \), namely the sphere with center \( pt(a) \) and squared radius \( wt(a) \). Consider now the constant function \( \hat{b}: \mathbb{R}^d \to \mathbb{R} \) defined by \( \hat{b}(pt(b)) = \hat{a}(pt(b)) \). The corresponding weighted point is \( b = (pt(b), wt(b)) \), with

\[
wt(b) = \| pt(b) \|^2 - 2\hat{a}(pt(b)) = \| pt(b) - pt(a) \|^2 - wt(a) \tag{5}
\]

In words, the sum of the two weights equals the squared Euclidean distance between the two centers. Since the weights are squared radii, Pythagoras’ theorem implies that the two spheres that are the zero-sets of \( \pi_a \) and \( \pi_b \) intersect orthogonally. It will be convenient to use the same suggestive language when one of the weights is non-positive. We get a special case when \( pt(a) = pt(b) \). Then \( \| pt(a) - pt(b) \|^2 = 0 \) so \( wt(b) = -wt(a) \). Translating this back to the language of affine maps, we get

\[
\frac{1}{2} [\hat{a}(pt(a)) + \hat{a}(pt(a))] = \varpi(pt(a)) \tag{6}
\]

compare with (1) and (2).

Next, we generalize the relations between points and hyperplanes to collections \( A \subseteq \mathbb{R}^d \times \mathbb{R} \) whose projections to \( \mathbb{R}^d \) are injective and affinely independent. Write \( \text{sol}(A) \) for the set of points \( x \in \mathbb{R}^d \) that satisfy \( \hat{a}(x) = \hat{b}(x) \) for all \( a, b \in A \), and \( \text{flt}(A) \) for the affine hull of the \( pt(a), a \in A \). For example, if \( A = \{ a \} \), then \( \text{sol}(A) = \mathbb{R}^d \) and \( \text{flt}(A) = pt(a) \). Writing \( q + 1 = \# A \) and \( p = d - q \), we have
Revisiting Alexander Duality with Tessellations and Mosaics

Indeed, if all weights are zero, then \( \text{sol}(A) \) is the set of centers of spheres that pass through all points of \( A \). This set is a \( p \)-dimensional affine subspace of \( \mathbb{R}^d \) orthogonal to the \( q \)-dimensional affine hull of \( A \). When we adjust the weight of an \( a \in A \), this affine subspace does not change other than move parallel to its initial position. So \( \text{flt}(A) \) and \( \text{sol}(A) \) retain the two properties stated above.

In addition to the two affine subspaces, we introduce two affine functions, \( \bar{A} : \mathbb{R}^d \to \mathbb{R} \) and \( \check{A} : \mathbb{R}^d \to \mathbb{R} \), that generalize \( \check{a} \) and \( \bar{a} \) as defined in (1) and (2). Specifically, \( \bar{A} \) agrees with \( \check{a} \) at \( \text{pt}(a) \) for every \( a \in A \) and its restriction to \( \text{sol}(A) \) is constant. Similarly, \( \check{A} \) agrees with \( \bar{a} \) within \( \text{sol}(A) \) for every \( a \in A \) and its restriction to \( \text{flt}(A) \) is constant. Recall that \( y(A) = \text{sol}(A) \cap \text{flt}(A) \).

**Lemma 2.1 (Common Maximum).** Let \( A \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and affinely independent projection to \( \mathbb{R}^d \). Then \( y = y(A) \) is the common maximum of

- (i) the restriction of \( \bar{A} - \bar{w} \) to \( \text{flt}(A) \),
- (ii) the restriction of \( \check{A} - \check{w} \) to \( \text{sol}(A) \),
- (iii) the average, \( \frac{1}{2} [\bar{A} + \check{A}] - \check{w} \), and in this case the value of the maximum vanishes.

We note that (iii) implies that the graph of \( \frac{1}{2} [\bar{A} + \check{A}] \) is the unique hyperplane in \( \mathbb{R}^{d+1} \) that touches the graph of \( \check{w} \) in the point \( (y, \check{w}(y)) \).

### 2.2 Projection of Envelopes

Since the Voronoi tessellation is defined in terms of minimum power distance, it can equally well be defined in terms of maximum affine function values. Specifically, let \( \text{env} : \mathbb{R}^d \to \mathbb{R} \) be the upper envelope of the affine maps defined by \( \text{env}(x) = \max_{a \in B} \check{a}(x) \), and call the linear pieces of this envelope the faces of \( \text{env} \), it is not difficult to see that there is a bijection between the faces of \( \text{env} \) and the cells of \( \text{Vor}(B) \) such that every cell is the vertical projection of the corresponding face to \( \mathbb{R}^d \). This property was known already to Voronoi [19].

A similar construction exists for Delaunay mosaics, which is usually phrased in terms of the convex hull of the points \( \text{pt}(a), \check{a}(\text{pt}(a)) \) in \( \mathbb{R}^{d+1} \). We call a face of this convex polytope **lower** if there is a non-vertical hyperplane in \( \mathbb{R}^{d+1} \) such that the face lies in and the rest of the polytope above the hyperplane. It is not difficult to see that there is a bijection between the lower faces of this polytope and the cells of \( \text{Del}(B) \) such that every cell is the vertical projection of the corresponding lower face to \( \mathbb{R}^d \). In this paper, it is convenient to add arbitrarily steep “ramps” around the polytope whose vertical projections decompose the rest of \( \mathbb{R}^d \) into convex cells. In other words, we introduce \( \text{end} : \mathbb{R}^d \to \mathbb{R} \) as the upper envelope of all affine maps \( \bar{c} : \mathbb{R}^d \to \mathbb{R} \) that satisfy \( \bar{c}(x) \leq y \) for every point \( (x, y) \in \mathbb{R}^d \times \mathbb{R} \) of the polytope. Most of these maps are redundant, except for those whose graphs support facets and the ramps that support \( (d-1) \)-dimensional faces on the silhouette of the polytope. Then there is a set of weighted points \( C \subseteq \mathbb{R}^d \times \mathbb{R} \), possibly including points at infinity, whose projection to \( \mathbb{R}^d \) is locally finite such that \( \text{end}(x) = \max_{c \in C} \bar{c}(x) \). Now we have complete symmetry and can write \( \text{Del}(B) = \text{Vor}(C) \) as well as \( \text{Vor}(B) = \text{Del}(C) \). We call \( C \) the **polar set** of \( B \) and, symmetrically, \( B \) the polar set of \( C \).
3 Continuous Functions

In this section, we consider two piecewise quadratic functions, whose pieces define the Voronoi tessellation and its dual Delaunay mosaic. The main result is that these two functions and their piecewise linear average have the same critical points.

3.1 Piecewise Quadratic and Piecewise Linear Functions

Recall that $\text{env}, \text{end} : \mathbb{R}^d \rightarrow \mathbb{R}$ are piecewise linear convex functions. Comparing them with $\varpi$, we get two piecewise quadratic functions, $\text{vor}, \text{del} : \mathbb{R}^d \rightarrow \mathbb{R}$, and one piecewise linear function, $\text{sd} : \mathbb{R}^d \rightarrow \mathbb{R}$, defined by

\begin{align*}
\text{vor}(x) &= \varpi(x) - \text{env}(x), \\
\text{del}(x) &= \text{end}(x) - \varpi(x), \\
\text{sd}(x) &= \frac{1}{2}[\text{end}(x) - \text{env}(x)] = \frac{1}{2}[\text{del}(x) + \text{vor}(x)].
\end{align*}

As illustrated in Figure 1, $\text{del}$ dominates $\text{vor}$, which implies that their average, $\text{sd}$, is sandwiched between them. To prove this formally, we introduce the common subdivision of the tessellation and the mosaic, denoted $\text{Sd}(B)$, which consists of all cells $\gamma = \tau \cap \sigma^*$ with $\tau \in \text{Vor}(B)$ and $\sigma^* \in \text{Del}(B)$. Since $\tau$ and $\sigma^*$ are convex, so is $\gamma$. The restrictions of $\text{del}$ and $\text{vor}$ to $\gamma$ are quadratic, while the restriction of $\text{sd}$ to $\gamma$ is linear.

**Lemma 3.1 (Dominance).** Let $B \subseteq \mathbb{R}^d \times \mathbb{R}$ have an injective and locally finite projection to $\mathbb{R}^d$. Then $\text{del}(x) \geq \text{sd}(x) \geq \text{vor}(x)$ for every $x \in \mathbb{R}^d$.

The inequalities in Lemma 3.1 imply that the sublevel sets and the superlevel sets of the three functions are nested:

\begin{align*}
\text{del}^{-1}(-\infty, t] &\subseteq \text{sd}^{-1}(-\infty, t] \subseteq \text{vor}^{-1}(-\infty, t], \\
\text{del}^{-1}[t, \infty) &\supseteq \text{sd}^{-1}[t, \infty) \supseteq \text{vor}^{-1}[t, \infty).
\end{align*}

The sublevel set of $\text{del}$ and the superlevel set of $\text{vor}$, for same value $t$, are illustrated in Figure 2 together with the channel in between. We will see shortly that the three functions share the critical points, at which they all agree.
3.2 Two Auxiliary Lemmas

We need three auxiliary results to prove that the functions defined in (7), (8), (9) share the critical points, two of which will be presented in this subsection. The first result is a new combinatorial statement about Voronoi tessellations and Delaunay mosaics.

\[\text{Lemma 3.2 (Excluded Crossing).} \quad \text{Let } B \subseteq \mathbb{R}^d \times \mathbb{R} \text{ have an injective and locally finite projection to } \mathbb{R}^d, \text{ let } \mu \neq \nu \text{ be cells in } \text{Vor}(B) \text{ and note that } \mu^*, \nu^* \text{ are cells in } \text{Del}(B). \text{ If } \text{int } \mu \setminus \nu^* \neq \emptyset, \text{ then } \text{int } \nu \setminus \mu^* = \emptyset.\]

We remark that we take the interiors of \(\mu\) and \(\nu\) so that the two hypothesized intersection points are different. This detail is a crucial aspect of the proof. Indeed, it is possible to have \(\mu \setminus \nu^* \neq \emptyset\) and \(\nu \setminus \mu^* \neq \emptyset\): let \(\nu^*\) be a right-angled triangle in \(\mathbb{R}^2\) and \(\mu^*\) its longest edge. Then \(\nu\) is the circumcenter of the triangle, which lies on \(\mu^*\), and \(\mu\) has \(\nu\) as an endpoint.

