Average and Expected Distortion of Voronoi Paths and Scapes *

Herbert Edelsbrunner and Anton Nikitenko

IST Austria (Institute of Science and Technology Austria), Am Campus 1, 3400 Klosterneuburg, Austria, edels@ist.ac.at, anton.nikitenko@ist.ac.at

Abstract

The approximation of a circle with a fine square grid distorts the perimeter by a factor of $\frac{4}{\pi}$. We prove that this factor is the same *on average* for approximations of any curve with any Delaunay mosaic (known as *Voronoi path*), and extend the results to all dimensions, generalizing Voronoi paths to *Voronoi scapes*.

2010 AMS Mathematics Subject Classification: 60D05 Geometric probability and stochastic geometry, 68U05 Computer graphics; computational geometry.

Keywords and phrases: Distortion, Voronoi tessellations, Delaunay mosaics, Voronoi paths, Grassmannians, mixed volume, Poisson point processes, average, expectation.

1 Introduction

Given a locally finite set $A \subseteq \mathbb{R}^d$ and a line segment, the *Voronoi path* of the line segment is the dual of the Voronoi tessellation of A intersected with the segment. In other words, it consists of all Delaunay edges dual to Voronoi cells of dimension $d-1$ crossed by the line segment. We generalize it to the *Voronoi scape* of A and a p-dimensional set $\Omega \subseteq \mathbb{R}^d$, which contains a p-cell in the Delaunay mosaic of A with multiplicity μ, if Ω intersects the corresponding Voronoi $(d-p)$-cell in μ points. We are interested in the *distortion*, which is the ratio of the p-dimensional volume of the Voronoi scape over the volume of Ω.

Considering the Voronoi tessellation of a stationary Poisson point process and a line segment in $\mathbb{R}^2$, [2] proves that the expected distortion is $\frac{4}{\pi}$. Extending this work to $d > 2$ dimensions, [5] proves that the expected distortion is $\sqrt{2d/\pi} + O(1/\sqrt{d})$. We remove the ambiguity in this answer by proving that the expected distortion in $\mathbb{R}^d$ is $d!!/(d-1)!!$, if d is odd, and $2d!!/(\pi(d-1)!!)$, if d is even, in which !! is the double factorial. Furthermore, we generalize the result to any dimension and prove that for p-dimensional sets the expected distortion is the binomial coefficient $\binom{d/2}{p/2}$, in which non-integer parameters are understood in the way the Gamma function extends the factorial:

$$\mathcal{D}_{p,d} = \binom{d/2}{p/2} = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{p}{2}+1)\,\Gamma(\frac{d-p}{2}+1)} = \begin{cases} \frac{d!!}{p!!\,(d-p)!!}\frac{2}{\pi} & \text{if } d \text{ is even and } p \text{ is odd,} \\ \frac{d!!}{p!!\,(d-p)!!} & \text{otherwise.} \end{cases} \quad (1)$$

The binomial interpretation also provides the asymptotics for $\mathcal{D}_{p,d}$. Table 1 shows the values for small dimensions. More precisely, we prove that (1) is the *average distortion* for

* This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme, grant no. 788183, from the Wittgenstein Prize, Austrian Science Fund (FWF), grant no. Z 342-N31, and from the DFG Collaborative Research Center TRR 109, 'Discretization in Geometry and Dynamics', Austrian Science Fund (FWF), grant no. I 02979-N35.

sufficiently regular p-dimensional sets and Voronoi tessellations. The claim for stationary Poisson point processes follows because they are invariant under rotations and translations. The proof is based on a decomposition of $\mathbb{R}^d \times \mathbb{G}r_{p,d}$ that is related to the *mixed complex* [6]. As a byproduct, we get an expression for the volumes of the cells in the mixed complex; see Corollary 5.1.

