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CEGAR for Compositional Analysis of Qualitative Properties in Markov Decision Processes

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Abstract We consider Markov decision processes (MDPs) which are a standard model for probabilistic systems. We focus on qualitative properties for MDPs that can express that desired behaviors of the system arise almost-surely (with probability 1) or with positive probability. We introduce a new simulation relation to capture the refinement relation of MDPs with respect to qualitative properties, and present discrete graph algorithms with quadratic complexity to compute the simulation relation. We present an automated technique for assume-guarantee style reasoning for compositional analysis of two-player games by giving a counterexample guided abstraction-refinement approach to compute our new simulation relation. We show a tight link between two-player games and MDPs, and as a consequence the results for games are lifted to MDPs with qualitative properties. We have implemented our algorithms and show that the compositional analysis leads to significant improvements.

Keywords Probabilistic systems · Games · ATL and ATL* · CEGAR · Simulation and alternating simulation

1 Introduction

Markov decision processes. *Markov decision processes (MDPs)* are standard models for analysis of probabilistic systems that exhibit both probabilistic and non-deterministic behavior [61,54]. In verification of probabilistic systems, MDPs have been adopted as models for concurrent probabilistic systems [47], probabilistic systems operating in open environments [76], under-specified probabilistic systems [10], and applied in diverse domains [7, 67] such as analysis of randomized communication and security protocols, stochastic distributed systems, biological systems, etc.

Compositional analysis and CEGAR. One of the key challenges in analysis of probabilistic systems (as in the case of non-probabilistic systems) is the *state explosion* problem [44], as the size of concurrent systems grows exponentially in the number of components. One key technique to combat the state explosion problem is the *assume-guarantee* style composition reasoning [73], where the analysis problem is decomposed into components and the

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results for components are used to reason about the whole system, instead of verifying the whole system directly. For a system with two components, the compositional reasoning can be captured as the following simple rule: consider a system with two components G_1 and G_2, and a specification G' to be satisfied by the system; if A is an abstraction of G_2 (i.e., G_2 refines A) and G_1 in composition with A satisfies G', then the composite systems of G_1 and G_2 also satisfies G'. Intuitively, A is an assumption on G_1's environment that can be ensured by G_2. This simple, yet elegant asymmetric rule is very effective in practice, specially with a *counterexample guided abstraction-refinement* (CEGAR) loop [45]. There are many symmetric [71] as well as circular compositional reasoning [50,71,68] rules; however the simple asymmetric rule is most effective in practice and extensively studied, mostly for non-probabilistic systems [71,53,14,59].

Compositional analysis for probabilistic systems. There are many works that have studied the abstraction-refinement and compositional analysis for probabilistic systems [13,60,66, 52]. Our work is most closely related to and inspired by [65] where a CEGAR approach was presented for analysis of MDPs (or labeled probabilistic transition systems); and the refinement relation was captured by *strong simulation* that captures the logical relation induced by safe-pCTL [56,4,10].

Qualitative analysis and its importance. In this work we consider the fragment of pCTL* [56,4,10] that is relevant for *qualitative analysis*, and refer to this fragment as QCTL^*. The qualitative analysis for probabilistic systems refers to *almost-sure* (resp. *positive*) properties that are satisfied with probability 1 (resp. positive probability). The qualitative analysis for probabilistic systems is an important problem in verification that is of interest independent of the quantitative analysis problem. There are many applications where we need to know whether the correct behavior arises with probability 1. For instance, when analyzing a randomized embedded scheduler, we are interested in whether every thread progresses with probability 1 [22]. Even in settings where it suffices to satisfy certain specifications with probability $\lambda < 1$, the correct choice of λ is a challenging problem, due to the simplifications introduced during modeling. For example, in the analysis of randomized distributed algorithms it is quite common to require correctness with probability 1 (see, e.g., [74,78]). Furthermore, in contrast to quantitative analysis, qualitative analysis is robust to numerical perturbations and modeling errors in the transition probabilities. The qualitative analysis problem has been extensively studied for many probabilistic models, such as for MDPs [47,80,34,35,37,36,42], perfect-information stochastic games [40,41,23,15, 16], concurrent stochastic games [51,26,25,24,17,39,38], partial-observation MDPs [6,43, 20,30,19], partial-observation stochastic games [32,9,27,31,70,33,28], and real-timed systems [11,5].

Our contributions. In this work we focus on the compositional reasoning of probabilistic systems with respect to qualitative properties, and our main contribution is a CEGAR approach for qualitative analysis of probabilistic systems. The details of our contributions are as follows:

1. To establish the logical relation induced by QCTL^* we consider the logic ATL^* for two-player games and the two-player game interpretation of an MDP where the probabilistic choices are resolved by an adversary. In case of non-probabilistic systems and games there are two classical notions for refinement, namely, *simulation* [69] and *alternating simulation* [1]. We first show that the logical relation induced by QCTL^* is *finer* than the intersection of simulation and alternating simulation. We then introduce a new notion of simulation, namely, *combined simulation*, and show that it captures the logical relation induced by QCTL^*.

2. We show that our new notion of simulation, which captures the logic relation of QCTL^*, can be computed using discrete graph algorithms in quadratic time. In contrast, the current best known algorithm for strong simulation is polynomial of degree seven and requires numerical algorithms. The other advantage of our approach is that it can be applied uniformly both to qualitative analysis of probabilistic systems as well as analysis of two-player games (that are standard models for open non-probabilistic systems).
3. We present a CEGAR approach for the computation of combined simulation, and the counterexample analysis and abstraction refinement is achieved using the ideas of [58] proposed for abstraction-refinement for games.
4. We have implemented our approach both for qualitative analysis of MDPs as well as games, and experimented on a number of well-known examples of MDPs and games. Our experimental results show that our method achieves significantly better performance as compared to the non-compositional verification as well as compositional analysis of MDPs with strong simulation.

Related works. Compositional and assume-guarantee style reasoning has been extensively studied mostly in the context of non-probabilistic systems [71,53,14,59]. Game-based abstraction refinement has been studied in the context of probabilistic systems [66]. The CEGAR approach has been adapted to probabilistic systems for reachability [60] and safe-pCTL [13] under monolithic (non-compositional) abstraction refinement. The work of [65] considers CEGAR for compositional analysis of probabilistic system with strong simulation. The main difference w.r.t. [65] is that strong simulation preserves exact probabilities and therefore the algorithm of [65] requires numerical algorithms whereas our algorithm requires only discrete graph algorithms. Moreover, our approach can be applied uniformly both to MDPs and two-player games. An abstraction-refinement algorithm for a class of quantitative properties was studied in [48,49] and also implemented [64]. Our logical characterization of the simulation relation is similar in spirit to [46], which shows how a fragment of the modal μ-calculus can be used to efficiently decide behavioral preorders between components. Our work focuses on CEGAR for compositional analysis of probabilistic systems for qualitative analysis: we characterize the required simulation relation; present a CEGAR approach for the computation of the simulation relation; and show the effectiveness of our approach both for qualitative analysis of MDPs and games.

Organization of the paper. In Section 2 we present the basic definitions of games and logic for games. In Section 3 we introduce a new simulation relation for games, show that it is finer than both simulation and alternating simulation, and present algorithms to compute the relation. In Section 4 we present the definitions of MDPs and qualitative logics, and in Section 5 show that the logical relation induced by the qualitative logics on MDPs can be obtained through our simulation relation introduced in Section 3. In Section 6 we present a CEGAR approach for our simulation relation and present experimental results in Section 7.

2 Game Graphs and Alternating-time Temporal Logics

Notations. Let AP denote a non-empty finite set of atomic propositions. Given a finite set S we will denote by S^* (respectively S^ω) the set of finite (resp. infinite) sequences of elements from S, and let $S^+ = S^* \setminus \{\epsilon\}$, where ϵ is the empty string.

2.1 Two-player Games

Definition 1 (Two-player games.) A *two-player* game is a tuple $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$, where

- S is a finite set of states.
- A is a finite set of actions.
- $\mathsf{Av} : S \to 2^A \setminus \emptyset$ is an *action-available* function that assigns to every state $s \in S$ the set $\mathsf{Av}(s)$ of actions available in s.
- $\delta : S \times A \to 2^S \setminus \emptyset$ is a non-deterministic *transition* function that given a state $s \in S$ and an action $a \in \mathsf{Av}(s)$ gives the set $\delta(s, a)$ of successors of s given action a.
- $\mathcal{L} : S \to 2^{\mathsf{AP}}$ is a *labeling* function that labels the states $s \in S$ with the set $\mathcal{L}(s)$ of atomic propositions true at s.
- $s_0 \in S$ is an initial state.

Definition 2 (Turn-based games.) A two-player game G is *turn-based* if in every state either Player 1 or Player 2 can make choices. Formally, for all $s \in S$ we have either (i) $|\mathsf{Av}(s)| = 1$ (then we refer to s as a Player-2 state); or (ii) for all $a \in \mathsf{Av}(s)$ we have $|\delta(s, a)| = 1$ (then we refer to s as a Player-1 state). For technical convenience we consider that in the case of turn-based games, there is an atomic proposition $\mathsf{turn} \in \mathsf{AP}$ such that for every Player-1 state s we have $\mathsf{turn} \in \mathcal{L}(s)$, and for every Player 2 state s' we have $\mathsf{turn} \notin \mathcal{L}(s')$.

Definition 3 (Plays.) A two-player game is played for infinitely many rounds as follows: the game starts in the initial state, and in every round Player 1 chooses an available action from the current state and then Player 2 chooses a successor state, and the game proceeds to the successor state for the next round. Formally, a *play* in a two-player game is an infinite sequence $\omega = s_0 a_0 s_1 a_1 s_2 a_2 \cdots$ of states and actions such that for all $i \geq 0$ we have that $a_i \in \mathsf{Av}(s_i)$ and $s_{i+1} \in \delta(s_i, a_i)$. We denote by Ω the set of all plays.