Write \(S^{d-1}\) for the unit sphere in \(\mathbb{R}^d\). The second result is a geometric statement about the common intersection of hemispheres, which are closed subsets of \(S^{d-1}\) that are bounded by great-spheres of dimension \(d-2\). Note that a unit vector, \(e \in S^{d-1}\), defines both a point as well as a hemisphere, namely the one whose points \(y \in S^{d-1}\) satisfy \(\langle e, y \rangle \leq 0\).

\[\text{Lemma 3.3 (Hemispheres).} \quad \text{The common intersection of a collection of hemispheres of } S^{d-1} \text{ is either contractible or a } (p-1)\text{-dimensional great-sphere with } 0 \leq p \leq d.\]

3.3 Down and Up Links

The third result is a topological statement about vector fields defined by two convex polytopes, \(P, Q \subseteq \mathbb{R}^d\), whose dimensions are complementary, \(p = \dim P\) and \(q = \dim Q\) with \(p + q = d\), and whose affine hulls intersect in a single point. The product, \(P \times Q\), is a convex polytope of dimension \(d\). Its boundary is a topological \((d-1)\)-sphere that decomposes into a thickened \((p-1)\)-sphere and a thickened \((q-1)\)-sphere: \(\partial(P \times Q) = (\partial P \times Q) \cup (P \times \partial Q)\). Indeed, for every \(s \in \partial(P \times Q)\), there are unique points \(y \in P\) and \(z \in Q\) such that \(s = y + z\), and at least one of \(y\) and \(z\) belongs to the respective boundary. We are interested in \(\psi: \partial(P \times Q) \to S^{d-1}\) defined by mapping \(s = y + z\) to \(\psi(s) = \frac{1}{2}(y - z)\); see Figure 3 for an illustration. To study
ψ, we introduce the down link and up link of P and Q:

\[ d\text{Lk}(P,Q) = \{ s \in \partial(P \times Q) \mid \langle \psi(s), n(s) \rangle \leq 0 \} \]

(12)

\[ u\text{Lk}(P,Q) = \{ s \in \partial(P \times Q) \mid \langle \psi(s), n(s) \rangle \geq 0 \} \]

(13)

in which \( n(s) \) is the unit outward directed normal at \( s \). This normal is unique for every facet, which we recall is a face of dimension \( d - 1 \), but it is not unique for faces of dimension \( d - 2 \) or less. We remedy this difficulty by writing \( n(s) \) for the collection of normals that interpolate between the normals of the incident facets, and by including \( s \) in the down or up link if the respective inequality is satisfied for at least one vector in \( n(s) \). In the left panel of Figure 3, the down link consists of the left edge and the right edge of the product, while the up link consists of the remaining two edges. Both have the homotopy type of the 0-sphere.

In the right panel, the down link consists of three edges, which the up link containing the remaining, top edge. Both links are contractible. The important difference is that \( P \) and \( Q \) intersect in the left panel while they are disjoint in the right panel.

**Lemma 3.4 (Down and Up Link).** Let \( P, Q \subseteq \mathbb{R}^d \) be convex polyhedra with orthogonal affine hulls of complementary dimensions: \( p = \dim P \), \( q = \dim Q \), and \( p + q = d \). Then \( \text{int } P \cap \text{int } Q \neq \emptyset \) implies \( d\text{Lk}(P,Q) \simeq S^{p-1} \), \( u\text{Lk}(P,Q) \simeq S^{q-1} \), and \( P \cap Q = \emptyset \) implies that both links are contractible.

We remark that the above lemma avoids the borderline case, when the interiors of \( P \) and \( Q \) are disjoint but \( P \) and \( Q \) are not. Then the two links have the homotopy type of nearby generic configurations, and one of the two is contractible.

### 3.4 Lower and Higher Links

Since the continuous functions we study are not smooth, it is necessary to define what we mean by a critical point. We need a definition that is general enough to apply to piecewise linear and to piecewise quadratic functions. Letting \( f : \mathbb{R}^d \to \mathbb{R} \) be such a function and \( x \in \mathbb{R}^d \), we write \( S_r = S_r(x) \) for the \((d - 1)\)-sphere with radius \( r > 0 \) and center \( x \). Letting \( S^r \) contain all \( y \in S_r \) with \( f(y) \leq 0 \), we note that its homotopy type is the same for all sufficiently small radii. Fixing a sufficiently small \( \varepsilon > 0 \), we call \( S_{\varepsilon}^r \) the lower link of \( x \) and \( f \). Symmetrically, \( S_{\varepsilon}^r \) contains all points \( y \in S_r \) with \( f(y) \geq 0 \), and we call \( S_{\varepsilon}^r \) the higher link of \( x \) and \( f \). Writing \( d\text{Lk}(x,f) \) and \( u\text{Lk}(x,f) \) for the lower and higher links, we call \( x \) a non-critical point of \( f \) if at least one of the two links is contractible. All points with
We will prove that taking the minimum or maximum over all points of a cell, we turn the continuous functions of Section 3 into discrete functions. In particular, we introduce \( \text{vor}: \text{Vor}(B) \to \mathbb{R} \), where \( \text{vor} \) is the Voronoi tessellation, and \( \text{sd} \) is the squared radii of the balls of \( B \) centered at \( b \) for \( b \in B \) and \( c \in C \) such that \( \tau = \text{cell}(b) \) and \( \sigma^* = \text{cell}(c) \). Hence, \( \text{sd} \) is twice the gradient of \( \text{sd} \) at every point in \( \text{int} \gamma \). We use this insight to prove the main result of this section.

**Theorem 3.5 (Critical Points).** Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and locally finite projection to \( \mathbb{R}^d \). Then \( x \in \mathbb{R}^d \) is a critical point of \( \text{vor} \) iff it is a critical point of \( \text{del} \) iff it is a critical point of \( \text{sd} \), and its index is the same for all three functions.

## 4 Discrete Functions

Parallel to the continuous functions studied in Section 3, we introduce discrete functions: on the Voronoi tessellation, the Delaunay mosaic, and their common subdivision. The main new concept is the channel between complementing subcomplexes of the tessellation and the mosaic.

### 4.1 Min and Max Functions

Taking the minimum or maximum over all points of a cell, we turn the continuous functions of Section 3 into discrete functions. In particular, we introduce \( \text{vor}: \text{Vor}(B) \to \mathbb{R} \), where \( \text{vor} \) is the Voronoi tessellation, and \( \text{sd} \) is the squared radii of the balls of \( B \) centered at \( b \) for \( b \in B \) and \( c \in C \) such that \( \tau = \text{cell}(b) \) and \( \sigma^* = \text{cell}(c) \). Hence, \( \text{sd} \) is twice the gradient of \( \text{sd} \) at every point in \( \text{int} \gamma \). We use this insight to prove the main result of this section.
\[ \text{del}(B) \rightarrow \mathbb{R}, \text{ and } \text{sdx}, \text{sdn} : \text{Sd}(B) \rightarrow \mathbb{R} \text{ defined by} \]

\[ \text{vor}(\tau) = \max_{x \in \tau} \text{del}(x), \]
\[ \text{del}(\sigma^*) = \min_{x \in \sigma} \text{vor}(x), \]
\[ \text{sdn}(\gamma) = \min_{x \in \gamma} \text{sd}(x), \]
\[ \text{sdx}(\gamma) = \max_{x \in \gamma} \text{sd}(x). \]

We note that \text{vor} is defined in terms of \text{del} and \text{del} in terms of \text{vor}. This is not a mistake but motivated by our desire to remain consistent with the standard literature on alpha shapes, where \text{del} is the (squared) radius function; see [7, 9]. It is also possible to define \text{vor} in terms of \text{vor} and \text{del} in terms of \text{del}, which gives slightly different discrete functions with essentially the same properties. It will often be convenient to apply the discrete Voronoi and Delaunay functions to the common subdivision. Technically, these are different functions, \text{sdv}, \text{sdd} : \text{Sd}(B) \rightarrow \mathbb{R}, defined by \text{sdv}(\gamma) = \text{vor}(\tau) \text{ and } \text{sdd}(\gamma) = \text{del}(\sigma^*), in which } \gamma = \tau \cap \sigma^*.

### 4.2 Sub- and Superlevel Sets

Observe that for \text{del} and \text{sdx}, the value of a cell is larger than or equal to the values of its faces, and for \text{vor} and \text{sdn}, it is less than or equal to the values of its faces. It follows that the following sub- and superlevel sets are complexes:

\[ \text{Vor}^t(B) = \text{vor}^{-1}[t, \infty), \]
\[ \text{Del}^t(B) = \text{del}^{-1}(-\infty, t], \]
\[ \text{Sd}^t(B) = \text{sdn}^{-1}[t, \infty), \]
\[ \text{Sd}^t(B) = \text{sdx}^{-1}(-\infty, t]. \]

We extend (10) and (11) from the continuous to the discrete setting.