	$d=1$	2	3	4	5	6	7	8	9	10
$p=1$	1	$\frac{4}{\pi}$	$\frac{3}{2}$	$\frac{16}{3\pi}$	$\frac{15}{8}$	$\frac{32}{5\pi}$	$\frac{35}{16}$	$\frac{256}{35\pi}$	$\frac{315}{128}$	$\frac{512}{63\pi}$
2		1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	4	$\frac{9}{2}$	5
3			1	$\frac{16}{3\pi}$	$\frac{5}{2}$	$\frac{32}{3\pi}$	$\frac{35}{8}$	$\frac{256}{15\pi}$	$\frac{105}{16}$	$\frac{512}{21\pi}$
4				1	$\frac{15}{8}$	3	$\frac{35}{8}$	6	$\frac{63}{8}$	10
5					1	$\frac{32}{5\pi}$	$\frac{7}{2}$	$\frac{256}{15\pi}$	$\frac{63}{8}$	$\frac{512}{15\pi}$
6						1	$\frac{35}{16}$	4	$\frac{105}{16}$	10
7							1	$\frac{256}{35\pi}$	$\frac{9}{2}$	$\frac{512}{21\pi}$
8								1	$\frac{315}{128}$	5
9									1	$\frac{512}{63\pi}$
10										1

Table 1: The average resp. expected distortion in small dimensions. Note that even rows and columns form the Pascal triangle.

Outline. Section 2 prepares the proof of our main result by computing the first and second moments of the p-dimensional volume of the projection of a unit p-cube in $\mathbb{R}^d$. Section 3 studies the space of point-direction pairs. Section 4 introduces a mild regularity condition for Voronoi tessellations. Section 5 computes the volume of the cells in the mixed complex. Section 6 proves that $\mathcal{D}_{p,d}$ is the average distortion for p-dimensional shapes in $\mathbb{R}^d$, and the expected distortion if the tessellation is of a stationary Poisson point process. Section 7 concludes the paper.

2 Random Projections

We need some preliminary computations. Let $\mathbb{G}r_{p,d}$ be the *linear Grassmannian manifold*, whose points are the p-planes that pass through the origin of $\mathbb{R}^d$. Given a p-dimensional unit cube, $E \subseteq \mathbb{R}^d$, and a p-plane, $L \in \mathbb{G}r_{p,d}$, we write $E|_L$ for the projection of the cube onto the plane, and $\|E|_L\|_p$ for its p-dimensional volume. The *j-th projection moment* is the average j-th power of the volume of the projection. We express this moment as an integral over the Grassmannian equipped with the uniform probability measure in (2) and convert it to two equivalent expressions involving the angle to a fixed plane in (3) and (4):

$$\mathbf{m}_{p,d}^{(j)} = \int_{L \in \mathbb{G}r_{p,d}} \|E|_L\|_p^j \,\mathrm{d}L, \tag{2}$$

$$= \int_{L \in \mathbb{G}r_{p,d}} \cos^j \varphi(L, L_0) \,\mathrm{d}L \tag{3}$$

$$= \int_{F \in \mathbb{S}t_{p,d}} \|F|_{L_0}\|_p^j \,\mathrm{d}F. \tag{4}$$

To explain (3) and (4), we fix the plane $L_0 \in \mathbb{G}r_{p,d}$ containing E. The *angle* between two p-planes, $\varphi(L, L_0) \in [0, \frac{\pi}{2}]$, is defined as the arc-cosine of the ratio of $\|B|_L\|_p$ over $\|B\|_p$

for any compact set with non-empty interior, $B \subseteq L_0$. The angle is symmetric, so we can instead consider the integrand in (3) as the projection of a unit p-cube in a random p-plane onto L_0. Formally, we write $\mathbb{S}t_{p,d}$ for the *Stiefel manifold* of orthonormal p-frames in $\mathbb{R}^d$, we identify a frame with the unit p-cube it spans, and we integrate using the uniform probability measure of $\mathbb{S}t_{p,d}$ to arrive at (4).

By construction, the 0-th projection moment is equal to 1, independent of p and d. We compute the 1-st and 2-nd projection moments, which curiously both have intuitive geometric interpretations.