Definition 4 (Strategies.) Strategies are recipes that describe how to extend finite prefixes of plays. Formally, a *strategy* for Player 1 is a function $\sigma : (S \times A)^* \times S \to A$, that given a finite history $w \cdot s \in (S \times A)^* \times S$ of the game gives an action from $\mathsf{Av}(s)$ to be played next. We write Σ for the set of all Player-1 strategies. A strategy for Player 2 is a function $\theta : (S \times A)^+ \to S$, that given a finite history $w \cdot s \cdot a$ of a play selects a successor state from the set $\delta(s, a)$. We write Θ for the set of all Player-2 strategies. *Memoryless* strategies are independent of the history, but depend only on the current state for Player 1 (resp. the current state and action for Player 2) and hence can be represented as functions $S \to A$ for Player 1 (resp. as functions $S \times A \to S$ for Player 2).

Definition 5 (Outcomes.) Given a strategy σ for Player 1 and θ for Player 2 the *outcome* is a unique play, denoted as $\mathsf{Play}(s, \sigma, \theta) = s_0 a_0 s_1 a_1 \cdots$, which is defined as follows: (i) $s_0 = s$; and (ii) for all $i \geq 0$ we have $a_i = \sigma(s_0 a_0 \ldots s_i)$ and $s_{i+1} = \theta(s_0 a_0 \ldots s_i a_i)$. Given a state $s \in S$ we denote by $\mathsf{Plays}(s, \sigma)$ (resp. $\mathsf{Plays}(s, \theta)$) the set of possible plays given σ (resp. θ), i.e., $\bigcup_{\theta' \in \Theta} \mathsf{Play}(s, \sigma, \theta')$ (resp. $\bigcup_{\sigma' \in \Sigma} \mathsf{Play}(s, \sigma', \theta)$).

Definition 6 (Parallel composition of two-player games.) Given games $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and $G' = (S', A, \mathsf{Av}', \delta', \mathcal{L}', s'_0)$ the *parallel composition* of the games $G \parallel G' = (\overline{S}, A, \overline{\mathsf{Av}}, \overline{\delta}, \overline{\mathcal{L}}, \overline{s}_0)$ is defined as follows:

- The states of the composition are $\overline{S} = S \times S'$.

– The set of actions does not change with the composition.
– For all (s, s') we have $\overline{\mathsf{Av}}((s, s')) = \mathsf{Av}(s) \cap \mathsf{Av}'(s')$.
– The transition function for a state $(s, s') \in \overline{S}$ and an action $a \in \overline{\mathsf{Av}}((s, s'))$ is defined as $\overline{\delta}((s, s'), a) = \{(t, t') \mid t \in \delta(s, a) \wedge t' \in \delta'(s', a)\}$.
– The labeling function $\overline{\mathcal{L}}((s, s'))$ is defined as $\mathcal{L}(s) \cup \mathcal{L}'(s')$.
– The initial state is $\overline{s}_0 = (s_0, s'_0)$.

Remark 1 For simplicity we assume that the set of actions in both components is identical, and for every pair of states the intersection of their available actions is non-empty. Parallel composition can be extended to cases where the sets of actions are different [2].

2.2 Alternating-time Temporal Logic

We consider the Alternating-time Temporal Logic (ATL*) [3] as a logic to specify properties for two-player games.

Syntax. The syntax of the logic is given in positive normal form by defining the set of *path formulas* (φ) and *state formulas* (ψ) according to the following grammar:

$$\begin{array}{ll} \text{state formulas:} & \psi ::= q \mid \neg q \mid \psi \vee \psi \mid \psi \wedge \psi \mid \mathrm{PQ}(\varphi) \\ \text{path formulas:} & \varphi ::= \psi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \bigcirc\varphi \mid \varphi\,\mathcal{U}\varphi \mid \varphi\mathcal{W}\varphi; \end{array}$$

where $q \in \mathsf{AP}$ is an atomic proposition and PQ is a path quantifier. The operators $\bigcirc$ (next), $\mathcal{U}$ (until), and $\mathcal{W}$ (weak until) are the temporal operators. We will use $\mathtt{true}$ as a shorthand for $q \vee \neg q$ and false for $q \wedge \neg q$ for some $q \in \mathsf{AP}$. The path quantifiers PQ are as follows:

$$\text{ATL}^* \text{ path quantifiers: } \langle\langle 1 \rangle\rangle, \langle\langle 2 \rangle\rangle, \langle\langle 1,2 \rangle\rangle, \text{ and } \langle\langle \emptyset \rangle\rangle.$$

Semantics. Given a play $\omega = s_0 a_0 s_1 a_1 \cdots$ we denote by $\omega[i]$ the suffix starting at the i-th state element of the play ω, i.e., $\omega[i] = s_i a_i s_{i+1} a_{i+1} \cdots$. The semantics of path formulas is defined inductively as follows:

$$\begin{array}{ll} \omega \models \psi & \text{iff } \omega[0] \models \psi \\ \omega \models \varphi_1 \vee \varphi_2 & \text{iff } \omega \models \varphi_1 \text{ or } \omega \models \varphi_2 \\ \omega \models \varphi_1 \wedge \varphi_2 & \text{iff } \omega \models \varphi_1 \text{ and } \omega \models \varphi_2 \\ \omega \models \bigcirc\varphi & \text{iff } \omega[1] \models \varphi \\ \omega \models \varphi_1 \mathcal{U} \varphi_2 & \text{iff } \exists j \in \mathbb{N} : \omega[j] \models \varphi_2 \text{ and } \forall 0 \leq i < j : \omega[i] \models \varphi_1 \\ \omega \models \varphi_1 \mathcal{W} \varphi_2 & \text{iff } \omega \models \varphi_1 \mathcal{U} \varphi_2 \text{ or } \forall j \in \mathbb{N} : \omega[j] \models \varphi_1. \end{array}$$

Given a path formula φ, we denote by $[\![\varphi]\!]_G$ the set of plays ω such that $\omega \models \varphi$. We omit the G lower script when the game is clear from context. The semantics of state formulas for ATL* is defined as follows:

$$\begin{array}{ll} s \models q & \text{iff } q \in \mathcal{L}(s) \\ s \models \neg q & \text{iff } q \notin \mathcal{L}(s) \\ s \models \psi_1 \vee \psi_2 & \text{iff } s \models \psi_1 \text{ or } s \models \psi_2 \\ s \models \psi_1 \wedge \psi_2 & \text{iff } s \models \psi_1 \text{ and } s \models \psi_2 \\ s \models \langle\langle 1 \rangle\rangle(\varphi) & \text{iff } \exists \sigma \in \Sigma, \forall \theta \in \Theta : \mathsf{Play}(s, \sigma, \theta) \in [\![\varphi]\!] \\ s \models \langle\langle 2 \rangle\rangle(\varphi) & \text{iff } \exists \theta \in \Theta, \forall \sigma \in \Sigma : \mathsf{Play}(s, \sigma, \theta) \in [\![\varphi]\!] \\ s \models \langle\langle 1,2 \rangle\rangle(\varphi) & \text{iff } \exists \sigma \in \Sigma, \exists \theta \in \Theta : \mathsf{Play}(s, \sigma, \theta) \in [\![\varphi]\!] \\ s \models \langle\langle \emptyset \rangle\rangle(\varphi) & \text{iff } \forall \sigma \in \Sigma, \forall \theta \in \Theta : \mathsf{Play}(s, \sigma, \theta) \in [\![\varphi]\!]; \end{array}$$

where $s \in S$ and $q \in \mathsf{AP}$. Given an ATL^* state formula ψ and a two-player game G, we denote by $[\![\psi]\!]_G = \{s \in S \mid s \models \psi\}$ the set of states that satisfy the formula ψ. We omit the G lower script when the game is clear from context.

Logic fragments. We define several fragments of the logic ATL^*:

- *Restricted temporal operator use.* An important fragment of ATL^* is ATL where every temporal operator is immediately preceded by a path quantifier.
- *Restricting path quantifiers.* We also consider fragments of ATL^* (resp. ATL) where the path quantifiers are restricted. We consider (i) 1-fragment (denoted 1-ATL^*) where only $\langle\!\langle 1 \rangle\!\rangle$ path quantifier is used; (ii) the $(1,2)$-fragment (denoted $(1,2)$-ATL^*) where only $\langle\!\langle 1,2 \rangle\!\rangle$ path quantifier is used; and (iii) the combined fragment (denoted C-ATL^*) where both $\langle\!\langle 1 \rangle\!\rangle$ and $\langle\!\langle 1,2 \rangle\!\rangle$ path quantifiers are used. We use a similar notation for the respective fragments of ATL formulas.

Logical characterization of states. Given two games G and G', and a logic fragment $\mathcal{F}$ of ATL^*, we consider the following relations on the state space induced by the logic fragment $\mathcal{F}$:

$$\preccurlyeq_{\mathcal{F}} (G, G') = \{(s, s') \in S \times S' \mid \forall \psi \in \mathcal{F} : \text{ if } s \models \psi \text{ then } s' \models \psi\};$$

and when the games are clear from context we simply write $\preccurlyeq_{\mathcal{F}}$ for $\preccurlyeq_{\mathcal{F}} (G, G')$. We will use the following notations for the relation induced by the logic fragments we consider: (i) $\preccurlyeq_1^*$ (resp. $\preccurlyeq_1$) for the relation induced by the 1-ATL^* (resp. 1-ATL) fragment; (ii) $\preccurlyeq_{1,2}^*$ (resp. $\preccurlyeq_{1,2}$) for the relation induced by the $(1,2)$-ATL^* (resp. $(1,2)$-ATL) fragment; and (iii) $\preccurlyeq_C^*$ (resp. $\preccurlyeq_C$) for the relation induced by the C-ATL^* (resp. C-ATL) fragment. Given G and G' we can also consider G'' which is the disjoint union of the two games, and consider the relations on G''; and hence we will often consider a single game as input for the relations.

3 Combined Simulation Relation Computation

In this section we first recall the notion of simulation [69] and alternating simulation [1]; and then present a new notion of *combined simulation*.

