We are given an input $A \in \mathbb{Q}^{m \times n}$, $b \in \mathbb{Q}^{m}$, $c \in \mathbb{Q}^{n}$ and we search for $x \in \mathbb{Q}^{n}$ that optimizes...
We are given an input $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$ and we search for $x \in \mathbb{Z}^n$ that optimizes...

- **minimize** $c'x$
  - s.t. $Ax \geq b$
  - where $x \geq 0$

- **maximize** $c'x$
  - s.t. $Ax \leq b$
  - where $x \geq 0$

- **minimize** $c'x$
  - s.t. $Ax = b$
  - where $x \geq 0$

- **maximize** $c'x$
  - s.t. $Ax = b$
  - where $x \geq 0$
Example of LP
Separating points

We are searching for a line \( y = ax + b \) that separates "disks" (top) from "squares" (bottom).

Variables: \( a \leq 0, \ b \leq 0 \)
Point \((x_1, y_1)\) above the line: \( x_1 \cdot a + b \leq y_1 \) "disk"
Point \((x_2, y_2)\) below the line: \( x_2 \cdot a + b \geq y_2 \) "square"
Objective: minimize 0
Example of ILP  
Separating points partially

We are searching for a line $y = ax + b$ that separates given points — "disks" (top) from "squares" (bottom) — with as few exceptions as possible.

Let us ignore all vertical and nearly-vertical solutions.

$M$ is a very large integer.

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Variables: \( a \leq 0, b \leq 0; \forall i : p_i \in \{0, 1\} \)
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Objective: minimize $\sum_i p_i$
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Unfortunately, there is no general algorithm for solving ILP in polynomial time
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Unfortunately, there is no general algorithm for solving ILP in polynomial time (we don’t know any such algorithm — but most importantly, it cannot exist unless $P = NP$).
Does it mean, however, that our problem cannot be solved in polynomial time?
Combinations of vectors (linear, affine, convex)

Definition

Let $V$ be a vector space over $F$. Consider vectors $x_1, \ldots, x_n \in V$.

We say that $y \in V$ is a linear combination of $x_1, \ldots, x_n$ if:

$\exists \alpha_1, \ldots, \alpha_n \in F : \sum_{i=1}^{n} \alpha_i x_i = y$

We say that $y \in V$ is an affine combination of $x_1, \ldots, x_n$ if:

$\exists \alpha_1, \ldots, \alpha_n \in F : \sum_{i=1}^{n} \alpha_i = 1 \land \sum_{i=1}^{n} \alpha_i x_i = y$

Suppose now that $F$ is a totally ordered field.

Denote the set of $\phi \in F$ such that $\phi \geq 0$ by the symbol $F^+_0$.

We say that $y \in V$ is a convex combination of $x_1, \ldots, x_n$ if:

$\exists \alpha_1, \ldots, \alpha_n \in F^+_0 : \sum_{i=1}^{n} \alpha_i = 1 \land \sum_{i=1}^{n} \alpha_i x_i = y$
Combinations of vectors (linear, affine, convex)

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$$

Suppose now that $F$ is a totally ordered field. Denote the set of $\varphi \in F$ such that $\varphi \geq 0$ by the symbol $F^+$. We say that $y \in V$ is a **convex combination** of $x_1, \ldots, x_n$ if:

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\exists \alpha_1, \ldots, \alpha_n \in F^+ : \sum_{i=1}^{n} \alpha_i x_i = y
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- Suppose now that $F$ is a totally ordered field. Denote the set of $\varphi \in F$ such that $\varphi \geq 0$ by the symbol $F_0^+$. We say that $y \in V$ is a **convex combination** of $x_1, \ldots, x_n$ if:
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Combinations of vectors (linear, affine, convex)

Example
Combinations of vectors (linear, affine, convex)
Quiz

Let $d \in \mathbb{Z}$ such that $d \geq 2$.
What is the maximum possible amount (largest set) of:

- **Linearly independent vectors in** $\mathbb{Z}_2^d$ **over** $\mathbb{Z}_2$?

- **Affinely independent vectors in** $\mathbb{C}^d$ **over** $\mathbb{C}$?

- **Convexly independent vectors in** $\mathbb{Q}^d$ **over** $\mathbb{Q}$?
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Combinations of vectors (linear, affine, convex)

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Combinations of vectors (linear, affine, convex)

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- Convexly independent vectors in $\mathbb{Q}^d$ over $\mathbb{Q}$? $\infty$
- Affinely independent vectors in $\mathbb{C}^d$ over $\mathbb{R}$?
- Linearly independent vectors in $\mathbb{R}^d$ over $\mathbb{Q}$?
Let $d \in \mathbb{Z}$ such that $d \geq 2$.