▶ **Lemma 2.1** (Projection Moments). *Let $d \geq 0$ and $0 \leq p \leq d$. Then*

$$\mathbf{m}_{p,d}^{(1)} = \frac{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{d-p+1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\frac{d+1}{2})}, \tag{5}$$

$$\mathbf{m}_{p,d}^{(2)} = 1/\binom{d}{p} = \frac{p!\,(d-p)!}{d!}. \tag{6}$$

Proof. The 1-st projection moment appears in the classic Crofton formula of integral geometry, which says that the volume of a convex body is proportional to the average volume of its projections. The constant of proportionality given in (5) can be found in [8, Formula (5.8)]. We use (4) together with a generalization of the Pythagorean theorem to compute the 2-nd moment. By Pythagoras, the squared length of a line segment is the sum of squared lengths of its projections onto the coordinate axes. The Cauchy–Binet formula [4, §4.6] can be used to generalize this to the squared volume of a p-dimensional parallelepiped in $\mathbb{R}^d$. Let P be such a parallelepiped, and write P_i for its projection onto the i-th coordinate p-plane (in which the numbering is arbitrary). There are $\binom{d}{p}$ coordinate p-planes, and the Cauchy–Binet formula asserts

$$\|P\|_p^2 = \sum_{i=1}^{\binom{d}{p}} \|P_i\|_p^2. \tag{7}$$

Letting $P = F \in \mathbb{S}t_{p,d}$ be the uniformly random unit p-cube, we can take the expectation on both sides of (7). We get 1 on the left-hand side and the sum of $\binom{d}{p}$ identical terms on the right-hand side. Hence, the average squared p-dimensional volume of the projection is $1/\binom{d}{p}$, as claimed. ∎

We set $\mathcal{D}_{p,d} = \mathbf{m}_{p,d}^{(1)}/\mathbf{m}_{p,d}^{(2)}$ and leave it to the reader to verify that this agrees with (1), where $\mathcal{D}_{p,d}$ is given in terms of Gamma functions as well as double factorials.

3 Tiling the Space of Point-Directions

We use the Delaunay mosaic to tile the space of *point-direction pairs*, $\mathbb{R}^d \times \mathbb{G}r_{p,d}$. Given a Delaunay mosaic of a set $A \subseteq \mathbb{R}^d$, denoted $\mathrm{Del}(A)$, consider a p-dimensional cell, $\gamma \in \mathrm{Del}(A)$, and its dual $(d-p)$-dimensional Voronoi cell, $\gamma^* \in \mathrm{Vor}(A)$. We define the *$p$-tile* of γ to consist of all pairs $(x, L) \in \mathbb{R}^d \times \mathbb{G}r_{p,d}$ such that $L + x$ has a non-empty intersection with γ^*, and x lies in the projection of γ onto $L + x$:

$$J(\gamma, \gamma^*) = \{(x, L) \in \mathbb{R}^d \times \mathbb{G}r_{p,d} \mid x \in \gamma|_{L+x} \text{ and } (L + x) \cap \gamma^* \neq \emptyset\}. \tag{8}$$

The *tiles* decompose the space $\mathbb{R}^d \times \mathbb{G}r_{p,d}$ in the sense that they cover the space of point-direction pairs and their interiors are pairwise disjoint. Since the detailed analysis of the boundaries is irrelevant for the current work, we only prove a weaker statement.

▶ **Lemma 3.1** (Uniqueness of Tile). *Let $A \subseteq \mathbb{R}^d$ be locally finite with* $\mathrm{conv} A = \mathbb{R}^d$*, and let* $0 \leq p \leq d$*. Then for almost every point-direction pair,* $(x, L) \in \mathbb{R}^d \times \mathbb{G}r_{p,d}$*, there exists a unique p-tile,* $J(\gamma, \gamma^*)$*, that contains* (x, L)*.*

Proof. Take any point-direction pair, (x, L). Assume without loss of generality that $x = 0$ is the origin and $L = \mathbb{R}^p$ is a coordinate p-plane in $\mathbb{R}^d$. Map each point $a \in A$ to the point $a' = a|_L \in \mathbb{R}^p$, and let $a'' = -\|a - a'\|^2 \in \mathbb{R}$ be its *weight*. The weighted points define a weighted Voronoi tessellation and the corresponding weighted Delaunay mosaic; see e.g. [1]. The mosaic is generically a simplicial complex and generally a polyhedral complex, which is geometrically realized in $\mathbb{R}^p$ by drawing each cell, γ, as the convex hull of the points that generate the p-dimensional Voronoi cells sharing γ^*. Consider the cells in $\mathrm{Vor}(A)$ that have a non-empty intersection with L, write $\mathcal{V}_L(A)$ for the collection of dual cells in $\mathrm{Del}(A)$, and observe that $\mathcal{V}_L(A)$ is the Voronoi scape of L.