Definition 7 (Simulation.) Given two-player games $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and $G' = (S', A', \mathsf{Av}', \delta', \mathcal{L}', s_0')$, a relation $\mathcal{S} \subseteq S \times S'$ is a *simulation* from G to G' if for all $(s, s') \in \mathcal{S}$ the following conditions hold:

1. *Proposition match:* The atomic propositions match, i.e., $\mathcal{L}(s) = \mathcal{L}'(s')$.
2. *Step-wise simulation condition:* For all actions $a \in \mathsf{Av}(s)$ and states $t \in \delta(s, a)$ there exists an action $a' \in \mathsf{Av}'(s')$ and a state $t' \in \delta(s', a')$ such that $(t, t') \in \mathcal{S}$.

We denote by $\mathcal{S}_{\max}^{G,G'}$ the largest simulation relation between the two games (we write $\mathcal{S}_{\max}$ instead of $\mathcal{S}_{\max}^{G,G'}$ when G and G' are clear from the context). We write $G \leqslant_{\mathcal{S}} G'$ when $(s_0, s_0') \in \mathcal{S}_{\max}$. The largest simulation relation characterizes the logic relation of $(1,2)$-ATL and $(1,2)$-ATL^*: the $(1,2)$-ATL-fragment interprets a game as a transition system and the formulas coincide with existential CTL, and hence the logic characterization follows from the classical results on simulation and CTL [69,2].

Proposition 1 *For all games G and G′ we have* $\mathcal{S}_{\max}^{G,G'} = \preccurlyeq_{1,2}^* = \preccurlyeq_{1,2}$.

Definition 8 (Alternating simulation.) Given two games $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and $G' = (S', A', \mathsf{Av}', \delta', \mathcal{L}', s_0')$, a relation $\mathcal{A} \subseteq S \times S'$ is an *alternating simulation* from G to G' if for all $(s, s') \in \mathcal{A}$ the following conditions hold:

1. *Proposition match:* The atomic propositions match, i.e., $\mathcal{L}(s) = \mathcal{L}'(s')$.
2. *Step-wise alternating-simulation condition:* For all actions $a \in \mathsf{Av}(s)$ there exists an action $a' \in \mathsf{Av}'(s')$ such that for all states $t' \in \delta'(s', a')$ there exists a state $t \in \delta(s, a)$ such that $(t, t') \in \mathcal{A}$.

We denote by $\mathcal{A}_{\max}^{G,G'}$ the largest alternating-simulation relation between the two games (we write $\mathcal{A}_{\max}$ instead of $\mathcal{A}_{\max}^{G,G'}$ when G and G' are clear from the context). We write $G \leqslant_{\mathcal{A}} G'$ when $(s_0, s_0') \in \mathcal{A}_{\max}$. The largest alternating-simulation relation characterizes the logic relation of 1-ATL and 1-ATL* [1].

Proposition 2 *For all games G and G' we have $\mathcal{A}_{\max}^{G,G'} = \preccurlyeq_1^* = \preccurlyeq_1$.*

Definition 9 (Combined simulation.) We present a new notion of combined simulation that extends both simulation and alternating simulation, and we show how the combined simulation characterizes the logic relation induced by C-ATL* and C-ATL. Intuitively, the requirements on the combined-simulation relation combine the requirements imposed by alternating simulation and simulation in a step-wise fashion. Given two-player games $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and $G' = (S', A', \mathsf{Av}', \delta', \mathcal{L}', s_0')$, a relation $\mathcal{C} \subseteq S \times S$ is a *combined simulation* from G to G' if for all $(s, s') \in \mathcal{C}$ the following conditions hold:

1. *Proposition match:* The atomic propositions match, i.e., $\mathcal{L}(s) = \mathcal{L}'(s')$.
2. *Step-wise simulation condition:* For all actions $a \in \mathsf{Av}(s)$ and states $t \in \delta(s, a)$ there exists an action $a' \in \mathsf{Av}'(s')$ and a state $t' \in \delta(s', a')$ such that $(t, t') \in \mathcal{C}$.
3. *Step-wise alternating-simulation condition:* For all actions $a \in \mathsf{Av}(s)$ there exists an action $a' \in \mathsf{Av}'(s')$ such that for all states $t' \in \delta'(s', a')$ there exists a state $t \in \delta(s, a)$ such that $(t, t') \in \mathcal{C}$.

We denote by $\mathcal{C}_{\max}^{G,G'}$ the largest combined-simulation relation between the two games (and write $\mathcal{C}_{\max}$ when G and G' are clear from the context). We also write $G \leqslant_{\mathcal{C}} G'$ when $(s_0, s_0') \in \mathcal{C}_{\max}$. We first illustrate with an example that the logic relation $\preccurlyeq_C$ induced by C-ATL is finer than the intersection of simulation and alternating-simulation relation; then present a game theoretic characterization of $\mathcal{C}_{\max}$; and finally show that $\mathcal{C}_{\max}$ gives the relations $\preccurlyeq_C^*$ and $\preccurlyeq_C$.

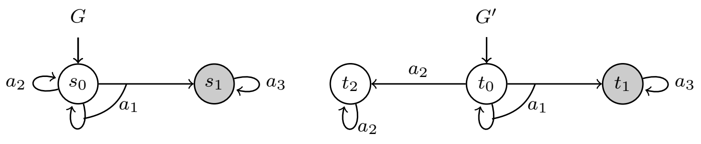


Fig. 1 Games G, G' such that $G \leqslant_{\mathcal{S}} G'$ and $G \leqslant_{\mathcal{A}} G'$, but $G \not\leqslant_{\mathcal{C}} G'$.

Example 1 Consider the games G and G' shown in Figure 1. White nodes are labeled by an atomic proposition p and gray nodes by q. The largest simulation and alternating-simulation relations between G and G' are: $\mathcal{S}_{\max}^{G,G'} = \{(s_0, t_0), (s_1, t_1)\}, \mathcal{A}_{\max}^{G,G'} = \{(s_0, t_0), (s_0, t_2), (s_1, t_1)\}$. However, consider the formula $\psi = \langle\langle 1 \rangle\rangle(\bigcirc(p \wedge \langle\langle 1, 2 \rangle\rangle(\bigcirc q)))$. We have that $s_0 \models \psi$, but $t_0 \not\models \psi$. It follows that $(s_0, t_0) \not\in \preccurlyeq_C$. $\square$

Definition 10 (Combined-simulation games.) The simulation and the alternating-simulation relation can be obtained by solving two-player safety games [57,1,18]. We now define a two-player game for the combined-simulation relation characterization. The game is played on the synchronized product of the two input games. Given a state (s, s'), first Player 2 decides whether to check for the step-wise simulation condition or the step-wise alternating-simulation condition. The step-wise simulation condition is checked by playing a two-step game, and the step-wise alternating-simulation condition is checked by playing a four-step game. Consider two games $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and $G' = (S', A', \mathsf{Av}', \delta', \mathcal{L}', s'_0)$. We construct the *combined-simulation game* $G^{\mathcal{C}} = (S^{\mathcal{C}}, A^{\mathcal{C}}, \mathsf{Av}^{\mathcal{C}}, \delta^{\mathcal{C}}, \mathcal{L}^{\mathcal{C}}, s_0^{\mathcal{C}})$ as follows:

– *The set of states.* The set of states $S^{\mathcal{C}}$ is:

$$\begin{aligned} S^{\mathcal{C}} = \quad & (S \times S') \cup (S \times S' \times \{\mathsf{Sim}\} \times \{1,2\}) \cup (S \times S' \times \{\mathsf{Alt}\} \times \{2\}) \\ & \cup (S \times S' \times \{\mathsf{Alt}\} \times A \times \{1\}) \cup (S \times S' \times \{\mathsf{Alt}\} \times A \times A' \times \{1,2\}) \end{aligned}$$

  Intuitively, in states in $S \times S'$ and in states where the last component is 2 it is Player 2's turn to make the choice of successors, and in all other states Player 1 makes the choice of actions.

– *The set of actions.* The set of actions is as follows: $A^{\mathcal{C}} = \{\bot\} \cup S \cup S' \cup A'$.

– *The transition function and the action-available function.*

  1. *Choice of simulation or alternating-simulation.* For a state (s, s') we have only one action $\bot$ available for Player 1 and we have $\delta^{\mathcal{C}}((s, s'), \bot) = \{(s, s', \mathsf{Alt}, 2), (s, s', \mathsf{Sim}, 2)\}$, i.e., Player 2 decides whether to check for step-wise simulation or step-wise alternating-simulation conditions.
  2. *Checking step-wise simulation conditions.* We describe the transitions for checking the simulation conditions:
     (a) For a state $(s, s', \mathsf{Sim}, 2)$ we have only one action $\bot$ available for Player 1 and we have $\delta^{\mathcal{C}}((s, s', \mathsf{Sim}, 2), \bot) = \{(t, s', \mathsf{Sim}, 1) \mid \exists a \in \mathsf{Av}(s) : t \in \delta(s, a)\}$.
     (b) For a state $\overline{s} = (t, s', \mathsf{Sim}, 1)$ we have $\mathsf{Av}^{\mathcal{C}}(\overline{s}) = \{t' \mid \exists a' \in \mathsf{Av}(s') : t' \in \delta'(s', a')\}$ and $\delta^{\mathcal{C}}(\overline{s}, t') = \{(t, t')\}$.