![Figure 4: The four discrete functions on the common subdivision, which dominate each other from left to right. All indicated sub- and superlevel sets are for the same value, \( t \).](image)

**Lemma 4.1 (Nested Spaces).** Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and locally finite projection to \( \mathbb{R}^d \). Then \( |\text{Del}^t(B)| \subseteq |\text{Sd}^t(B)| \) and \( |\text{Vor}^t(B)| \subseteq |\text{Sd}^t(B)| \).
Let \( t \in \mathbb{R} \) be a value different from \( sd(x) \) for all vertices \( x \) of \( Sd(B) \). Then \( Sd_t(B) \cap Sd^t(B) = \emptyset \), and similarly their underlying spaces are disjoint. Combining the two relations in Lemma 4.1, we therefore have \( |\text{Del}_t(B)| \cap |\text{Vor}^t(B)| = \emptyset \), which we illustrated in Figure 5.

On the other hand, if \( t \) is the value of a vertex, \( x \), then \( x \) belongs to \( Sd_t(B) \) as well as to \( Sd^t(B) \). If \( x \) is furthermore a critical point of \( sd \), then \( x \) belongs also to \( |\text{Del}_t(B)| \) and to \( |\text{Vor}^t(B)| \).

4.3 Channels

Since the sub- and superlevel sets of \( \text{del} \) and \( \text{vor} \) considered in Lemma 4.1 have disjoint underlying spaces, it makes sense to study the space in between. For each value \( t \in \mathbb{R} \), this is the underlying space of an open collection of cells in the common subdivision of the tessellation and the mosaic. For each cell \( \gamma = \tau \setminus \sigma^* \) in \( Sd(B) \), the relevant values are

\[
\begin{align*}
t_0(\gamma) &= sdv(\gamma) = \text{vor}(\tau), \\
t_1(\gamma) &= sdd(\gamma) = \text{del}(\sigma^*).
\end{align*}
\]

(25)

(26)

Moving from \(-\infty\) to \( \infty \) along the real numbers, \( \tau \) is dropped from \( \text{Vor}^t(B) \) at \( t = t_0(\gamma) \) and \( \sigma^* \) is added to \( \text{Del}_t(B) \) at \( t = t_1(\gamma) \). If \( \gamma \) is a critical cell of \( \text{vor} \) and \( \sigma^* = \tau^* \) is the corresponding critical cell of \( \text{del} \), then \( \gamma \) is a point that belongs to both underlying spaces at \( t = t_0(\gamma) = t_1(\gamma) \), and to exactly one of these underlying spaces for all other values of \( t \). For all other cells \( \gamma = \tau \cap \sigma^* \), Lemma 4.1 implies \( t_0(\gamma) < t_1(\gamma) \). In all cases, \( \gamma \) belongs to the space in between \( \text{Del}_t(B) \) and \( \text{Vor}^t(B) \) for all \( t_0(\gamma) < t < t_1(\gamma) \). More formally, we define the channel of \( B \) at \( t \):

\[
\text{Ch}_t(B) = \{ \gamma \mid \tau \notin \text{Vor}^t(B), \sigma^* \notin \text{Del}_t(B) \},
\]

(27)

in which \( \gamma = \tau \cap \sigma^* \); see Figure 6. This is the complement of the union of two subcomplexes of \( Sd(B) \) or, equivalently, the intersection of two complementary open sets:

\[
\begin{align*}
\text{Ch}_t(B) &= Sd(B) \setminus [sdd^{-1}(-\infty, t) \cup sdv^{-1}[t, \infty)] \\
&= sdd^{-1}(t, \infty) \cap sdv^{-1}(-\infty, t);
\end{align*}
\]

(28)

(29)
Recall that $sdd(\gamma)$ is at least the maximum and $sdv(\gamma)$ is at most the minimum $sd(x)$ over all points $x \in \gamma$. It follows that $sd^{-1}(t)$ is disjoint of the underlying spaces of $sdd^{-1}(\mathbb{R}, t]$ and $sdv^{-1}(t, \mathbb{R})$, unless $t$ is a critical value of $sd$, in which case the corresponding critical points belong to all three. Hence, $sd^{-1}(t)$ is contained in the underlying space of the channel, unless $t$ is a critical value, in which case the level set passes through the corresponding critical points. We state this insight together with a straightforward related property more formally.

\begin{theorem}[Split Channel] Let $B \subseteq \mathbb{R}^d \times \mathbb{R}$ have an injective and locally finite projection to $\mathbb{R}^d$, and let $t \in \mathbb{R}$ be different from all critical values of $sd$. Then

\[ sd^{-1}(t) \subseteq \{|Ch(B)|, \]

\[ sd^{-1}(t) \text{ is an orientable $(d-1)$-manifold.} \]

On the other hand, if $t$ is a critical value of $sd$, then both of these properties are violated, but only at the corresponding critical points, where the level set and the channel go through topological reorganization.

\section{Internal Retraction}

In this section, we deformation retract the channel, both to the corresponding Voronoi complex and Delaunay complex. We begin by studying the structure of the steps, which we classify depending on their effect on the homology of the sublevel set.

\subsection{Classification with Homology}

Recall the concepts introduced in Appendix A.2 and note that the step graph of a monotonic function defines a partial order. We can construct the complex by adding the steps one at a time according to a linear extension. To determine the effect of adding a step on the homology of the subcomplex, we compute its relative homology, as we now explain.

Let $J_0, J_1, \ldots, J_m$ be a linear extension of the partial order defined by the step graph of $f: K \rightarrow \mathbb{R}$. This order may or may not be consistent with the sublevel sets of $f$, in the sense that the corresponding values listed in the same order may or may not be sorted. Write $K_j = \bigcup_{0 \leq i < j} J_i$, note that $K_j$ is closed, and get $K_{j+1} = K_j \cup J_{j+1}$ by adding the next
step. To describe how the addition of \( J = J_{j+1} \) affects the homology of the complex, we consider the pair \((J, J)\), in which \( J = \text{cl} J \) is the closure and \( \bar{J} = J \setminus J \). Since \( K_{j+1} \sqcup J \) is a complex, we have \( \bar{J} = K_j \cap J \), which is the intersection of two complexes and therefore a complex itself. We are interested in the relative homology of \((\bar{J}, J)\), since it will allow us to deduce the homology of \( K_{j+1} \) from that of \( K_j \). Fixing a field to compute homology, we classify the steps according to the ranks of the relative homology groups, which we denote as \( \beta_p = \text{rank} H_p(\bar{J}, J) \) for all dimensions \( p \).

**Definition 5.1 (Critical Steps).** We call \( J \) a non-critical step of \( f \) if \( \beta_p = 0 \) for all \( p \geq 0 \). Otherwise, \( J \) is a critical step. It is a simple critical step if \( \beta_p = 1 \) for all ranks except in a single dimension, \( p \), in which \( \beta_p = 1 \).

We now explain how to deduce the homology of a complex from the homology of its predecessor and the relative homology of the step. We get the homology of \( K_{j+1} = K_j \cup J \) using the long exact sequence of a pair:

\[
\cdots \to H_p(K_j) \to H_p(K_{j+1}) \to H_p(K_{j+1}, K_j) \to H_{p-1}(K_j) \to \cdots \tag{30}
\]

Note that \( H_p(K_{j+1}, K_j) \) is isomorphic to \( H_p(\bar{J}, J) \) for every dimension \( p \) by excision. Assuming the ranks of the homology groups of \( K_j \) and of \( (\bar{J}, J) \) are given, there are very few options for the ranks of \( K_{j+1} \) that make the sequence exact. For example, if \( J \) is a non-critical step, then \( \text{rank} H_p(K_{j+1}) = \text{rank} H_p(K_j) \) for every \( p \). If \( J \) is a simple critical step with index \( p \), then either \( \text{rank} H_p(K_{j+1}) = \text{rank} H_p(K_j) + 1 \) or \( \text{rank} H_{p-1}(K_{j+1}) = \text{rank} H_{p-1}(K_j) - 1 \), with equal ranks in all other dimensions.

### 5.2 Critical and Non-critical Steps

Note that for a discrete or generalized discrete Morse function, every critical step is simple and indeed consists of only a single cell. In contrast, the discrete version of a generic piecewise linear map can have non-simple critical steps, such as monkey saddles, etc. However, these steps are still special since each has a unique lower bound, which is a vertex.

Similarly, the discrete functions in this paper are special cases within the general framework introduced in Appendix A.2. In particular, each step of the Delaunay function, \( \text{del} : \text{Del}(B) \to \mathbb{R} \), has a unique upper bound, as we will prove shortly. To include the discrete Voronoi function in this discussion, we note that \( \text{vor} : \text{Vor}(B) \to \mathbb{R} \) is anti-monotonic, so \(-\text{vor}\) is monotonic, the above discussion applies, and every step of \( \text{vor} \) has a unique upper bound as well. Furthermore, the critical steps of \( \text{del} \) and \( \text{vor} \) are simple, in the sense that they contain a single cell each, as we now prove.

**Lemma 5.2 (Steps).** Every step of \( \text{vor} \) and of \( \text{del} \) has a unique upper bound, and if it is critical, then it is simple and consists of a single cell whose dimension is equal to the index of the step.