As proved in [9], the weighted Voronoi tessellation is the intersection of L with $\mathrm{Vor}(A)$ and, by duality, the weighted Delaunay mosaic is the orthogonal projection of $\mathcal{V}_L(A)$ to L. If L and $\mathrm{Del}(A)$ are in general position, then all Delaunay cells in $\mathcal{V}_L(A)$ project injectively to L, and the cells of dimension less than p form a set of zero measure. If $\mathrm{Del}(A)$ covers $\mathbb{R}^d$, then the weighted Delaunay mosaic covers $\mathbb{R}^p$. Hence, for almost all point-direction pairs, (x, L), there is a unique Delaunay p-cell γ, such that $(x, L) \in J(\gamma, \gamma^*)$, as claimed. ∎

The proof of the lemma gives some insight into the motivation for choosing this particular tiling of the space of point-direction pairs. We now compute the measure of a tile.

▶ **Lemma 3.2** (Volume of Tile). *The measure of* $J = J(\gamma, \gamma^*)$ *is* $\|J\| = \|\gamma\|_p \, \|\gamma^*\|_{d-p} / \binom{d}{p}$*.*

Proof. The measure of the tile is the integral of 1 over its pairs. Setting $x = y + z$, in which $y \in L$ and $z \in L^\perp$, the integral is

$$\|J\| = \int_{L \in \mathbb{G}r_{p,d}} \int_{y \in L} \mathbf{1}_{y \in \gamma|_L} \int_{z \perp L} \mathbf{1}_{(L+z) \cap \gamma^* \neq \emptyset} \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}L \tag{9}$$

$$= \|\gamma\|_p \, \|\gamma^*\|_{d-p} \int_{L \in \mathbb{G}r_{p,d}} \cos^2 \varphi(L, \gamma) \, \mathrm{d}L, \tag{10}$$

where we get (10) by noticing that the innermost integral in (9) is the $(d-p)$-dimensional volume of the projection of γ^* to $L^\perp$, which is $t \, \|\gamma^*\|_{d-p}$ with $t = \cos \varphi(L, \gamma)$, and the middle integral is the p-dimensional volume of the projection of γ to L, which is $t \, \|\gamma\|_p$. Using (3) and (6), we see that the integral in (10) is $\mathbf{m}_{p,d}^{(2)}$. ∎

We take a closer look at the projection of a tile to $\mathbb{R}^d$. Let (x, L) be a point-direction pair in $J = J(\gamma, \gamma^*)$ with $\dim \gamma = p$. There are points $u \in \gamma$ and $v \in \gamma^*$ such that $x = u|_{L+x}$ and $v = (L + x) \cap \gamma^*$. Because of the right angle between the direction and the projection, we have $\|x - u\|^2 + \|x - v\|^2 = \|u - v\|^2$, so x lies on the smallest sphere that passes through u and v. Indeed, u and v define a $(d-1)$-dimensional set of point-direction pairs, and the points of these pairs all lie on the mentioned sphere. Let $z_0 = z(\gamma, \gamma^*) = \mathrm{aff}\, \gamma \cap \mathrm{aff}\, \gamma^*$ and observe that the sphere defined by u and v also passes through z_0. Let $R_0 = R(\gamma, \gamma^*)$ be the maximum distance between a point of γ and a point of γ^* and note that R_0 is the radius of every largest empty sphere that passes through the vertices of γ. Since the diameter of the sphere spanned by u and v is $\|u - v\| \leq R_0$, it follows that the ball with center z_0 and radius R_0 contains this sphere. This implies an upper bound on the size of the projection of the tile to $\mathbb{R}^d$, which we state formally for later reference.

▶ **Lemma 3.3** (Projection of Tile). *The projection of $J = J(\gamma, \gamma^*)$ to $\mathbb{R}^d$ is contained in the closed ball with center $z_0 = z(\gamma, \gamma^*)$ and radius $R_0 = R(\gamma, \gamma^*)$.*

It follows that the volume of the projection of the tile to $\mathbb{R}^d$ is at most R_0^d times the volume of the unit ball in $\mathbb{R}^d$. Since we assume the uniform probability measure on $\mathbb{G}r_{p,d}$, the same upper bound holds for the measure of the tile itself.