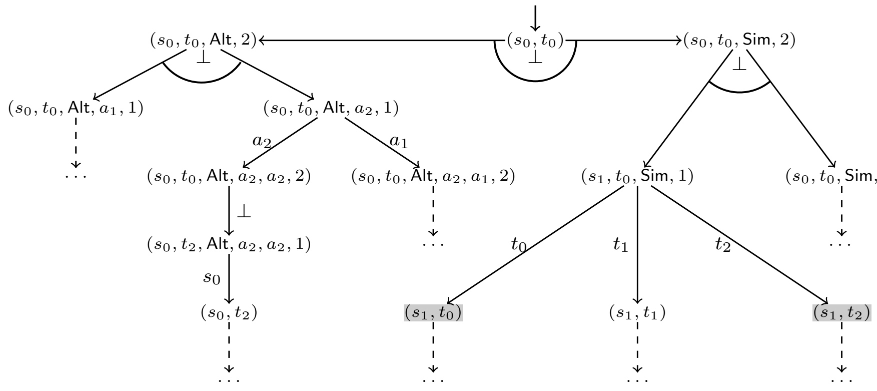
     Intuitively, first Player 2 chooses an action $a \in \mathsf{Av}(s)$ and a successor $t \in \delta(s, a)$ and challenges Player 1 to match, and Player 1 responds with an action $a' \in \mathsf{Av}'(s')$ and a state $t' \in \delta'(s', a')$.
  3. *Checking step-wise alternating-simulation conditions.* We describe the transitions for checking the alternating-simulation conditions:
     (a) For a state $(s, s', \mathsf{Alt}, 2)$ we have only one action $\bot$ available for Player 1 and we have $\delta^{\mathcal{C}}((s, s', \mathsf{Alt}, 2), \bot) = \{(s, s', \mathsf{Alt}, a, 1) \mid a \in \mathsf{Av}(s)\}$.
     (b) For a state $\overline{s} = (s, s', \mathsf{Alt}, a, 1)$ we have $\mathsf{Av}^{\mathcal{C}}(\overline{s}) = \mathsf{Av}'(s')$ and $\delta^{\mathcal{C}}(\overline{s}, a') = \{(s, s', \mathsf{Alt}, a, a', 2)\}$.
     (c) For a state $(s, s', \mathsf{Alt}, a, a', 2)$ we have only one action $\bot$ available for Player 1 and we have $\delta^{\mathcal{C}}((s, s', \mathsf{Alt}, a, a', 2), \bot) = \{(s, t', \mathsf{Alt}, a, a', 1) \mid t' \in \delta'(s', a')\}$.
     (d) For a state $\overline{s} = (s, t', \mathsf{Alt}, a, a', 1)$ we have $\mathsf{Av}^{\mathcal{C}}(\overline{s}) = \delta(s, a)$ and $\delta^{\mathcal{C}}(\overline{s}, t) = \{(t, t')\}$.

     Intuitively, first Player 2 chooses an action a from $\mathsf{Av}(s)$ and Player 1 responds with an action $a' \in \mathsf{Av}'(s')$ (in the first two-steps); then Player 2 chooses a successor t' from $\delta'(s', a')$ and Player 1 responds by choosing a successor t in $\delta(s, a)$.

– *The labeling function.* The set of atomic proposition AP contains a single proposition $p \in \mathsf{AP}$. The labeling function $\mathcal{L}^{\mathcal{C}}$ given a state $\overline{s} \in S^{\mathcal{C}}$ is defined as follows: $\mathcal{L}^{\mathcal{C}}(\overline{s}) = p$

iff $\overline{s} = (s, s')$ and $\mathcal{L}(s) \neq \mathcal{L}'(s')$. Intuitively, Player 2's goal is to reach a state (s, s') where the propositional labeling of the original games do not match, i.e., to reach a state labeled p by $\mathcal{L}^{\mathcal{C}}$.
– *The initial state.* The state $s_0^{\mathcal{C}}$ is (s_0, s_0').

In the combined simulation game we refer to Player 1 as the *proponent* (trying to establish the combined simulation) and Player 2 as the *adversary* (trying to violate the combined simulation).



(s_0, t_0)
$(s_0, t_0, \mathsf{Alt}, 2)$
$(s_0, t_0, \mathsf{Sim}, 2)$
$\perp$
$(s_0, t_0, \mathsf{Alt}, a_1, 1)$
$(s_0, t_0, \mathsf{Alt}, a_2, 1)$
a_2
a_1
$(s_0, t_0, \mathsf{Alt}, a_2, a_2, 2)$
$(s_0, t_0, \mathsf{Alt}, a_2, a_1, 2)$
$(s_0, t_2, \mathsf{Alt}, a_2, a_2, 1)$
s_0
(s_0, t_2)
$(s_1, t_0, \mathsf{Sim}, 1)$
$(s_0, t_0, \mathsf{Sim}, 1)$
t_0
t_1
t_2
(s_1, t_0)
(s_1, t_1)
(s_1, t_2)
$\cdots$

Fig. 2 Part of the combined-simulation game of G and G' from Figure 1.

Example 2 A part of the combined-simulation game of G and G' from Figure 1 is shown in Figure 2. Dashed arrows indicate that the successors of a given state are omitted in the figure. Gray states are labeled by an atomic proposition p, hence are the goal states for the adversary. □

Shorthand for safety objectives. We will use the following shorthand for *safety* objectives: $\Box\,\varphi \equiv \varphi\,\mathcal{W}\,\mathsf{false}$; i.e., the formula $\Box\varphi$ is satisfied by paths where φ is always true.

Theorem 1 *For all games G and G' we have* $\mathcal{C}_{\max} = [\![\langle\!\langle 1 \rangle\!\rangle(\Box\neg p)]\!]_{G^{\mathcal{C}}} \cap (S \times S')$.

Proof The statement follows directly from the definition of combined simulation, and the fact that the game construction mimics the definition of combined simulation (as in the case of simulation and alternating simulation [57, 1, 18]). □

Winning strategies. Given a combined-simulation game $G^{\mathcal{C}}$ we say that a strategy σ for the proponent is *winning* from a state s if for all strategies θ of the adversary we have $\mathsf{Play}(s, \sigma, \theta) \models \Box(\neg p)$. A strategy θ for the adversary is *winning* from state s if for all strategies σ of the proponent we have $\mathsf{Play}(s, \sigma, \theta) \models \mathsf{true}\,\mathcal{U} p$. Whenever the proponent (resp. adversary) has a winning strategy, the proponent (resp. adversary) also has memoryless winning strategy [55].

Combined simulation logical characterization. Our next goal is to establish that combined simulation gives the logical characterization of C-ATL* and C-ATL. To prove the result we first introduce the following relation between plays: Given two plays $\omega = s_0 a_0 s_1 a_1 s_2 \cdots$ and $\omega' = s_0' a_0' s_1' a_1' s_2' \cdots$ we write $\omega \leqslant_{\mathcal{C}} \omega'$ if for all $i \geq 0$ we have $(s_i, s_i') \in \mathcal{C}_{\max}$.

Lemma 1 *Given two games G and G', let $\mathcal{C}_{\max}$ be the combined simulation. For all $(s, s') \in \mathcal{C}_{\max}$ the following assertions hold:*

- *For all Player 1 strategies σ in G, there exists a Player 1 strategy σ' in G' such that for every play $\omega' \in \mathsf{Plays}(s', \sigma')$ there exists a play $\omega \in \mathsf{Plays}(s, \sigma)$ such that $\omega \leqslant_{\mathcal{C}} \omega'$.*
- *For all pair of strategies σ and θ in G, there exists a pair of strategies σ' and θ' in G' such that $\mathsf{Play}(s, \sigma, \theta) \leqslant_{\mathcal{C}} \mathsf{Play}(s', \sigma', \theta')$,*

Proof We present the details of the first item.

- Consider a winning strategy $\sigma^{\mathcal{C}}$ for the proponent in $G^{\mathcal{C}}$ such that for all $(s, s') \in \mathcal{C}_{\max}$ and against all strategies $\theta^{\mathcal{C}}$ we have $\mathsf{Play}(s, \sigma^{\mathcal{C}}, \theta^{\mathcal{C}}) \in [\![\Box(\neg p)]\!]$. Given the Player 1 strategy σ in G we construct σ' in G' using the strategy $\sigma^{\mathcal{C}}$. Consider a history $w \cdot s$ in G and $w' \cdot s' \in G'$ such that $(s, s') \in \mathcal{C}_{\max}$. Let $\sigma(w \cdot s) = a$. We define $\sigma'(w' \cdot s')$ as follows. Let h be an arbitrary history in $G^{\mathcal{C}}$ that only visits states in $\mathcal{C}_{\max}$ and ends in (s, s'). Let $a' = \sigma^{\mathcal{C}}(h \cdot (s, s', \mathsf{Alt}, 2) \cdot (s, s', \mathsf{Alt}, a, 2))$; (i.e., the action played by the strategy $\sigma^{\mathcal{C}}$ in response to the choice of checking alternating simulation and the action a by Player 2 in $G^{\mathcal{C}}$). Then the strategy σ' plays accordingly, i.e., $\sigma'(w' \cdot s') = a'$. In the next step for every choice t' of the adversary there exists a choice t of the proponent such that $\mathcal{L}(t) = \mathcal{L}'(t')$ and $(t, t') \in \mathcal{C}_{\max}$ and the matching can proceed.
- The proof is similar to the first item, and instead of using the step-wise alternating-simulation gadget for strategy construction (of the first item) we use the step-wise simulation gadget from $G^{\mathcal{C}}$ to construct the strategy pairs.

The desired result follows. $\square$

In the following theorem we establish the relation between combined simulation and the C-ATL* fragment of ATL*.

Theorem 2 *For all games G and G' we have $\mathcal{C}_{\max} = \preccurlyeq^*_C = \preccurlyeq_C$.*

Proof First implication. We first prove the implication $\mathcal{C}_{\max} \subseteq \preccurlyeq^*_C$. We will show the following assertions:

- For all states s and s' such that $(s, s') \in \mathcal{C}_{\max}$, we have that every C-ATL* state formula satisfied in s is also satisfied in s'.
- For all plays ω and ω' such that $\omega \leqslant_{\mathcal{C}} \omega'$, we have that every C-ATL* path formula satisfied in ω is also satisfied in ω'.

We will prove the theorem by induction on the structure of the formulas. The interesting cases for the induction step are formulas $\langle\!\langle 1 \rangle\!\rangle(\varphi)$ and $\langle\!\langle 1, 2 \rangle\!\rangle(\varphi)$, where φ is a path formula.