We observe that our definition of a critical step is consistent with that of a critical point. An interesting detail are the borderline non-critical points, which we recall have a contractible lower link and a non-contractible upper link, or the other way round. Correspondingly in the discrete setting, we call \( \tau \in \text{Vor}(B) \) a **borderline non-critical cell** if \( \tau \cap \tau^* \neq \emptyset \) but \( \text{int} \tau \cap \text{int} \tau^* = \emptyset \). A borderline critical cell is not critical, but there are arbitrarily small perturbations of the weighted points in \( B \) that render such a cell critical. Note that \( \tau \) is a borderline non-critical cell of \( \text{vor} \) iff \( \tau^* \) is a borderline non-critical cell of \( \text{del} \). To bring such cases in focus, we introduce a condition that avoids them.
**Definition 5.3** (General Position II). A set \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) with injective and locally finite projection to \( \mathbb{R}^d \) satisfies Condition II of general position if \( \text{vor} \) has no borderline non-critical cell or, equivalently, if \( \text{del} \) has no borderline non-critical cell.

Note that Condition II is independent of Condition I of general position introduced in Appendix A.1.

5.3 Evolution of Channel

For every non-critical value \( t \in \mathbb{R} \), we have a partition of \( \mathbb{R}^d \) into the underlying space of the superlevel set of \( \text{vor} \), of the sublevel set of \( \text{del} \), and of the level in between. We are interested in the evolution of this partition as \( t \) goes from \(-\infty\) to \( \infty \). It is convenient to study the corresponding partition of the common subdivision,

\[
\text{Sd}(B) = \text{add}^{-1}(-\infty, t] \cup \text{Ch}_t(B) \cup \text{sdv}^{-1}[t, \infty),
\]

as \( t \) goes from \(-\infty\) to \( \infty \). At the beginning, the only non-empty set in the partition is the superlevel set of \( \text{sdv} \), and step by step the cells migrate first to the channel and second to the sublevel set of \( \text{add} \), until at the end the latter is the only non-empty subset in the partition. As illustrated in Figure 7, we distinguish between non-critical steps and critical steps of index \( q \), with \( 0 \leq q \leq d \). By Lemma 5.2, the cells of an index \( q \) critical step subdivide an open \( q \)-cell in \( \text{Del}(B) \) or in \( \text{Vor}(B) \). Write \( J_i \) and \( t_i \) for the steps of \( \text{sdv} \) and \( \text{add} \) and their values, for \( 0 \leq i \leq m \). We assume the indexing satisfies \( t_i \leq t_{i+1} \) for \( 0 \leq i < m \), and in case of a tie, the steps of \( \text{sdv} \) precede those of \( \text{add} \). Write \( V_i \) and \( D_i \) for the two complexes after processing steps \( J_0 \) through \( J_i \), and let \( C_i = \text{Sd}(B) \setminus [V_i \sqcup D_i] \) be the third set in the partition. We get the next partition as

\[
\begin{align*}
V_{i+1} &= V_i \setminus J_{i+1}, & C_{i+1} &= C_i \cup J_{i+1}, & D_{i+1} &= D_i, \\
V_{i+1} &= V_i, & C_{i+1} &= C_i \setminus J_{i+1}, & D_{i+1} &= D_i \cup J_{i+1},
\end{align*}
\]

in which the first row describes the change if the step belongs to \( \text{sdv} \) and the second row if the step belongs to \( \text{add} \). To avoid discussing the homology of unbounded spaces, we add a point at infinity to compactify \( \mathbb{R}^d \) to \( S^d \).
Revisiting Alexander Duality with Tessellations and Mosaics

CASE \( J_{i+1} \) is non-critical. Then the \( p \)-th homology groups of \( V_i \) and \( V_{i+1} \) are isomorphic, and so are the \( p \)-th homology groups of \( D_i \) and \( D_{i+1} \), for every \( p \).

CASE \( J_{i+1} \) is an index \( q \) critical step of \( \text{sdv} \). Then either \( \beta_q(V_{i+1}) = \beta_q(V_i) - 1 \) or \( \beta_{q-1}(V_{i+1}) = \beta_{q-1}(V_i) + 1 \), with equality for the ranks in all other dimensions.

CASE \( J_{i+1} \) is an index \( q \) critical step of \( \text{sd} \). Then either \( \beta_q(D_{i+1}) = \beta_q(D_i) + 1 \) or \( \beta_{q-1}(D_{i+1}) = \beta_{q-1}(D_i) - 1 \), with equality for the ranks in all other dimensions.

Recall that the critical steps come in pairs of complementary indices \( p + q = d \). Assuming \( J_{i+1} \) and \( J_{i+2} \) are such a pair of critical steps, one of \( \text{sdv} \) and the other of \( \text{sd} \), we get either \( \beta_q(V_{i+2}) = \beta_q(V_i) - 1 \) or \( \beta_{q-1}(V_{i+2}) = \beta_{q-1}(V_i) + 1 \) for the ranks on one side of the channel, and either \( \beta_q(D_{i+2}) = \beta_q(D_i) + 1 \) or \( \beta_{q-1}(D_{i+2}) = \beta_{q-1}(D_i) - 1 \) for the ranks on the other side of the channel. This is consistent with Alexander duality but fails to imply it as we did not yet couple the events on the two sides.

5.4 Crushing the Channel

This subsection addresses the missing step in the proof of Alexander duality for \( V_i \) and \( D_i \). To this end, we show that the channel that separates the two complexes can be deformation retracted. Let \( t \in \mathbb{R} \) such that \( D_i = \text{Del}_t(B) \) and \( V_i = \text{Vor}^t(B) \), and recall that \( |D_i| \supseteq \text{vor}^{-1}(\mathbb{R},t) \) and \( |V_i| \subseteq \text{del}^{-1}(t,\infty) \). Since the situation is symmetric, it suffices to talk about \( D_i \). By definition, a boundary cell of \( D_i \) is contained in \( \partial |D_i| \), and by construction, \( \sigma^* \in D_i \) is a boundary cell iff the intersection of the corresponding spheres has a non-empty contribution to the boundary of \( \text{vor}^{-1}(\mathbb{R},t) \). Letting \( p \) be the dimension of the dual cell, \( \sigma \in \text{Vor}(B) \), and \( pt(b) \) be one of the vertices of \( \sigma^* \), this contribution is \( A_{\sigma} = \sigma \cap C_{\sigma}(pt(b)) \), in which the squared radius of the sphere is \( r^2 = wt(b) + t \). Hence, \( A_{\sigma} \) is a subset of a \((p-1)\)-sphere, which may or may not be connected. An important part of the construction is the join of \( \sigma^* \) and \( A_{\sigma} \), which is the union of line segments connecting the two sets:

\[
\sigma^* \times A_{\sigma} = \{(1-\lambda)y + \lambda z | y \in \sigma^*, z \in A_{\sigma}, 0 \leq \lambda \leq 1\}.
\] (32)

Writing \( U_t = \text{vor}^{-1}(\mathbb{R},t) \) and following [4], we decompose \( U_t \setminus |D_i| \) into such joins. The deformation retraction will happen along the fibers of this decomposition, which are the line segments in the joins. We therefore need that the fibers cover \( U_t \setminus |D_i| \) and that they do not intersect except at shared endpoints. But this is clear because the entire decomposition can be obtained by projecting pieces of a convex surface in \( \mathbb{R}^{d+1} \) to \( \mathbb{R}^d \). This surface is the boundary of the convex hull of the graphs of \( \text{end} \) and \( \varpi + t \). The pieces that belong to the graph of \( \text{end} \) project to cells in \( D_i \), the pieces that bridge the gap between the two graphs project to the joins, and the rest belongs to the graph of \( \varpi \), which we do not project.

We now return to splitting the channel along the middle, by which we mean that we split it along \( \text{sd}^{-1}(t) \). It is important that each fiber intersect this level set in exactly one point.

Lemma 5.4 (Fiber Integrity). Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and locally finite projection to \( \mathbb{R}^d \), let \( t \in \mathbb{R} \) be a non-critical value, and let \( y, z \) be endpoints of a fiber in the decomposition of \( U_t \setminus |\text{Del}_t(B)| \). Then there is a unique \( 0 \leq \lambda \leq 1 \) such that \( \text{sd}((1-\lambda)y + \lambda z) = t \).

To construct the deformation retraction, we clip every fiber where it intersects \( \text{sd}^{-1}(t) \) and retract the remaining piece to its endpoint in \( |D_i| \). To describe this formally, we let \( \lambda' \) be the solution to \( \text{sd}((1-\lambda')y + \lambda' z) = t \) and write \( M_t = \text{sd}^{-1}(-\infty, t] \) and \( \tilde{M} = \text{sd}^{-1}[t, \infty) \). The deformation retraction is \( D: M_t \times [0,1] \to M_t \), defined by mapping \( x \in |D_i| \) to \( D(x,s) = x \), and mapping \( x = (1-\lambda')y + \lambda' z \) to \( D(x,s) = (1-s\lambda')y + s\lambda' z \), for every \( s \in [0,1] \).

Symmetrically, we deformation retract \( \tilde{M} \) to \( |V_i| = |\text{Vor}^t(B)| \). We formally state the implications.
Theorem 5.5 (Crushing). Let $B \subseteq \mathbb{R}^d \times \mathbb{R}$ have an injective and locally finite projection to $\mathbb{R}^d$ and $t \in \mathbb{R}$ be non-critical. Then $|\text{Del}_t(B)| \simeq M_t$ and $|\text{Vor}_t(B)| \simeq M_t$.