4 Mixed Regularity

Taking the union of progressively more tiles, we eventually cover all of $\mathbb{R}^d \times \mathbb{G}r_{p,d}$. However, at each step some of the points miss some of the directions, and which directions are covered depends on the mosaic. For what follows, we require a mild regularity condition for this tiling. For a set $\Omega \subseteq \mathbb{R}^d$ we call a tile a *boundary tile of* Ω if its projection to $\mathbb{R}^d$ contains at least one point inside and at least one point outside Ω.

▶ **Definition 4.1** (Mixed Regularity). Let $A \subseteq \mathbb{R}^d$ be locally finite. We say that A has the property of *mixed regularity* if, for any p, the total measure of the boundary p-tiles of a d-ball of radius R centered at the origin is $o(R^d)$.

Note that $\mathrm{conv}A = \mathbb{R}^d$ is necessary for A to have the mixed regularity property. Indeed, if $\mathrm{conv}A$ does not cover $\mathbb{R}^d$, then there exists an unbounded Voronoi cell and thus a tile with infinite measure. Motivated by the analysis in Section 3, we give some sufficient conditions for a set $A \subseteq \mathbb{R}^d$ to satisfy the mixed regularity property.

▶ **Lemma 4.2** (Sufficient Conditions). *A locally finite set $A \subseteq \mathbb{R}^d$ has the mixed regularity property if one of the following holds:*

1. *The radii of all circumspheres of top-dimensional Delaunay cells are bounded.*
2. *Each ball in $\mathbb{R}^d$ of radius greater than R_0 contains a point of A.*
3. *There is a function $g(R) = o(R)$ such that every ball of radius $g(R)$ that intersects the d-ball of radius R centered at the origin contains at least one point of A.*

Conditions 1 and 2 are equivalent, while the last one is weaker. We finish the section with an application to Poisson point processes.

▶ **Lemma 4.3** (Mixed Regularity in Expectation). *A stationary Poisson point process $A \subseteq \mathbb{R}^d$ satisfies the mixed regularity property in expectation; that is: the total* expected *measure of the boundary tiles of a d-ball of radius R centered at the origin is $o(R^d)$.*

Proof. For each boundary tile, its Delaunay cell (which is almost surely a simplex) is a face of a top-dimensional Delaunay simplex whose circumsphere intersects the boundary of the ball. In [7, Appendix A] it is proved that the total number of such spheres is $o(R^d)$. A minor modification of that proof suffices to show that the total volume of these balls is $o(R^d)$. Together with Lemma 3.3, this implies the claimed bound. ■

5 Mixed Volume

Call $\|\gamma\|_p \|\gamma^*\|_{d-p}$ the *mixed volume* of a p-cell $\gamma \in \mathrm{Del}(A)$ and its dual $(d-p)$-cell $\gamma^* \in \mathrm{Vor}(A)$. We note that this concept is related to a particular decomposition of $\mathbb{R}^d$, as we now explain. Given $A \subseteq \mathbb{R}^d$, the d-dimensional cells of the *mixed complex* defined in [6] are translates of the products $\frac{1}{2}\gamma \times \frac{1}{2}\gamma^*$. The d-dimensional volume of this cell is $\|\frac{1}{2}\gamma \times \frac{1}{2}\gamma^*\|_d = \|\gamma\|_p \|\gamma^*\|_{d-p}/2^d$. As proved in [6], the cells in the mixed complex have pairwise disjoint interiors and they cover $\mathbb{R}^d$. Assuming the mixed regularity property, this implies that, up to a lower order term, the cells for $p = 0$ cover a fraction of $1/2^d$ of the ball of radius R centered at the origin. By symmetry, this is also true for $p = d$. We continue with a generalization of these bounds to dimension p between 0 and d.