- Assume $s \models \langle\!\langle 1 \rangle\!\rangle(\varphi)$ and $(s, s') \in \mathcal{C}_{\max}$. It follows that there exists a strategy $\sigma \in \Sigma$ that ensures the path formula φ from state s against any strategy $\theta \in \Theta$. We want to show that $s' \models \langle\!\langle 1 \rangle\!\rangle(\varphi)$. By Lemma 1(item 1) we have that there exists a strategy σ' for Player 1 from s' such that for every play $\omega' \in \mathsf{Plays}(s', \sigma')$ there exists a play $\omega \in \mathsf{Plays}(s, \sigma)$ such that $\omega \leqslant_{\mathcal{C}} \omega'$. By inductive hypothesis we have that $s' \models \langle\!\langle 1 \rangle\!\rangle(\varphi)$.
- Assume $s \models \langle\!\langle 1, 2 \rangle\!\rangle(\varphi)$ and $\mathcal{C}(s, s')$. It follows that there exist strategies $\sigma \in \Sigma, \theta \in \Theta$ that ensure the path formula φ from state s. By Lemma 1(item 2) we have that there exist strategies σ' and θ' such that the two plays $\omega' = \mathsf{Play}(s', \sigma', \theta')$ and $\omega = \mathsf{Play}(s, \sigma, \theta)$ satisfy $\omega \leqslant_{\mathcal{C}} \omega'$. By inductive hypothesis we have that $s' \models \langle\!\langle 1, 2 \rangle\!\rangle(\varphi)$.
- Consider a path formula φ. If $\omega \leqslant_{\mathcal{C}} \omega'$, then by inductive hypothesis for every subformula φ' of φ we have that if $\omega \models \varphi'$ then $\omega' \models \varphi'$. It follows that if $\omega \models \varphi$ then $\omega' \models \varphi$.

Second implication. It remains to prove the second implication $\preccurlyeq_C^* \subseteq \preccurlyeq_C \subseteq \mathcal{C}_{\max}$. Assume that given states s and s' we have that $(s,s') \notin \mathcal{C}_{\max}$, then there exists a winning strategy in the corresponding combined-simulation game for the adversary from state (s,s'), i.e., there exists a strategy $\theta^{\mathcal{C}}$ such that against all strategies $\sigma^{\mathcal{C}}$ we have $\mathsf{Play}((s,s'),\sigma^{\mathcal{C}},\theta^{\mathcal{C}})$ reaches a state labeled p. As memoryless strategies are sufficient for both players in $G^{\mathcal{C}}$ [55], there also exists a bound $i \in \mathbb{N}$, such that the proponent fails to match the choice of the adversary in at most i turns. We sketch the inductive proof that there exists a formula with i nested operators $\langle\!\langle 1 \rangle\!\rangle \bigcirc$ or $\langle\!\langle 1,2 \rangle\!\rangle \bigcirc$ that is satisfied in s but not in s'. For i equal to 0 the states can be distinguished by atomic propositions. For the inductive step one can express the simulation turns by a $\langle\!\langle 1,2 \rangle\!\rangle(\bigcirc \ldots)$ formula and alternating simulation turns by a $\langle\!\langle 1 \rangle\!\rangle(\bigcirc \ldots)$ formula. It follows that $(s,s') \notin \preccurlyeq_C$. The result follows. □

Remark 2 Lemma 1 and Theorem 2 also hold for alternating games, see Theorem 7 in Appendix A. Note that in most cases the action set is constant and the state space of the games are huge. Then the combined simulation game construction is quadratic, and solving safety games on them can be achieved in linear time (in the size of the game) using discrete graph algorithms [62, 8].

Theorem 3 *Given two-player games G and G', the $\mathcal{C}_{\max}$, $\preccurlyeq_C^*$, and $\preccurlyeq_C$ relations can be computed in quadratic time using discrete graph algorithms.*

4 MDPs and Qualitative Logics

In this section we consider Markov decisions processes (MDPs) and logics to reason qualitatively about them. We consider MDPs which can be viewed as a variant of two-player games defined in Section 2. First, we fix some notation: a probability distribution f on a finite set X is a function $f : X \to [0,1]$ such that $\sum_{x \in X} f(x) = 1$, and we denote by $\mathcal{D}(X)$ the set of all probability distributions on X. For $f \in \mathcal{D}(X)$ we denote by $\mathrm{Supp}(f) = \{x \in X \mid f(x) > 0\}$ the *support of f*.

4.1 MDPs

Definition 11 (Markov decision processes.) A *Markov decision process* (MDP) is a tuple $G = (S,(S_1,S_P),A,\mathsf{Av},\delta_1,\delta_P,\mathcal{L},s_0)$; where (i) S is a finite set of states with a partition of S into Player-1 states S_1 and probabilistic states S_P; (ii) A is a finite set of actions; (iii) $\mathsf{Av} : S_1 \to 2^A \setminus \emptyset$ is an action-available function that assigns to every Player-1 state the non-empty set $\mathsf{Av}(s)$ of actions available in s; (iv) $\delta_1 : S_1 \times A \to S$ is a deterministic transition function that given a Player-1 state and an action gives the next state; (v) $\delta_P : S_P \to \mathcal{D}(S)$ is a probabilistic transition function that given a probabilistic state gives a probability distribution over the successor states (i.e., $\delta_P(s)(s')$ is the transition probability from s to s'); (vi) the function $\mathcal{L}$ is the proposition labeling function as for two-player games; and (vii) s_0 is the initial state. Strategies for Player 1 are defined as for games. In this work we will consider MDPs with qualitative properties, and hence not consider reward-based MDP models.

Interpretations. We interpret an MDP in two distinct ways: (i) as a $1\frac{1}{2}$-player game and (ii) as a turn-based two-player game, where we regard the probabilistic states as Player-2 states. We will use the two-player interpretation to relate logical characterizations of MDPs and logical characterization of two-player games with fragments of ATL^*.

$1\frac{1}{2}$-Player Interpretation. Once a strategy $\sigma \in \Sigma$ for Player 1 is fixed, the outcome of the MDP is a random walk for which the probabilities of *events* are uniquely defined, where an *event* $\Phi \subseteq \Omega$ is a measurable set of plays [55]. For a state $s \in S$ and an event $\Phi \subseteq \Omega$, we write $\Pr_s^\sigma(\Phi)$ for the probability that a play belongs to Φ if the game starts from the state s and Player 1 follows strategy σ.

Two-player Interpretation. The two-player interpretation corresponds to turn-based two-player games introduced in Section 2, where the probabilistic aspect of the MDP is replaced by a second player. Formally, given an MDP $G = (S, (S_1, S_P), A, \mathsf{Av}, \delta_1, \delta_P, \mathcal{L}, s_0)$ we define a turn-based two-player game $\widehat{G} = (\widehat{S}, \widehat{A}, \widehat{\mathsf{Av}}, \widehat{\delta}, \widehat{\mathcal{L}}, \widehat{s_0})$ as follows: (i) the states are $\widehat{S} = S$; (ii) the set of actions contains a new action $\perp$ not present in A, i.e., $\widehat{A} = A \cup \{\perp\}$; (iii) the action-available function for states $s \in S_1$ is defined as $\widehat{\mathsf{Av}}(s) = \mathsf{Av}(s)$ and for states $s_p \in S_P$ as $\widehat{\mathsf{Av}}(s_p) = \{\perp\}$; (iv) for $s \in S_1$ and a in $\widehat{\mathsf{Av}}(s)$ we have $\widehat{\delta}(s, a) = \{\delta_1(s, a)\}$, and for $s_p \in S_P$ we have $\widehat{\delta}(s_p, \perp) = \mathrm{Supp}(\delta_p(s_p))$; (v) the labeling function for a Player-1 state s is $\widehat{\mathcal{L}}(s) = \mathcal{L}(s) \cup \{\mathsf{turn}\}$ and for a Player-2 state s' coincides with $\mathcal{L}(s')$; and (vi) the initial state is the same $\widehat{s}_0 = s_0$. Given an MDP G we denote by $\widehat{G}$ the two-player interpretation of the MDP. Note that for all Player-1 states $s \in S_1$ we have $|\widehat{\delta}(s)| = 1$ and for all Player-2 states $s_p \in S_P$ we have $|\mathsf{Av}(s_p)| = 1$. Therefore for any MDP the corresponding two-player interpretation is a turn-based game.

Example 3 In Figure 3 we present three MDPs G_1, G_2, and G' that we use as running examples. We thoroughly describe only MDP $G' = (S, (S_1, S_P), A, \mathsf{Av}, \delta_1, \delta_P, \mathcal{L}, s_0)$. Player-1 states, depicted as circles, are $S_1 = \{s'_0, s'_2, s'_3\}$ and probabilistic states, depicted as rectangles, are $S_P = \{s'_1, s'_4\}$. The set of actions is $A = \{a, b\}$. Action a is available in states s'_0, s'_2 and action b is available only in states s'_0, s'_3. The deterministic transition function is $\delta_1(s'_0, a) = s'_1, \delta_1(s'_0, b) = s'_4, \delta_1(s'_2, a) = s'_4, \delta_1(s'_2, b) = s'_4, \delta_1(s'_3, b) = s'_4$. The probabilistic transition function δ_P gives the following probability distributions over possible successor states: $\delta_P(s'_1)(s'_2) = \frac{1}{2}, \delta_P(s'_1)(s'_3) = \frac{1}{2}, \delta_P(s'_4)(s'_3) = 1$. There is a single atomic proposition $p \in \mathsf{AP}$ and the states labeled by p are depicted in gray. The initial state is s'_0. $\square$