In words, the channel can be split into halves, each half can be decomposed into line segments called fibers, and by retracting the fibers, we glue the boundaries of $|\text{Del}_t(B)|$ and $|\text{Vor}_t(B)|$ without altering the homotopy type, which is that of $\mathbb{R}^d$ or, after compactification, that of $S^d$. Hence, Alexander duality applies, so we get $\beta_{q-1}(\text{Vor}_t(B)) = \beta_p(\text{Del}_t(B))$ for all dimensions $p + q = d$, except when $p = 0$ or $q = 0$ in which case the two ranks differ by 1. Recalling the parallel change of the two complexes discussed above, we know conclude that we see the birth of a $p$-dimensional homology class in $\text{Vor}_t(B)$ iff we see the birth of a $(q - 1)$-dimensional homology class in $\text{Del}_t(B)$ at the same threshold, and similarly for the death of such classes.

6 Discussion

Motivated by challenges caused by data in non-general position, this paper presents a geometric approach to Alexander duality based on complementary subcomplexes of Voronoi tessellations and Delaunay mosaics. Along the way, we gain new insights into an old subject, and we encounter questions we have not been able to answer:

- The piecewise linear $s_d: \mathbb{R}^d \to \mathbb{R}$ can be defined for sets $B, C \subseteq \mathbb{R}^d \times \mathbb{R}$ that do not satisfy the polar relationship assumed in this paper. What are its properties, and what additional features does it enjoy when $B$ and $C$ are polar, as assumed in this paper?
- The level sets of $s_d: \mathbb{R}^d \to \mathbb{R}$ suggest themselves as easy-to-use yet topologically correct PL surfaces of unions of balls, which in $\mathbb{R}^3$ are popular models of biomolecules. What are their combinatorial and geometric properties, and how fast can they be computed?
- We prove in this paper that the channel deformation retracts to the Voronoi complex as well as the complementary Delaunay complex. Can the same result be obtained with discrete methods, for example by collapsing the steps of the discrete versions of $s_d$?

The geometric approach to Alexander duality described in this paper gives rise to a one-parameter family of complementary complexes, which may be studied with persistent homology [6]. It would be interesting to further develop this approach while connecting it to applications. Is Raleigh–Bénard convection with its family of bi-partitions of space [13] an opportune candidate?
References


A Background

We review Voronoi tessellations and the dual Delaunay mosaics, which we introduce for points with real weights in Euclidean space. In addition, we introduce discrete Morse theory for polyhedral complexes.

A.1 Voronoi Tessellations and Delaunay Mosaics

We refer to \( b \in \mathbb{R}^d \times \mathbb{R} \) as a weighted point, with location \( \text{pt}(b) \in \mathbb{R}^d \) and weight \( \text{wt}(b) \in \mathbb{R} \). It is common to interpret \( b \) as a sphere, with center \( \text{pt}(b) \) and squared radius \( \text{wt}(b) \), but for this we have to allow for spheres with non-positive squared radii. The power distance of a point \( x \in \mathbb{R}^d \) from \( b \) is \( \pi_b(x) = \|x - \text{pt}(b)\|^2 - \text{wt}(b) \). It is positive outside the sphere, zero on the sphere, and negative inside the sphere. Of course, for a sphere with negative squared radius, all points are outside. Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) be a set of weighted points whose projection to \( \mathbb{R}^d \) is injective and locally finite. In other words, for every location there is an open neighborhood that separates it from the other locations. For a subset \( A \subseteq B \), consider all points \( x \in \mathbb{R}^d \) with equal power distance from the weighted points in \( A \) and strictly larger power distance from the other weighted points, and call its closure the (Voronoi) cell of \( A \), denoted \( \text{cell}(A) \). Each non-empty cell is a convex polyhedron in \( \mathbb{R}^d \), and its dimension depends on \( A \). The (weighted) Voronoi tessellation of \( B \), denoted \( \text{Vor}(B) \), is the collection of non-empty cells. It is a polyhedral complex in the sense that every cell is a convex polyhedron, every face of a cell is again a cell, and any two cells are either disjoint or intersect in a common face, which is therefore a cell in the tessellation. A cell of dimension \( p \) has faces of dimension from 0 to \( p \), and we call the faces of dimension \( p - 1 \) its facets. Define the dual cell of \( A \) as the convex hull of the points, denoted \( \text{cell}^*(A) \), which is again a convex polyhedron. The dimension of a cell and its dual cell are necessarily complementary: if \( p = \dim \text{cell}(A) \) and \( q = \dim \text{cell}^*(A) \), then \( p + q = d \). The (weighted) Delaunay mosaic of \( B \), denoted \( \text{Del}(B) \), is the collection of dual cells. Figure 8 illustrates the concepts by drawing a Voronoi tessellation and the corresponding Delaunay mosaic on top of each other.

![Figure 8: The overlay of a Voronoi tessellation and its dual Delaunay mosaic. The former is not simple because of the vertex incident to four edges, and the latter is not simplicial because of the region with four edges. We add half-lines to the mosaic to decompose the complement of the convex hull into convex cells.](image)

We consider two independent notions of general position, the first of which we introduce now. In \( \mathbb{R}^d \), we call a Voronoi tessellation simple if every \( p \)-dimensional cell is face of exactly \( q + 1 = d - p + 1 \) top-dimensional cells, and we call a Delaunay mosaic simplicial if every
q-dimensional dual cell is the convex hull of \( q + 1 \) points. Clearly, a Voronoi tessellation is simple iff the corresponding Delaunay mosaic is simplicial.

**Definition A.1 (General Position I).** A set \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) with injective and locally finite projection to \( \mathbb{R}^d \) satisfies Condition I of general position if \( \text{Vor}(B) \) is simple or, equivalently, if \( \text{Del}(B) \) is simplicial.

We stress that this paper does not assume \( B \) is in general position and introduces the notion mostly to clarify the difference between the generic and the non-generic situation.

Besides \( \text{Vor}(B) \) and \( \text{Del}(B) \), we will be interested in subcomplexes and subsets of these complexes. To stress the difference, we note that a subcomplex is closed under taking faces, while a subset does not necessarily enjoy this property. We call a subset open if it is closed under taking cofaces. As an example consider a subset \( K \subseteq \text{Vor}(B) \) and let \( K^* \subseteq \text{Del}(B) \) contain cell*(A) iff cell(A) \( \in K \). Clearly, \( K \) is a subcomplex of the Voronoi tessellation iff \( K^* \) is an open subset of \( \text{Del}(B) \), and vice versa. While the cells in a complex may intersect, their (relative) interiors are disjoint. Indeed, for every \( x \in \mathbb{R}^d \) there is a unique cell \( \tau \in \text{Vor}(B) \) whose interior contains \( x \). The same is true for the Delaunay mosaic if we restrict ourselves to points \( x \) in the convex hull of the \( \text{pt}(b), b \in B \). As suggested in Figure 8, we will extend the Delaunay mosaic artificially so that this restriction can be removed. We define the underlying space of a subset \( K \) of a polyhedral complex as the union of interiors of its cells:

\[
|K| = \{ x \in \mathbb{R}^d \mid x \in \text{int} \tau \text{ for some } \tau \in K \}.
\]

If \( K \) is a complex, then this is just the union of cells, but if \( K \) is not a complex, then the union of interiors is a strict subset of the union of cells.

## A.2 Discrete Morse Theory

Letting \( K \) be a polyhedral complex in \( \mathbb{R}^d \), we call \( f : K \rightarrow \mathbb{R} \) a discrete function. It is monotonic if \( f(\nu) \leq f(\mu) \) whenever \( \nu \) is a face of \( \mu \) in \( K \), and it is anti-monotonic if \(-f\) is monotonic. For every \( t \in \mathbb{R} \), we call \( f^{-1}(t) \) a level set, \( f^{-1}(-\infty, t] \) a sublevel set, and \( f^{-1}(t, \infty) \) a superlevel set of \( f \). A subset \( L \subseteq K \) is closed if \( \mu \in L \) and \( \nu \subseteq \mu \) implies \( \nu \in L \), and it is open if \( \nu \in L \) and \( \nu \subseteq \mu \) implies \( \mu \in L \). Closed subsets are usually referred to as subcomplexes of \( K \). The following three statements are equivalent:
- \( f \) is monotonic,
- every sublevel set of \( f \) is closed,
- every superlevel set of \( f \) is open.

The Hasse diagram of \( K \) is the directed graph whose nodes are the cells of \( K \), with an arc from \( \nu \) to \( \mu \) if \( \nu \subseteq \mu \) and \( \dim \nu = \dim \mu - 1 \). We note that \( f : K \rightarrow \mathbb{R} \) is monotonic iff the values along every directed path of the Hasse diagram are non-decreasing. A step of \( f \) is a connected component of the Hasse diagram restricted to a level set of \( f \), and we write \( \nabla f \) for the collection of steps, which partitions \( K \). We construct the step graph by taking the steps in \( \nabla f \) as nodes and drawing an arc from \( I \) to \( J \) if there are cells \( \nu \in I \) and \( \mu \in J \) such that the Hasse diagram has an arc from \( \nu \) to \( \mu \). In other words, the step graph is obtained from the Hasse diagram by contracting every arc whose end-cells share the function value. It follows that that the values along every directed path of the step graph are strictly increasing.