▶ **Corollary 5.1** (Mixed Volumes). *Let $A \subseteq \mathbb{R}^d$ have the mixed regularity property. For any $0 \leq p \leq d$, the sum of the mixed volumes over all p-dimensional Delaunay cells contained in a ball of radius R is $\nu_d \binom{d}{p} R^d + o(R^d)$, in which ν_d is the volume of the unit ball in $\mathbb{R}^d$.*

Proof. Let $B(R)$ be the ball with radius R centered at the origin. Set $\mathcal{B}_p(R) = B(R) \times \mathbb{G}r_{p,d}$, let $M_p(R)$ be the smallest collection of p-tiles whose union contains $\mathcal{B}_p(R)$, and let $\partial M_p(R)$ be the boundary tiles of $B(R)$. Clearly,

$$M_p(R) \setminus \partial M_p(R) \subseteq \mathcal{B}_p(R) \subseteq M_p(R). \tag{11}$$

We note, that if a tile, $J = J(\gamma, \gamma^*)$, contains a point inside the ball, then either γ is inside the ball, or J is a boundary tile. Indeed, for every point $x \in \gamma \setminus B(R)$, there is a direction L, such that $L + x$ intersects γ^*, hence $(x, L) \in J(\gamma, \gamma^*)$. By Lemma 3.2, the measure of this tile is $\|\gamma\|_p \|\gamma^*\|_{d-p} / \binom{d}{p}$ and, by the mixed regularity property, the measures of the tiles corresponding to Delaunay cells inside the ball sum up to $\|B(R)\|_d (1 + o(1))$. Multiplying by $\binom{d}{p}$ concludes the proof. ■

6 Average and Expected Distortion

Recall that the Voronoi scape of a set $\Omega \subseteq \mathbb{R}^d$ and a locally finite set $A \subseteq \mathbb{R}^d$ consists of all Delaunay cells whose dual Voronoi cells have a non-empty intersection with Ω. Assuming Ω is p-dimensional, only intersections with $(d-p)$-dimensional Voronoi cells are relevant. Generically, such an intersection has a finite *multiplicity*, and we define the *Voronoi scape*, denoted $\mathcal{V}_\Omega(A)$, as the multiset of p-cells, $\gamma \in \mathrm{Del}(A)$, in which γ appears the multiplicity of the intersection between Ω and γ^* times. We are ready to prove the main result of the paper.

▶ **Theorem 6.1** (Average Volume). *Let $A \subseteq \mathbb{R}^d$ have the mixed regularity property, and let Ω be a p-dimensional rectifiable set in $\mathbb{R}^d$. The average volume of $\mathcal{V}_\Omega(A)$ over all congruent copies of Ω inside the d-ball with radius R centered at the origin is $\|\Omega\|_p (\mathcal{D}_{p,d} + o(1))$ as R goes to infinity.*

Proof. We start with the Crofton formula [8, Formula (5.7)], which states that

$$\int_{Q \in \mathbb{G}r_{d-p,d}} \int_{y \in Q^\perp} \chi((Q+y) \cap \Omega) \,\mathrm{d}y \,\mathrm{d}Q = \mathbf{m}^{(1)}_{p,d} \|\Omega\|_p, \tag{12}$$

in which $\chi((Q+y) \cap \Omega)$ is the almost always finite multiplicity of the intersection between $Q + y$ and Ω. Fixing a $(d-p)$-dimensional moving polyhedron, P, we notice that for each intersection point of Q and Ω, the total measure of congruent copies of P that intersect Ω at this point is $\|P\|_{d-p}$. Hence, the total measure of intersection points over all congruent copies of P is

$$\int_{P' \cong P} \chi(P' \cap \Omega) \,\mathrm{d}P' = \|P\|_{d-p} \int_{Q \in \mathbb{G}r_{d-p,d}} \int_{y \in Q^\perp} \chi((Q+y) \cap \Omega) \,\mathrm{d}y \,\mathrm{d}Q \tag{13}$$

$$= \|P\|_{d-p} \|\Omega\|_p \mathbf{m}^{(1)}_{p,d}. \tag{14}$$

Taking $P = \gamma^*$ and moving Ω instead of the polyhedron, we see that the total measure of intersection points of congruent copies of Ω with γ^* is $\|\gamma^*\|_{d-p}\|\Omega\|_p \mathbf{m}^{(1)}_{p,d}$.