Parallel composition of MDPs. An MDP is said to be alternating if the initial state is a Player-1 state, all the successors of Player-1 states are probabilistic states, and vice versa. Given two alternating MDPs $G = (S, (S_1, S_P), A, \mathsf{Av}, \delta_1, \delta_P, \mathcal{L}, s_0)$ and $G' = (S', (S'_1, S'_P), A, \mathsf{Av}', \delta'_1, \delta'_P, \mathcal{L}', s'_0)$, the parallel composition is an MDP $G \parallel G' = (\overline{S}, (\overline{S}_1, \overline{S}_P), A, \overline{\mathsf{Av}}, \overline{\delta}_1, \overline{\delta}_P, \overline{\mathcal{L}}, \overline{s}_0)$ defined as follows: (i) the states are $\overline{S} = \overline{S}_1 \cup \overline{S}_P$, where $\overline{S}_1 = S_1 \times S'_1$ and $\overline{S}_P = S_P \times S'_P$; (ii) for a state $(s, s') \in \overline{S}_1$ we have $\overline{\mathsf{Av}}((s, s')) = \mathsf{Av}(s) \cap \mathsf{Av}'(s')$; (iii) for a state $(s, s') \in \overline{S}_1$ and an action $a \in \overline{\mathsf{Av}}((s, s'))$ we have $\overline{\delta}_1((s, s'), a) = (\delta_1(s, a), \delta'_1(s', a))$; (iv) for a state $(s_p, s'_p) \in \overline{S}_P$ we have $\overline{\delta}((s_p, s'_p))(t, t') = \delta_P(s_p)(t) \cdot \delta'_P(s'_p)(t')$; (v) for a state $(s, s') \in \overline{S}$ we have $\overline{\mathcal{L}}((s, s')) = \mathcal{L}(s) \cup \mathcal{L}'(s')$, and (vi) the initial state is (s_0, s'_0).

4.2 Qualitative Logics for MDPs

We consider the qualitative fragment of pCTL^* [56,4,10] and refer to the logic as *qualitative pCTL** (denoted as QCTL^*) as it can express qualitative properties of MDPs.

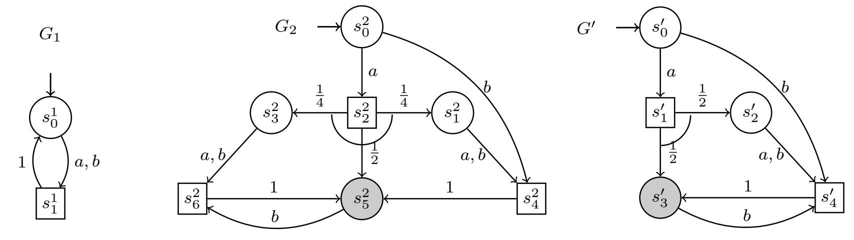


Fig. 3 Examples of MDPs.

Syntax and semantics. The syntax of the logic is given in positive normal form and is similar to the syntax of ATL^*. It has the same state and path formulas as ATL^* with the exception of path quantifiers. The logic QCTL^* comes with two path quantifiers (PQ), namely $\langle \mathsf{Almost} \rangle$ and $\langle \mathsf{Positive} \rangle$ (instead of $\langle\!\langle 1 \rangle\!\rangle$, $\langle\!\langle 2 \rangle\!\rangle$, $\langle\!\langle 1,2 \rangle\!\rangle$, and $\langle\!\langle \emptyset \rangle\!\rangle$).

$$\mathrm{QCTL}^* \text{ path quantifiers: } \langle \mathsf{Almost} \rangle, \langle \mathsf{Positive} \rangle.$$

The semantics of the logic QCTL^* is the same for the fragment shared with ATL^*, therefore we only give semantics for the new path quantifiers. Given a path formula φ, we denote by $[\![\varphi]\!]_G$ the set of plays ω such that $\omega \models \varphi$. For a state s and a path formula φ we have:

$$\begin{aligned} s &\models \langle \mathsf{Almost} \rangle(\varphi) && \text{iff } \exists \sigma \in \Sigma : \Pr_s^\sigma([\![\varphi]\!]) = 1 \\ s &\models \langle \mathsf{Positive} \rangle(\varphi) && \text{iff } \exists \sigma \in \Sigma : \Pr_s^\sigma([\![\varphi]\!]) > 0. \end{aligned}$$

As before, we denote by QCTL the fragment of QCTL^* where every temporal operator is immediately preceded by a path quantifier, and for a state formula ψ the set $[\![\psi]\!]_G$ denotes the set of states in G that satisfy the formula ψ.

Logical relation induced by QCTL **and** QCTL^***.** Given two MDPs G and G', the logical relation induced by QCTL^*, denoted as $\preccurlyeq_Q^*$, (resp. by QCTL, denoted as $\preccurlyeq_Q$), is defined as follows:

$$\preccurlyeq_Q^* = \{(s, s') \in S \times S' \mid \forall \psi \in \mathrm{QCTL}^* : \text{ if } s \models \psi \text{ then } s' \models \psi\}$$

(resp. $\forall \psi \in \mathrm{QCTL}$).

5 Characterization of Qualitative Simulation for MDPs

In this section we establish the equivalence of the $\preccurlyeq_Q^*$ relation on MDPs with the $\preccurlyeq_C^*$ relation on the two-player interpretation of MDPs, i.e., we prove that for all MDPs G and G' we have $\preccurlyeq_Q^* (G, G') = \preccurlyeq_C (\widehat{G}, \widehat{G}')$, where $\widehat{G}$ (resp. $\widehat{G}'$) is the two-player interpretation of the MDP G (resp. G'). In the first step we show how to translate some of the QCTL formulas into C-ATL formulas. We only need to translate the path quantifiers due to the similarity of path formulas in the logics.

Lemma 2 *For all atomic propositions q, r and for all MDPs G, we have:*

$$[\![\langle \mathsf{Almost} \rangle(\bigcirc q)]\!]_G = [\![\langle\!\langle 1 \rangle\!\rangle(\bigcirc q)]\!]_{\widehat{G}} \tag{1}$$

$$[\![\langle \mathsf{Almost} \rangle(q \mathcal{W} r)]\!]_G = [\![\langle\!\langle 1 \rangle\!\rangle(q \mathcal{W} r)]\!]_{\widehat{G}} \tag{2}$$

$$[\![\langle \mathsf{Positive} \rangle(\bigcirc q)]\!]_G = [\![\langle\!\langle 1,2 \rangle\!\rangle(\bigcirc q)]\!]_{\widehat{G}} \tag{3}$$

$$[\![\langle \mathsf{Positive} \rangle(q \mathcal{U} r)]\!]_G = [\![\langle\!\langle 1,2 \rangle\!\rangle(q \mathcal{U} r)]\!]_{\widehat{G}} \tag{4}$$

Proof Point 1. The inclusion $[\![\langle\mathsf{Almost}\rangle(\bigcirc q)]\!] \supseteq [\![\langle\!\langle 1\rangle\!\rangle(\bigcirc q)]\!]$ follows from the fact that there exists a strategy for Player 1 such that for all strategies of Player 2 the next state reached satisfies q. It follows that the same strategy for Player 1 ensures the formula with probability 1. For the second inclusion $[\![\langle\mathsf{Almost}\rangle(\bigcirc q)]\!] \subseteq [\![\langle\!\langle 1\rangle\!\rangle(\bigcirc q)]\!]$ we consider two cases: (i) let $s \in [\![\langle\mathsf{Almost}\rangle(\bigcirc q)]\!]$ be a Player-1 state. Then there exists an available action a that leads to a state that satisfies formula q. As s is a Player-1 state, the transition function under a has a unique successor. Therefore, playing the same action ensures q also in the two-player interpretation. The second case is that s is a probabilistic state. In that case all the successors in the support of the probabilistic transition function satisfy q. Therefore formula q is also satisfied in the two-player interpretation.

Point 2. As for the previous point the inclusion $[\![\langle\mathsf{Almost}\rangle(q\mathcal{W}r)]\!] \supseteq [\![\langle\!\langle 1\rangle\!\rangle(q\mathcal{W}r)]\!]$ follows easily from the definition. For the second inclusion let σ be a strategy that satisfies the formula $q\mathcal{W}r$ almost-surely in the $1\frac{1}{2}$-player interpretation. Assume towards contradiction that there exists a strategy θ for Player 2 in the two-player interpretation, such that the play $\mathsf{Play}(s,\sigma,\theta)$ violates $q\mathcal{W}r$. It follows, that the play $\mathsf{Play}(s,\sigma,\theta)$ satisfies $\neg r\,\mathcal{U}\neg q$. This is possible only if there exists a finite path to a $\neg q$ state that uses only $\neg r$ states, and the finite path has a positive probability in the $1\frac{1}{2}$-player interpretation of the MDP. The contradiction follows.

Point 3. and 4. Point 3 follows similarly to Point 1, and Point 4 follows the same arguments as in Point 2. □

Lemma 3 *For all atomic propositions r and for all MDPs G we have:* $[\![\langle\mathsf{Positive}\rangle(\Box r)]\!]_G = [\![\langle\mathsf{Positive}\rangle(r\,\mathcal{U}\langle\mathsf{Almost}\rangle(\Box\, r))]\!]_G$.