A monotonic \( f : K \rightarrow \mathbb{R} \) is a discrete Morse function if every step is either a pair or a singleton; see [10] but note that we inessentially simplified the setting by requiring that the cells in a pair share the same value. The singletons contain the critical cells and the pairs contain the non-critical cells of \( f \). Following the convention in smooth Morse theory [15],
where the index of a critical point is indicative of the effect of advancing the sublevel set beyond its value, we call the dimension of a critical cell its index. Indeed, adding a critical $p$-cell gives either birth to a $p$-cycle or death to a $(p-1)$-cycle, which affects the homology of the complex accordingly. In contrast, removing the two cells of a pair $\{\nu, \mu\}$ — which is allowed only if the result is still closed — has no effect on the homology of the complex. This operation is referred to as an elementary collapse, and denoted $K \searrow K \setminus \{\nu, \mu\}$. More generally, we write $K \searrow L$ if $L$ can be obtained from $K$ by a sequence of elementary collapses.

The main motivation for distinguishing between critical and non-critical cells is the following result in [10]:

**Proposition A.2 (Collapse).** Let $f: K \to \mathbb{R}$ be a discrete Morse function on a polyhedral complex, and let $t_0 < t_1$ such that no critical cell has its value in $(t_0, t_1]$. Then $f^{-1}(-\infty, t_1] \searrow f^{-1}(-\infty, t_0]$.

To generalize the concept, we call a subset $J \subseteq K$ an interval if there are cells $\alpha, \omega \in K$ such that $J = \{\nu \in K \mid \alpha \subseteq \nu \subseteq \omega\}$. In words, the interval has a unique lower bound, $\alpha$, and a unique upper bound, $\omega$, and consists of all faces of $\omega$ that have $\alpha$ as a face. A monotonic $f: K \to \mathbb{R}$ is a generalized discrete Morse function if every step is an interval; see [11]. The intervals of size one contain the critical cells and all other intervals contain the non-critical cells of $f$. Removing the cells of an interval of size larger than one from $K$ is referred to as a collapse, which is allowed only if the result is still closed.

In the simplicial case, the Hasse diagram restricted to an interval is isomorphic to the 1-skeleton of a cube of the appropriate dimension. Choosing a direction, we get a collection of parallel edges of the cube, which corresponds to a partition of the interval into pairs. In the polyhedral case, such a partition is not quite as obvious but it exists. In other words, every collapse can be decomposed into a sequence of elementary collapses. The proof of this claim reduces to the fact that every convex polytope allows for a discrete Morse function with a single critical cell, which is a vertex [2]. This motivates us to use the same notation, $K \searrow L$, if $L$ can be obtained from $K$ by a sequence of possibly non-elementary collapses. Similarly, the decomposition into elementary collapses implies that Proposition A.2 holds also for generalized discrete Morse functions.
In this appendix, we give proofs of lemmas and theorems presented in this paper, and we illustrate the geometric structures with a quad of figures. We restate each of the claims before presenting the proof. Here, B.1 is 2.1, B.2 is 3.1, B.3 is 3.2, B.4 is 3.3, B.5 is 3.4, B.6 is 3.5, B.7 is 4.1, B.8 is 5.2, and B.9 is 5.4.

**Lemma B.1** (Common Maximum). Let \( A \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and affinely independent projection to \( \mathbb{R}^d \). Then \( y = y(A) \) is the common maximum of

(i) the restriction of \( \bar{A} - \bar{\omega} \) to \( \text{flt}(A) \),
(ii) the restriction of \( \bar{A} - \omega \) to \( \text{sol}(A) \),
(iii) the average, \( \frac{1}{2} [\bar{A} + \bar{A}] - \bar{\omega} \), and in this case the value of the maximum vanishes.

**Proof.** We begin by mapping every location \( x \in \text{flt}(A) \) to a weighted point \( u \in \mathbb{R}^d \times \mathbb{R} \) with \( \text{wt}(u) = 2\bar{\omega}(\text{pt}(u)) - 2\bar{A}(\text{pt}(u)) \), noting that \( \bar{u}(\text{pt}(u)) = \bar{A}(\text{pt}(u)) \).

Similarly, we map every location \( v \in \text{sol}(A) \) to \( v \in \mathbb{R}^d \times \mathbb{R} \) with \( \text{wt}(v) = 2\bar{\omega}(\text{pt}(v)) - 2\bar{A}(\text{pt}(v)) \), noting that \( \bar{v}(\text{pt}(v)) = \bar{A}(\text{pt}(v)) \). By construction \( \bar{u} : \mathbb{R}^d \to \mathbb{R} \) agrees with \( \bar{A} \) on \( \text{sol}(A) \). Hence, relation (5) applies, which implies that the zero-sets of \( \pi_u \) and \( \pi_v \) intersect orthogonally. Observe that this is true for all pairs \( \langle \text{pt}(u), \text{pt}(v) \rangle \in \text{flt}(A) \times \text{sol}(A) \), so we have what for two lines in \( \mathbb{R}^2 \) is sometimes called a coaxal system [17].

If we now fix \( v \) with \( \text{pt}(v) \in \text{sol}(A) \), we get \( u \) with minimum weight by minimizing \( ||\text{pt}(v) - \text{pt}(u)||^2 \). This minimum is attained for \( \text{pt}(u) = y \), and since \( \text{wt}(u) = 2\bar{\omega}(\text{pt}(u)) - 2\bar{A}(\text{pt}(u)) \), this implies that \( y \) maximizes \( \bar{A} - \bar{\omega} \), as claimed in (i). The proof of (ii) is symmetric.

While we considered only the restrictions of \( \bar{A} \) and \( \bar{\omega} \) to affine subspaces, they are defined on the entire \( \mathbb{R}^d \). Hence, the map \( f : \mathbb{R}^d \to \mathbb{R} \) defined by \( f(x) = \frac{1}{2} [\bar{A}(x) + \bar{A}(x')] \) is well defined. It is affine since \( \bar{A} \) and \( \bar{\omega} \) are affine. Letting \( x' \) and \( x'' \) be the orthogonal projections of \( x \in \mathbb{R}^d \) onto \( \text{flt}(A) \) and \( \text{sol}(A) \), respectively, we have \( f(x) = \frac{1}{2} [\bar{A}(x') + \bar{A}(x'')] \). At the intersection of the two affine subspaces, we have \( f(y) - \bar{\omega}(y) = 0 \) by (6). At every other point \( x \in \mathbb{R}^d \), \( f(x) - \bar{\omega}(x) < 0 \), simply because \( \bar{A}(x') - \bar{\omega}(y) \leq \bar{A}(y) - \bar{\omega}(y) \) and \( \bar{A}(x'') - \bar{\omega}(x'') \leq \bar{A}(y) - \bar{\omega}(y) \), with strict inequality at least once. This implies (iii). □

**Lemma B.2** (Dominance). Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and locally finite projection to \( \mathbb{R}^d \). Then \( \text{del}(x) \geq s\text{d}(x) \geq \text{vor}(x) \) for every \( x \in \mathbb{R}^d \).

**Proof.** Let \( a \in \mathbb{R}^d \times \mathbb{R} \) such that \( \bar{a}(\text{pt}(a)) = \text{env}(\text{pt}(a)) \). Hence, \( \bar{a}(\text{pt}(a)) \geq b(\text{pt}(a)) \) for all \( b \in B \), with equality at least once. By (3), \( \bar{b}(\text{pt}(b)) \geq \bar{a}(\text{pt}(b)) \), for all \( b \in B \), and therefore \( \text{end}(y) \geq \bar{a}(y) \) for all \( y \in \mathbb{R}^d \), which includes \( y = \text{pt}(a) \). Writing \( x = \text{pt}(a) \), this implies

\[
\text{del}(x) - \text{vor}(x) = \text{end}(x) + \text{env}(x) - 2\bar{\omega}(x) \geq \bar{a}(x) + \bar{a}(x) - 2\bar{\omega}(x),
\]

in which the right-hand side vanishes because of (6). This implies the claimed inequalities. □

**Lemma B.3** (Excluded Crossing). Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) have an injective and locally finite projection to \( \mathbb{R}^d \), let \( \mu \neq \nu \) be cells in \( \text{Vor}(B) \) and note that \( \mu^*, \nu^* \) are cells in \( \text{Del}(B) \). If \( \text{int} \mu \cap \nu^* \neq \emptyset \), then \( \text{int} \nu \cap \mu^* = \emptyset \).
Proof. To reach a contradiction, assume that both intersections are non-empty, so we can choose points $x \in \text{int } \mu \cap \nu^*$ and $y \in \text{int } \nu \cap \mu^*$. Since the interiors of $\mu$ and $\nu$ are disjoint, we have $x \neq y$. Let $M, N \subseteq B$ be such that $\mu = \text{cell}(M)$ and $\nu = \text{cell}(N)$. By definition of a cell, $x$ has the same power distance from all $a \in M$, and a strictly larger power distance from all $b \in B \setminus M$. Write $R_M = \pi_a(x)$ with $a \in M$, and write $R_N = \pi_c(x)$ with $c \in N$.

Assume without loss of generality that $R_N \geq R_M$. Then every weighted point $a \in M$ satisfies $\pi_a(y) \geq R_N \geq R_M = \pi_a(x)$, so $\|y - \text{pt}(a)\| \geq \|x - \text{pt}(a)\|$. Drawing the perpendicular bisector of $x$ and $y$, this implies that all $\text{pt}(a)$ with $a \in M$ lie in the closed half-space that contains $x$. Since $y$ lies outside this half-space, it is not contained in the convex hull of the $\text{pt}(a)$ with $a \in M$, but this contradicts $y \in \mu^*$.

Lemma B.4 (Hemispheres). The common intersection of a collection of hemispheres of $\mathbb{S}^{d-1}$ is either contractible or a $(p-1)$-dimensional great-sphere with $0 \leq p \leq d$.