What is the \textbf{maximum} possible amount (largest set) of:

- \textbf{Linearly} independent vectors in $\mathbb{Z}_2^d$ over $\mathbb{Z}_2$? $d$
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- \textbf{Convexly} independent vectors in $\mathbb{Q}^d$ over $\mathbb{Q}$? $\infty$
- \textbf{Affinely} independent vectors in $\mathbb{C}^d$ over $\mathbb{R}$? $2d + 1$
- \textbf{Linearly} independent vectors in $\mathbb{R}^d$ over $\mathbb{Q}$?
Combinations of vectors (linear, affine, convex)
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  - $d$

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  - $d + 1$

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Combinations of vectors (linear, affine, convex)

Basic properties

\[
\{ \text{convex combin.} \} \subseteq \{ \text{affine combin.} \} \subseteq \{ \text{linear combin.} \}
\]

Let \( V \) be a vector space and \( X \subseteq V \).

- If \( X \) is convexly dependent, then \( X \) is affinely dependent.
- If \( X \) is affinely dependent, then \( X \) is linearly dependent.
- If \( X \) is linearly independent, then \( X \) is affinely independent.

Let us have a matrix \( A \in F^{m \times n} \) and a vector \( b \in F^{m} \).

We search for a solution \( x \in F^{n} \).

- If \( Ax = 0 \), any linear combination of solutions is a solution.
- If \( Ax = b \), any affine combination of solutions is a solution.
- If \( Ax \leq b \), any convex combination of solutions is a solution.

In this example, \( F \) must be a totally ordered field.
Combinations of vectors (linear, affine, convex)

Basic properties

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Let \( V \) be a vector space and \( X \subseteq V \).

- \( X \) is convexly dependent. \( \implies \) \( X \) is affinely dependent.
  \[ \implies X \text{ is linearly dependent.} \]
- \( X \) is linearly independent. \( \implies \) \( X \) is affinely independent.
  \[ \implies X \text{ is convexly independent.} \]
Combinations of vectors (linear, affine, convex)

Basic properties

\{\text{convex combin.}\} \subseteq \{\text{affine combin.}\} \subseteq \{\text{linear combin.}\}

Let $V$ be a vector space and $X \subseteq V$.

- $X$ is convexly dependent. $\implies$ $X$ is affinely dependent.
  $\implies$ $X$ is linearly dependent.

- $X$ is linearly independent. $\implies$ $X$ is affinely independent.
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Let us have a matrix $A \in F^{m \times n}$ and a vector $b \in F^m$.

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Convex hull  
Definition and exercises

*Def.* Let $V$ be a vector space over a totally ordered field. Consider a set $X \subseteq V$. We define a convex hull of $X$ as:

$$\text{conv}(X) = \{y \in V \mid y \text{ is a convex combin. of some } x_1, x_2, \ldots \in X\}$$
Def. Let $V$ be a vector space over a totally ordered field. Consider a set $X \subseteq V$. We define a convex hull of $X$ as:

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Ex. Prove this set identity:

$$\text{conv}(\text{conv}(X)) = \text{conv}(X)$$
Convex hull

Definition and exercises

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Ex. Let $v \in V$. If $M \subseteq V$, we define $M + v$ as $\{m + v \mid m \in M\}$. Prove this set identity:

$$\text{conv}(X) + v = \text{conv}(X + v)$$
**Def.** Let $V$ be a vector space over a totally ordered field. Consider a set $X \subseteq V$. We define a convex hull of $X$ as:

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**Ex.** Prove this set identity:

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**Ex.** Let $v \in V$. If $M \subseteq V$, we define $M + v$ as $\{ m + v \mid m \in M \}$. Prove this set identity:

$$\text{conv}(X) + v = \text{conv}(X + v)$$

**Ex.** Express a $d$-dimensional simplex (triangle, tetrahedron,...) as:

- A convex hull of $d + 1$ points.
- An intersection of $d + 1$ closed half-spaces.
Def. Let \( A \in \mathbb{R}^{m \times n} \), \( b \in \mathbb{R}^m \) where \( m \geq n \). Consider a set \( P \subseteq \mathbb{R}^n \) defined by \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \), that is, a polyhedron. Let \( p \in P \). The following conditions are equivalent formulations of \( p \) being a vertex of the polyhedron \( P \).

- No vector \( y \neq 0 \) satisfies both \( p + y \in P \) and \( p - y \in P \).
- We have \( p \notin \text{conv}(P \setminus \{ p \}) \).
- There exists a hyperplane \( H \) of dimension \( n - 1 \) such that \( P \cap H = \{ p \} \).
  Recall that \( H = \{ x \in \mathbb{R}^n \mid h'x = r \} \) for some \( h \in \mathbb{R}^n \), \( r \in \mathbb{R} \).
- There is a cost vector \( c \in \mathbb{R}^n \) such that \( p \) is the unique maximum of the corresponding cost function, that is, \( \forall x \in (P \setminus \{ p \}) \) we have \( c'p > c'x \).
- There are \( n \) linearly independent constraints tight (=) at \( p \).

Ex. Prove their equivalence.
Thanks for your attention!

Questions?