A p-cell $\gamma \in \mathrm{Del}(A)$ belongs to the Voronoi scape of a congruent copy Ω' of Ω precisely $\chi(\Omega' \cap \gamma^*)$ times, and we just computed the total value of this quantity. The total contribution of γ to the p-volume of the Voronoi scapes of the congruent copies of Ω is thus $\|\gamma\|_p\|\gamma^*\|_{d-p}\|\Omega\|_p \mathbf{m}^{(1)}_{p,d}$. We get the final result by dividing the total contribution of the Delaunay cells inside the ball—which we compute using Corollary 5.1—by the total measure of the congruent copies inside the ball:

$$\frac{\|B(R)\|_d\|\Omega\|_p \mathbf{m}^{(1)}_{p,d}/\mathbf{m}^{(2)}_{p,d}}{\|B(R)\|_d(1+o(1))} = \|\Omega\|_p(\mathcal{D}_{p,d} + o(1)), \tag{15}$$

in which we use (5) and (6) to get the expression on the right. ∎

We finish with stating the answer to the original question that motivated the work reported in this paper. We showed in Section 4 that the stationary Poisson point process has the mixed regularity property in expectation, which allows us to repeat all results while adding the expectation to all quantities. By the isometry invariance of the process, for any set Ω, the expected volume of $\mathcal{V}_\Omega(A)$ does not depend on the position of Ω. Exchanging the expectation and the average inside the ball of radius R centered at the origin and letting R go to infinity, we arrive at probabilistic versions of Theorem 6.1.

▶ **Theorem 6.2** (Expected Volume). *Let $A \subseteq \mathbb{R}^d$ be a stationary Poisson point process with intensity $\rho > 0$, and let Ω be a compact rectifyable p-manifold in $\mathbb{R}^d$. Then the expected p-dimensional volume of the Voronoi scape of Ω is $\mathcal{D}_{p,d}\|\Omega\|_p$.*

7 Discussion

The main contribution of this paper is a complete analysis of the average and expected distortion of p-dimensional Voronoi scapes in $\mathbb{R}^d$, for $0 \le p \le d$. For $p = 1$, these scapes are known as Voronoi paths, for which the expected distortion has been studied but was known only in $\mathbb{R}^2$; see [2]. A useful insight from our analysis is that the expected distortion for a stationary Poisson point process is the average distortion of a general locally finite point set. We make crucial use of this insight in the proof of our results. *Can these results be extended to other measures, such as notions of curvature, for example?* The proof of Theorem 6.1 suggests that this extension would require a detailed analysis of the Crofton formula. Insights in this direction could be helpful in using the Voronoi scape to measure otherwise difficult to measure shapes [3].

In our analysis, the properties that make a mosaic a Delaunay mosaic are not used other than in the quantification of the mixed regularity property for locally finite sets. Indeed, we only need a pair of dual complexes in which dual cells are orthogonal to each other, a property that holds also for the generalizations of Voronoi tessellations and Delaunay mosaics to points with real weights; see e.g. [1].

Acknowledgements

The authors thank Ranita Biswas and Tatiana Ezubova for the collaboration on an experimental project that motivated the work reported in this paper.

References

1 F. Aurenhammer. Power diagrams: properties, algorithms and applications. *SIAM J. Comput.* **16** (1987), 78–96.

2 F. Baccelli, K. Tchoumatchenko and S. Zuyev. Markov paths on the Poisson–Delaunay graph with applications to routing in mobile networks. *Adv. Appl. Probab.* **32** (2000), 1–18.

3 R. Biswas, H. Edelsbrunner, T. Ezubova and A. Nikitenko. Image analysis with isotropic random background. Manuscript, IST Austria, Klosterneuburg, Austria, 2020.

4 J.G. Broida and S.G. Williamson. *A Comprehensive Introduction to Linear Algebra.* Addison-Wesley, Reading, Massachusetts, 1989.

5 P.M.M. de Castro and O. Devillers. Expected length of the Voronoi path in a high dimensional Poisson–Delaunay triangulation. *Discrete Comput. Geom.* **60** (2018), 200–219.

6 H. Edelsbrunner. Deformable smooth surface design. *Discrete Comput. Geom.* **21** (1999), 87–115.

7 H. Edelsbrunner, A. Nikitenko and M. Reitzner. Expected sizes of Poisson–Delaunay mosaics and their discrete Morse functions. *Adv. Appl. Prob.* **49** (2017), 745–767.

8 R. Schneider and W. Weil. *Stochastic and Integral Geometry.* Springer, Berlin, Germany, 2008.

9 R. Sibson. A vector identity for Dirichlet tessellations. *Math. Proc. Cambridge* **87** (1980), 151–155.