Proof. Let $E \subseteq \mathbb{S}^{d-1}$ be the set of vectors defining the hemispheres in the given collection. If $E \neq \emptyset$ and there is a point $x \in \mathbb{S}^{d-1}$ with $\langle e, x \rangle > 0$ for all $e \in E$, then the hemispheres have a non-empty and contractible common intersection. Otherwise, let $x \in \mathbb{S}^{d-1}$ such that $\langle e, x \rangle \geq 0$, for all $e \in E$, with equality for a minimum number of vectors. If $x$ does not exist, then the intersection of hemispheres is empty, which is the case $p = 0$ in the claimed statement. When $x$ exists, it may not be unique, but the vectors $e$ for which the scalar product vanishes are unique. Similarly, the linear span of these vectors is unique, and letting $0 \leq d - p \leq d$ be its dimension, the common intersection of the hemispheres is a $(p-1)$-dimensional great-sphere. The case $p = d$ corresponds to an empty collection of hemispheres so that the common intersection is the entire $\mathbb{S}^{d-1}$.

Lemma B.5 (Down and Up Link). Let $P, Q \subseteq \mathbb{R}^d$ be convex polyhedra with orthogonal affine hulls of complementary dimensions: $p = \dim P$, $q = \dim Q$, and $p + q = d$. Then $\text{int } P \cap \text{int } Q \neq \emptyset$ implies $dLk(P, Q) \simeq \mathbb{S}^{q-1}$, $uLk(P, Q) \simeq \mathbb{S}^{p-1}$, and $P \cap Q = \emptyset$ implies that both links are contractible.

Proof. Assume that the affine hulls of $P$ and $Q$ intersect at $0 \in \mathbb{R}^d$. Every facet $E$ of $R = P \times Q$ is either of the form $F \times Q$ or $P \times G$, in which $F$ and $G$ are facets of $P$ and $Q$, respectively. Whether or not $E$ belongs to the down link or the up link depends on the relative position of $E$ and $0$, and the rule is opposite for the two forms. To explain, we call $E$ visible (from 0) if $\langle n(s), s \rangle \leq 0$ for every $s \in E$ and invisible (from 0) if $\langle n(s), s \rangle \geq 0$ for every $s \in E$. We observe that $dLk(P, Q)$ contains all visible facets $E$ of the form $E = F \times Q$ and all invisible facets of the form $E = P \times G$, while $uLk(P, Q)$ contains all invisible facets of the first type and all visible facets of the second type.

In the first case, when $\text{int } P \cap \text{int } Q \neq \emptyset$, 0 belongs to the interior of $R$. Hence all facets of $R$ are invisible, which implies that the down link is $P \times \partial Q$, which has the homotopy type of a $(q - 1)$-sphere. Symmetrically, the up link is $\partial P \times Q$, which has the homotopy type of the $(p - 1)$-sphere, as claimed.

In the second case, when $P \cap Q = \emptyset$, not all facets of $R$ are invisible. Which facets of $R$ are visible and which are invisible can be seen within the affine hulls of $P$ and of $Q$. Specifically, there is a bijection between the visible and invisible facets of $R$ and the visible and invisible facets of $P$ inside $\text{aff } P$ and of $Q$ inside $\text{aff } Q$. For the down link, we need the visible facets of $P$ and the invisible facets of $Q$, so we apply a projective transformation that maps $Q$ to another convex polytope $Q'$ and 0 to another point $0'$ such that a facet of $Q$ is invisible from 0 iff the corresponding facet of $Q'$ is visible from $0'$. This transformation does
not affect $P$. We get a new product, $R' = P \times Q'$ and we are interested in the part of the boundary that is visible from $O'$. By convexity, this part of $\partial R'$ is contractible, which implies that the corresponding part of $\partial R$ is also contractible. The latter is $d\text{Lk}(P, Q)$, which is therefore contractible. By a symmetric argument, $u\text{Lk}(P, Q)$ is also contractible, as claimed.

**Theorem B.6 (Critical Points).** Let $B \subseteq \mathbb{R}^d \times \mathbb{R}$ have an injective and locally finite projection to $\mathbb{R}^d$. Then $x \in \mathbb{R}^d$ is a critical point of $\text{vor}$ iff it is a critical point of $\text{del}$ iff it is a critical point of $sd$, and its index is the same for all three functions.

**Proof.** We prove that $x \in \mathbb{R}^d$ is a critical point (of $\text{vor}$, $\text{del}$, and $sd$) iff $x = \text{int } \nu \cap \text{int } \nu^*$ for a cell $\nu \in \text{Vor}(B)$ and its dual cell $\nu^* \in \text{Del}(B)$, and that in this case its index is $q = \dim \nu^*$.

We begin with $f = \text{vor}$, which maps every $x \in \mathbb{R}^d$ to half the smallest power distance to a weighted point in $B$. The restriction of $\text{vor}$ to a cell $\nu$ is also the restriction of a quadratic function on $\text{aff } \nu$ to $\nu$. This quadratic function has a unique minimum, namely at $y = \text{aff } \nu \cap \text{aff } \nu^*$. The only possibility for a point $x \in \text{int } \nu$ to be a critical point of $\text{vor}$ is therefore $x = y$. This implies that $\text{int } \nu \cap \text{aff } \nu^* \neq \emptyset$ is necessary for $x$ to be critical. To prove that $\text{aff } \nu \cap \text{int } \nu^* \neq \emptyset$ is also needed, we assume this intersection is empty. Then there is a plane of dimension $q - 1$ in $\text{aff } \nu^*$ that separates $\nu^*$ from $x = y$. Moving $x$ normal to this plane within $\text{aff } \nu^*$ increases $\text{vor}$ in one direction and decreases it in the other direction. This implies that $x$ is not a critical point, so we conclude that $x = \text{int } \nu \cap \text{int } \nu^*$ is necessary.

It is easy to see that it is also sufficient because $\text{vor}$ increases along all directions within $\text{aff } \nu$ and it decreases in all directions within $\text{aff } \nu^*$. The index is the dimension of the affine subspace within which $x$ is a maximum of $f$, which is $q = \dim \nu^*$, as claimed. The argument for $f = \text{del}$ is symmetric and therefore omitted. The index is still $q$, and not $p$ as suggested by symmetry, because $\text{del}$ maps every $x \in \mathbb{R}^d$ to the negative of the smallest power distance to a weighted point in $C$.

The argument for $f = sd$ is more involved. The only possible critical points are the vertices of $sd(B)$. To simplify, we assume that cells $\nu$ and $\mu^*$ have complementary dimensions have interiors that are either disjoint or intersect in a single point, which is therefore a vertex of $sd(B)$. Writing $x = \text{int } \nu \cap \text{int } \mu^*$, we let $S_r(x)$ be a sufficiently small sphere centered at $x$.

It intersects a cell of $sd(B)$ iff that cell is incident to $x$, and these cells define a cell complex embedded on the sphere. We observe that that this cell complex is dual to the boundary complex of $P \times Q$, in which $P = \mu$ and $Q = \nu^*$ are the duals of the cells that intersect in $x$. Every point $v \in S^{d-1}$ is a direction, and we write $sd_x(v)$ for the derivative of $sd$ at $x$ in the direction $v$. The goal is to prove that the links of $x$ and $sd$ are closely related to the links of $P$ and $Q$, namely

$$\ell\text{Lk}(x, sd) \simeq d\text{Lk}(P, Q) \quad \text{and} \quad h\text{Lk}(x, sd) \simeq u\text{Lk}(P, Q).$$  

By Lemma 3.4, the down and up links of $P$ and $Q$ either have the homotopy types of $S^{q-1}$ and $S^{p-1}$, if $\text{int } P \cap \text{int } Q \neq \emptyset$, and they are both contractible, if $\text{int } P \cap \text{int } Q = \emptyset$. Assuming (35), this implies that the links of $x$ and $sd$ have the homotopy types of $S^{q-1}$ and $S^{p-1}$, if $\nu = \mu$, and they are both contractible, if $\nu \neq \mu$. Indeed, $\nu \neq \mu$ together with $\text{int } \nu \cap \text{int } \mu^* \neq \emptyset$ implies $P \cap Q = \emptyset$ by Lemma 3.2.

We finally prove (35). Recall that every facet $E$ of $P \times Q$ corresponds to an edge of $sd(B)$ incident to $x$. As explained in the proof of Lemma 3.4, $\psi$: $\partial (P \times Q) \to S^{d-1}$ maps the vertices of $E$ to the gradients of $sd$ within the corresponding $d$-cells of $sd(B)$. Assuming without loss of generality that $x$ is the origin, we write $\bar{a}_i: \mathbb{R}^d \to \mathbb{R}$ defined by $\bar{a}_i(y) = \langle \nabla \bar{a}_i, y \rangle$ for the
linear function that corresponds to the $i$-th vertex of $E$. By construction, there is a single 
constant such that $\langle \nabla a_i, n_E \rangle = \text{const}$ for all $i$. The gradient of any affine combination of 
the functions is the affine combination of the gradients: $\nabla \sum_i \lambda_i a_i = \sum_i \lambda_i \nabla a_i$. There is a 
unique affine combination with shortest gradient, which we denote $\bar{a}_E$. It corresponds to 
the orthogonal projection of $x$ onto all $E$ or, equivalently, to the edge of $\text{Sd}(B)$ incident to $x$ 
that is dual to $E$. Its gradient satisfies 
\[
\langle \nabla \bar{a}_E, n_E \rangle = \langle \sum_i \lambda_i \nabla a_i, n_E \rangle = [\sum_i \lambda_i] \cdot \text{const} = \text{const},
\]
in which the $\lambda_i$ are the coefficients of $\bar{a}_E$, although this apparently does not matter. It 
follows that $n_E$ belongs to $\ell\text{Lk}(x, sd)$ iff $E$ belongs to $d\text{Lk}(P,Q)$. By the nerve theorem, 
the full subcomplex of the decomposition of $S_\ell(x)$ defined by the vertices with non-positive 
$\langle \nabla \bar{a}_E, n_E \rangle$ has the same homotopy type as $d\text{Lk}(P,Q)$. The rest of the lower link deformation 
retracts to this full subcomplex, which implies the left homotopy equivalence in (35). The 
symmetric argument relating the higher link of $x$ and $sd$ with the up link of $P$ and $Q$ implies 
the right homotopy equivalence in (35). This completes the proof.