Proof The result follows from [30, Lemma 1] (shown even for a more general class of partially observable MDPs) and from [29] showing that deterministic strategies are sufficient in POMDPs. □

Lemma 4 *For all atomic propositions q, r and for all MDPs G, we have:*

$$[\![\langle\mathsf{Positive}\rangle(q\mathcal{W}r)]\!]_G = [\![\langle\!\langle 1,2\rangle\!\rangle(q\mathcal{U}r)]\!]_{\widehat{G}} \cup [\![\langle\!\langle 1,2\rangle\!\rangle(q\mathcal{U}(\langle\!\langle 1\rangle\!\rangle(q\mathcal{W}\mathsf{false})))]\!]_{\widehat{G}}$$

Proof By definition we have that $[\![\langle\mathsf{Positive}\rangle(q\mathcal{W}r)]\!] = [\![\langle\mathsf{Positive}\rangle((q\mathcal{U}r) \vee (\Box q))]\!]$. We write the formula as follows: $[\![\langle\mathsf{Positive}\rangle((q\mathcal{U}r) \vee (\Box q))]\!] = [\![\langle\mathsf{Positive}\rangle(q\mathcal{U}r)]\!] \cup [\![\langle\mathsf{Positive}\rangle(\Box q)]\!]$. By Lemma 3 we have that $[\![\langle\mathsf{Positive}\rangle(\Box q)]\!] = [\![\langle\mathsf{Positive}\rangle(q\,\mathcal{U}\langle\mathsf{Almost}\rangle(\Box\, q))]\!]$. Note that $\Box\, q \equiv q\mathcal{W}\mathsf{false}$. All these facts together with the already established translations presented in Lemma 2 give us the desired result. □

To complete the translation of temporal operators it remains to express the QCTL formula $[\![\langle\mathsf{Almost}\rangle(q\mathcal{U}r)]\!]$ in terms of C-ATL. We first introduce the Apre function:

Apre. Given two sets of states $X, Y \subseteq S$ we define the predecessor operator Apre as follows:

$$\begin{aligned}\mathsf{Apre}(Y,X) = \{&s \in S_1 \mid \exists a \in \mathsf{Av}(s) \;:\; \delta_1(s,a) \in X \cap Y\} \cup \\ &\{s_p \in S_P \mid \mathrm{Supp}(\delta_P(s_p)) \subseteq Y \wedge \mathrm{Supp}(\delta_P(s_p)) \cap X \neq \emptyset\}.\end{aligned}$$

Intuitively, in the $1\frac{1}{2}$ interpretation the Apre function given two sets of states X and Y, selects Player-1 states from which Player 1 can enforce the next state to be in $X \cap Y$, and selects probabilistic states such that the next state is with probability one in Y and with positive probability in X. As is shown in [51] we can express the states $[\![\langle\mathsf{Almost}\rangle(q\mathcal{U}r)]\!]$

using the following μ-calculus notation, where μ (resp. ν) denotes the least (resp. greatest) fixpoint:

$$[\![\langle \mathsf{Almost} \rangle (q \mathcal{U} r)]\!] = \nu Y. \mu X. ([\![r]\!] \cup ([\![q]\!] \cap \mathsf{Apre}(Y, X))) \tag{5}$$

The fixpoint computation on an MDP with n states can be described as follows: the variable Y_0 is initialized to whole state space, and in each iteration i the variable $X_{i,0}$ is initialized to the empty set; the variable $X_{i,j+1}$ is computed from $X_{i,j}$ applying the one step Apre operator. Finally, the variable Y_i is set as the fixpoint of iteration i.

Formally, for $1 \leq i \leq n$ and $0 \leq j \leq n-1$ we have

$$Y_0 = [\![\mathsf{true}]\!]; \quad X_{i,0} = [\![\mathsf{false}]\!]; \quad X_{i,j+1} = ([\![r]\!] \cup ([\![q]\!] \cap \mathsf{Apre}(Y_{i-1}, X_{i,j}))); \quad Y_i = X_{i,n};$$

and then $Y_n = [\![\langle \mathsf{Almost} \rangle (q \mathcal{U} r)]\!]$. Next we show that the Apre function can be expressed in C-ATL. For C-ATL formulas ψ_1, ψ_2 we define:

$$F_{\mathsf{Apre}}(\psi_1, \psi_2) = \langle\!\langle 1 \rangle\!\rangle (\bigcirc \psi_1) \wedge \langle\!\langle 1, 2 \rangle\!\rangle (\bigcirc (\psi_1 \wedge \psi_2))$$

Lemma 5 *For all MDPs G and* C-ATL *state formulas ψ_1, ψ_2 we have:*

$$[\![F_{\mathsf{Apre}}(\psi_1, \psi_2)]\!]_{\widehat{G}} = \mathsf{Apre}([\![\psi_1]\!]_{\widehat{G}}, [\![\psi_2]\!]_{\widehat{G}})$$

Proof We prove the two inclusions. We start with $\mathsf{Apre}([\![\psi_1]\!], [\![\psi_2]\!]) \subseteq [\![F_{\mathsf{Apre}}(\psi_1, \psi_2)]\!]$. Let s be a state in $\mathsf{Apre}([\![\psi_1]\!], [\![\psi_2]\!])$, we consider two cases: (i) $s \in S_1$; and (ii) $s \in S_P$. For the case (i) it follows from the definition of Apre that there exists an action $a \in \mathsf{Av}(s)$ such that the unique state $\delta_1(s, a)$ satisfies $\psi_1 \wedge \psi_2$. It follows that $s \in [\![\langle\!\langle 1 \rangle\!\rangle (\bigcirc \psi_1) \wedge \langle\!\langle 1, 2 \rangle\!\rangle (\bigcirc (\psi_1 \wedge \psi_2))]\!]$ and therefore $s \in [\![F_{\mathsf{Apre}}(\psi_1, \psi_2)]\!]$. In case (ii) $s \in S_P$, we have by definition $\mathrm{Supp}(\delta_P(s)) \subseteq [\![\psi_1]\!]$, and $\mathrm{Supp}(\delta_P(s)) \cap [\![\psi_2]\!] \neq \emptyset$. It follows that $\mathrm{Supp}(\delta_P(s)) \cap [\![\psi_1 \wedge \psi_2]\!] \neq \emptyset$ and $s \in [\![\langle\!\langle 1 \rangle\!\rangle (\bigcirc \psi_1) \wedge \langle\!\langle 1, 2 \rangle\!\rangle (\bigcirc (\psi_1 \wedge \psi_2))]\!]$, and therefore $s \in [\![F_{\mathsf{Apre}}(\psi_1, \psi_2)]\!]$.

We continue with the second inclusion $[\![F_{\mathsf{Apre}}(\psi_1, \psi_2)]\!] \subseteq \mathsf{Apre}([\![\psi_1]\!], [\![\psi_2]\!])$. Let s be a state in $[\![F_{\mathsf{Apre}}(\psi_1, \psi_2)]\!]$, we again consider two cases: (i) $s \in S_1$; and (ii) $s \in S_P$. For case (i) when $s \in S_1$ assume $s \in [\![\langle\!\langle 1 \rangle\!\rangle (\bigcirc \psi_1) \wedge \langle\!\langle 1, 2 \rangle\!\rangle (\bigcirc (\psi_1 \wedge \psi_2))]\!]$, it follows that there exists an available action $a \in \mathsf{Av}(s)$ such that the unique state $\delta_1(s, a)$ is in $[\![\psi_1 \wedge \psi_2]\!]$. For the second case (ii) when $s \in S_P$ we again assume $s \in [\![\langle\!\langle 1 \rangle\!\rangle (\bigcirc \psi_1) \wedge \langle\!\langle 1, 2 \rangle\!\rangle (\bigcirc (\psi_1 \wedge \psi_2))]\!]$. The first part of the formula ensures that $\delta_P(s) \subseteq [\![\psi_1]\!]$ and the second part ensures that $\delta_P(s) \cap [\![\psi_1 \wedge \psi_2]\!] \neq \emptyset$. The desired result follows. $\square$

The following lemma shows the first of the two inclusions:

Lemma 6 *For all MDPs G and G' we have $\preccurlyeq_C (\widehat{G}, \widehat{G}') \subseteq \preccurlyeq_Q (G, G')$.*

Proof We prove the counterpositive, i.e., we construct a mapping of formulas f : QCTL $\rightarrow$ C-ATL such that given two states s, s' and a QCTL formula ψ we have that if $s \models \psi$ and $s' \not\models \psi$ then the C-ATL formula $f(\psi)$ is true in s and not true in s'. We proceed by structural induction on the QCTL formula and replace parts that are in scope of a path quantifier by their C-ATL version. The cases where ψ is an atomic proposition or a Boolean combination of formulas are straightforward. It remains to translate the formulas $\langle \mathsf{Almost} \rangle (\bigcirc \varphi_1)$, $\langle \mathsf{Almost} \rangle (\varphi_1 \mathcal{W} \varphi_2)$, and $\langle \mathsf{Almost} \rangle (\varphi_1 \mathcal{U} \varphi_2)$ for QCTL formulas φ_1, φ_2. The translation of the first two follows directly from Lemma 2, therefore it remains to translate the QCTL formula $\langle \mathsf{Almost} \rangle (\varphi_1 \mathcal{U} \varphi_2)$. We proceed by encoding the fixpoint computation of the $\langle \mathsf{Almost} \rangle (\varphi_1 \mathcal{U} \varphi_2)$ formula into nested C-ATL formulas. Let n be the number of states of the MDP. Let $\{\widetilde{\phi}_i,\ \phi_{i,j} \mid 0 \le i,j \le n\}$ be a set of formulas defined by the following clauses:

$$\begin{aligned}
&\widetilde{\phi}_0 = \mathsf{true};\\
\forall 1 \le i \le n :\ &\phi_{i,0} = \mathsf{false}\\
\forall 1 \le i \le n . \forall 0 \le j \le n-1 :\ &\phi_{i,j+1} = f(\varphi_2) \vee (f(\varphi_1) \wedge F_{\mathsf{Apre}}(\widetilde{\phi}_{i-1}, \phi_{i,j}))\\
\forall 1 \le i \le n :\ &\widetilde{\phi}_i = \phi_{i,n};
\end{aligned}$$

By Lemma 5 the set of nested formulas $\phi_{i,j}$ represents the computation of $X_{i,j}$ and $\widetilde{\phi}_i$ the computation of Y_i (for the computation of the fixpoint formula). It follows that we have $[\![\langle\mathsf{Almost}\rangle(\varphi_1 \mathcal{U} \varphi_2)]\!] = [\![\widetilde{\phi}_n]\!]$ and concludes the translation. The translation for formulas $\langle\mathsf{Positive}\rangle(\bigcirc\varphi_1)$, $\langle\mathsf{Positive}\rangle(\varphi_1 \mathcal{W} \varphi_2)$, and $\langle\mathsf{Positive}\rangle(\varphi_1 \mathcal{U} \varphi_2)$ to C-ATL formulas follows from Lemma 2 and Lemma 4. The desired result follows. □