Lemma B.7 (Nested Spaces). Let $B \subseteq \mathbb{R}^d \times \mathbb{R}$ have an injective and locally finite projection 
to $\mathbb{R}^d$. Then $|\text{Del}(B)| \subseteq |\text{Sd}(B)|$ and $|\text{Vor}(B)| \subseteq |\text{Sd}(B)|$.

Proof. Recall the functions $\text{adv, sdd}: \text{Sd}(B) \to \mathbb{R}$. By construction, the underlying spaces of 
their sub- and superlevel sets agree with those of vor and del. In particular, $|\text{sdd}^{-1}(-\infty, t]| = 
|\text{Del}(B)|$ and $|\text{adv}^{-1}[t, \infty)| = |\text{Vor}(B)|$. By Lemma 3.1, we have 
\[
\text{sdd}(\gamma) \geq \text{adv}(\gamma) \geq \text{snn}(\gamma) \geq \text{sdv}(\gamma),
\]
for every $\gamma \in \text{Sd}(B)$. As illustrated in Figure 4, this implies $\text{sdd}^{-1}(-\infty, t] \subseteq \text{adv}^{-1}(-\infty, t]$ 
and $\text{adv}^{-1}[t, \infty) \subseteq \text{sdd}^{-1}[t, \infty)$. The sequence of inequalities in (37) thus imply the two 
clubed containment relations.

Lemma B.8 (Steps). Every step of vor and of del has a unique upper bound, and if it is 
critical, then it is simple and consists of a single cell whose dimension is equal to the index 
of the step.

Proof. We first prove that every step of del has a unique upper bound, and we omit the 
proof for vor, which is symmetric. By definition, 
\[
\text{del}(\sigma^*) = \min_{x \in \sigma} [\varpi(x) - \text{env}(x)],
\]
in which $\text{env} = \varpi - \text{vor}$ is piecewise linear and convex. Because $\varpi$ is strictly convex, the 
minimum on the right-hand side of (38) is attained at a unique point, which we refer to as 
y = $y(\sigma)$. The step $J$ of del that contains $\sigma^*$ also contains every $\tau^* \in \text{Del}(B)$ with $y(\tau) = y$.
It contains no other cell, else there would be a cell with two points minimizing a strictly 
convex function. Without loss of generality, assume that $\sigma^*$ is the unique cell in $J$ such 
that $\sigma$ contains $y$ in its interior. It follows that $\sigma \subseteq \tau$ for all $\tau^* \in J$, which is equivalent to 
$\tau^* \subseteq \sigma^*$ for all $\tau^* \in J$. Hence, $\sigma^*$ is the unique upper bound of $J$.

We second prove that every step that contains two or more cells is non-critical. Such 
a step, $J$, has a unique upper bound, $\sigma^*$. Write $q = \dim \sigma^*$, and let $A \subseteq B$ contain the 
weighted points such that $\sigma^*$ is the convex hull of the pt(a), with $a \in A$. Let $S_\ell(x)$ be the
smallest sphere such that \( \pi_a(x) = r^2 \) for every \( a \in A \), and recall that this sphere is unique. Because \( \sigma^* \) is an upper bound, we have \( \pi_b(x) > r^2 \) for all \( b \in B \setminus A \). All cells \( \tau^* \in J \setminus \{ \sigma^* \} \) are faces of \( \sigma^* \) that are visible from \( x \). By this we mean that the line segment connecting \( x \) and a point \( z \in \interior \sigma^* \) is disjoint from \( \interior \sigma^* \), while the line that passes through \( x \) and \( z \) has a non-empty intersection with \( \interior \sigma^* \). This implies that the union of interiors of the cells in \( J \setminus \{ \sigma^* \} \) is an open \((q-1)\)-ball. As before, we define \( \vec{J} = \interior J \) and \( \dot{J} = \vec{J} \setminus J \). Since \( \vec{J} \) is a closed \( q \)-ball and \( \dot{J} \) is a closed \((q-1)\)-ball in its boundary, the rank of \( H_p(\vec{J}, \dot{J}) \) is 0 for every dimension \( p \). Hence, \( J \) is non-critical, which implies that every critical step consists of a single cell, as claimed. Adding a cell of dimension \( q \) to the appropriate sublevel set affects either the \( q \)-th or the \((q-1)\)-st homology group, which implies that the index of a critical step is the dimension of its cell, again as claimed.

\[ \text{(Fiber Integrity)} \]

Let \( B \subseteq \mathbb{R}^d \times \mathbb{R} \) be an injective and locally finite projection to \( \mathbb{R}^d \), let \( t \in \mathbb{R} \) be a non-critical value, and let \( y, z \) be endpoints of a fiber in the decomposition of \( U \setminus \Delta \ell_t(B) \). Then there is a unique \( 0 \leq \lambda \leq 1 \) such that \( \sd((1-\lambda)y + \lambda z) = t \).

**Proof.** We have \( \sd(y) < t < \sd(z) \) for the fiber with endpoints \( y \in \sigma^* \) and \( y \in A_\sigma \). It follows that it intersects \( \sd^{-1}(t) \) an odd number of times. To show that this number is 1, we recall that the sublevel set of \( \vor \) and the superlevel set of \( \del \) are both unions of balls:

\[ \vor^{-1}(-\infty, t] = \bigcup_{b \in B} b_t \quad \text{and} \quad \del^{-1}[t, \infty) = \bigcup_{c \in C} c_t, \]  

in which \( b_t \) is the ball with center \( \pt(b) \) and squared radius \( \wt(b) + t \), for \( b \in B \), and \( c_t \) is the ball with center \( \pt(c) \) and squared radius \( \wt(c) - t \), for \( c \in C \). By construction, we have \( \|\pt(b) - \pt(c)\| \geq \wt(b) + t + \wt(c) - t \); that is: \( b_t \) and \( c_t \) are orthogonal or further than orthogonal from each other. Assuming \( \wt(b) + t > 0 \), we set \( y = \pt(b) \) and let \( z \) be a point on the boundary of \( b_t \). Assuming \( c_t \) contains \( z \), there is a unique \( 0 < \lambda \leq 1 \) such that \( x = (1-\lambda)y + \lambda z \) belongs to \( c_t \) iff \( \lambda \leq \lambda_c \). Setting \( \lambda_c = \infty \) if \( c_t \) does not contain \( z \), we let \( \lambda_{\min} = \min_{c \in C} \lambda_c \). Hence, \( x \) belongs to \( \del^{-1}[t, \infty) \) iff \( \lambda_{\min} \leq \lambda \).

Recall that \( \del(x) \geq \sd(x) \geq \vor(x) \) for every \( x \in \mathbb{R}^d \), by Lemma 3.1. This implies that the two unions of balls cover the entire \( \mathbb{R}^d \), and that the level set is contained in their intersection; see Figure 2 and equations (10) and (11). Let now \( y \in \sigma^* \) and \( z \in A_\sigma \) be the endpoints of a fiber, and consider the ball with center \( y \) and squared radius \( \|z - y\|^2 \). It is not necessarily a ball \( b_t \) with \( b \in B \), but it is contained in the union of balls \( b_t \), with \( \pt(b) \) a vertex of \( \sigma^* \), and its boundary contains the intersection of the boundaries of these balls. It follows that it is orthogonal to or further than orthogonal from all balls \( c_t \), with \( c \in C \). By construction, \( z \in \del^{-1}[t, \infty) \), so there is a unique \( 0 < \lambda_{\min} \leq 1 \) such that a point \( x = (1-\lambda)y + \lambda z \) of the fiber belongs to \( \del^{-1}[t, \infty) \) iff \( \lambda_{\min} \leq \lambda \). In summary, the points at which the fiber intersects the level set all lie between \( y' = (1 - \lambda_{\min})y + \lambda_{\min}z \) and \( z \). Write \( [y', z] \) for this portion of the fiber, which we orient from \( y' \) to \( z \). It is not difficult to see that the restriction of \( \vor \) to \( [y', z] \) is a strictly increasing piecewise quadratic function. Similarly, the restriction of \( \del \) to \( [y', z] \) is a strictly increasing piecewise quadratic function. It follows that \( \sd \) restricted to \( [y', z] \) is a strictly increasing piecewise linear function, which implies that it crosses \( t \) exactly once. Hence, the fiber intersects \( \sd^{-1}(t) \) in exactly one point, as claimed.
Figure 9: Pictures of the same decomposition of the plane into “land” and “water”. All geometric structures are for the same value of $t$: (a) sub-, super-, and level sets of three continuous functions; (b) sub- and superlevel sets of the discrete functions on the Voronoi tessellation and the Delaunay mosaic; (c) channel divided by level set of piecewise linear function; (d) level sets of piecewise linear functions, with square boxes marking the neighborhoods of a non-critical point, a minimum, a saddle, and a maximum.