Lemma 7 *For all MDPs G and G' we have $\preccurlyeq_Q (G, G') \subseteq \preccurlyeq_C (\widehat{G}, \widehat{G}')$.*

Proof Given MDPs with n states in total, it follows from the proof of Theorem 2 for the combined-simulation game that the n-step approximation $\preccurlyeq_C^n$ is exactly the same as $\preccurlyeq_C$. We define a sequence $\Psi_0, \Psi_1, \ldots, \Psi_n$ of sets of formulas of QCTL with the property that $s \preccurlyeq_C^i t$ iff every formula $\psi \in \Psi_i$ that is true in s is also true in t. We denote by $\mathsf{BoolC}(\Psi)$ all the formulas that consist of disjunctions and conjunctions of formulas in Ψ. We assume that $\mathsf{BoolC}(\Psi)$ does not contain repeated elements, therefore from finiteness of Ψ follows finiteness of $\mathsf{BoolC}(\Psi)$. We define $\Psi_0 = \mathsf{BoolC}(\{q, \neg q \mid q \in \mathsf{AP}\})$, and for all $0 \le i < n$ we define $\Psi_{i+1} = \mathsf{BoolC}(\{\Psi_i \cup \{\langle\mathsf{Positive}\rangle(\bigcirc\psi), \langle\mathsf{Almost}\rangle(\bigcirc\psi) \mid \psi \in \Psi_i\}\})$. The formulas in $\Psi_0, \Psi_1, \ldots, \Psi_n$ provide witnesses that for all $0 \le i \le n$ we have that $\preccurlyeq_Q \subseteq \preccurlyeq_C^i$, in particular we have that $\preccurlyeq_Q \subseteq \preccurlyeq_C$. □

Theorem 4 *For all MDPs G and G' we have $\preccurlyeq_Q (G, G') = \preccurlyeq_C (\widehat{G}, \widehat{G}')$.*

Theorem 5 *For all MDPs G and G' we have $\preccurlyeq_Q^* (G, G') = \preccurlyeq_Q (G, G')$*

Proof We need to show that if a QCTL* formula distinguishes two states, then there is a QCTL formula that also distinguishes them. The basic idea is similar to the proof of [21, Theorem 7.1, assertion 2]. We first construct a deterministic parity automata given the formula in QCTL*, and the almost-sure or positive solutions for MDPs with parity objectives can be encoded as a μ-calculus formula [26]. The translation of μ-calculus formulas to a QCTL formula is done as in Lemma 6. □

Remark 3 The size of the formulas when translating from QCTL* to QCTL (Theorem 5) may be doubly exponential in the size of the input formula. Note, that the translation of LTL to deterministic parity automata is already doubly-exponential [75].

Theorem 6 *Given MDPs G and G' the relation $\preccurlyeq_Q^* (G, G')$ can be computed in quadratic time using discrete graph algorithms.*

Proof Follows directly from Theorems 3, 4, and 5. □

6 CEGAR for Combined Simulation

In this section we present a CEGAR approach for the computation of the combined simulation relation in two-player games.

Abstraction. An *abstraction* of a game consists of a partition of the game graph such that in each partition the atomic proposition labeling match for all states. Given an abstraction of a game, the abstract game can be defined by collapsing states of each partition and redefining the action-available and transition functions. The redefinition of the action-available and transition functions can either increase or decrease the power of the players. If we increase the power of Player 1 and decrease the power of Player 2, then the abstract game will be in alternating simulation with the original game, and if we increase the power of both players, then the abstract game will simulate the original game. We now formally define the partitions, and the two abstractions.

Partitions for abstraction. A *partition* of a game $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ is an equivalence relation $\Pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ on S such that: (i) for all $1 \le i \le k$ we have $\pi_i \subseteq S$ and for all $s, s' \in \pi_i$ we have $\mathcal{L}(s) = \mathcal{L}(s')$ (labeling match); (ii) $\bigcup_{1 \le i \le k} \pi_i = S$ (covers the state space); and (iii) for all $1 \le i, j \le k$, such that $i \ne j$ we have $\pi_i \cap \pi_j = \emptyset$ (disjoint). Note that in turn-based games Player 1 and Player 2 states are distinguished by proposition turn, so they belong to different partitions.

Simulation abstraction. Given a two-player game $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and a partition Π of G, we define the *simulation abstraction of* G as a two-player game $Abs_{\mathcal{S}}^{\Pi}(G) = (\overline{S}, A, \overline{\mathsf{Av}}, \overline{\delta}, \overline{\mathcal{L}}, \overline{s}_0)$, where

- $\overline{S} = \Pi$: the partitions in Π are the states of the abstract game.
- For all $\pi_i \in \Pi$ we have $\overline{\mathsf{Av}}(\pi_i) = \bigcup_{s \in \pi_i} \mathsf{Av}(s)$: the set of available actions is the union of the actions available to the states in the partition, and this gives more power to Player 1.
- For all $\pi_i \in \Pi$ and $a \in \overline{\mathsf{Av}}(\pi_i)$ we have $\overline{\delta}(\pi_i, a) = \{\pi_j \mid \exists s \in \pi_i : (a \in \mathsf{Av}(s) \wedge \exists s' \in \pi_j : s' \in \delta(s, a))\}$: there is a transition from a partition π_i given an action a to a partition π_j if some state $s \in \pi_i$ can make an a-transition to some state in $s' \in \pi_j$, and this gives more power to Player 2.
- For all $\pi_i \in \Pi$ we have $\overline{\mathcal{L}}(\pi_i) = \mathcal{L}(s)$ for some $s \in \pi_i$: the abstract labeling is well-defined, since all states in a partition are labeled by the same atomic propositions.
- $\overline{s}_0$ is the partition in Π that contains state s_0.

Alternating-simulation abstraction. Given a two-player game $G = (S, A, \mathsf{Av}, \delta, \mathcal{L}, s_0)$ and a partition Π of G, we define the *alternating-simulation abstraction of* G as a two-player game $Abs_{\mathcal{A}}^{\Pi}(G) = (\widetilde{S}, A, \widetilde{\mathsf{Av}}, \widetilde{\delta}, \widetilde{\mathcal{L}}, \widetilde{s}_0)$, where

- $\widetilde{S} = \Pi$;
- For all $\pi_i \in \Pi$ we have $\widetilde{\mathsf{Av}}(\pi_i) = \bigcup_{s \in \pi_i} \mathsf{Av}(s)$;
- For all $\pi_i \in \Pi$ and $a \in \widetilde{\mathsf{Av}}(\pi_i)$ we have $\widetilde{\delta}(\pi_i, a) = \{\pi_j \mid \forall s \in \pi_i : (a \in \mathsf{Av}(s) \wedge \exists s' \in \pi_j : s' \in \delta(s, a))\}$: there is a transition from a partition π_i given an action a to a partition π_j if all states $s \in \pi_i$ can make an a-transition to some state in $s' \in \pi_j$, and this gives less power to Player 2. For technical convenience we assume $\widetilde{\delta}(\pi_i, a)$ is non-empty.
- For all $\pi_i \in \Pi$ we have $\widetilde{\mathcal{L}}(\pi_i) = \mathcal{L}(s)$ for some $s \in \pi_i$;
- $\widetilde{s}_0$ is the partition in Π that contains state s_0 (as in the case of simulation abstraction).

The following proposition states that (alternating-)simulation abstraction of a game G is in (alternating-)simulation with G.

Proposition 3 *For all partitions Π of a two-player game G we have: (1) $G \leqslant_{\mathcal{A}} Abs_{\mathcal{A}}^{\Pi}(G)$; and (2) $G \leqslant_{\mathcal{S}} Abs_{\mathcal{S}}^{\Pi}(G)$.*

Example 4 Consider a two-player interpretation $\widehat{G}_2$ of the MDP G_2 from Figure 3. The coarsest partition of $\widehat{G}_2$ is $\Pi = \{\pi_0, \pi_1, \pi_2\}$, where $\pi_0 = \{s_0^2, s_1^2, s_3^2\}, \pi_1 = \{s_2^2, s_4^2, s_6^2\}, \pi_2 = \{s_5^2\}$. The alternating-simulation abstraction and the simulation abstraction of Π are depicted in Figure 4. □

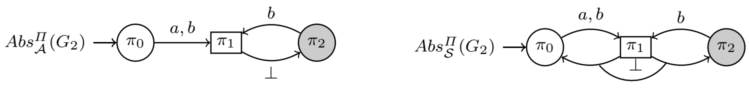


Fig. 4 Alternating-simulation and simulation abstractions of the two-player interpretation $\widehat{G}_2$ (MDP G_2 from Figure 3).

6.2 Sound Assume-Guarantee Rule

In this section we present the sound assume-guarantee rule for the combined-simulation problem. To achieve this we first need an extension of the notion of combined-simulation game.

Definition 12 (Modified combined-simulation game.) Consider games $G^{\mathsf{Alt}} = (S, A, \delta^{\mathsf{Alt}}, \mathsf{Av}^{\mathsf{Alt}}, \mathcal{L}, s_0)$, $G^{\mathsf{Sim}} = (S, A, \delta^{\mathsf{Sim}}, \mathsf{Av}^{\mathsf{Sim}}, \mathcal{L}, s_0)$ and $G' = (S', A, \delta', \mathsf{Av}', \mathcal{L}', s_0')$. The *modified simulation game* $G^{\mathcal{M}} = (S^{\mathcal{M}}, A^{\mathcal{M}}, \mathsf{Av}^{\mathcal{M}}, \delta^{\mathcal{M}}, \mathcal{L}^{\mathcal{M}}, s_0^{\mathcal{M}})$ is defined exactly like the combined simulation game given G^{Alt} and G', with the exception that the step-wise simulation gadget is defined using the transitions of G^{Sim} instead of G^{Alt}. Formally, we change the transitions as follows:

– *Checking step-wise simulation conditions.* Transition (a) from Definition 10 is redefined: for a state $(s, s', \mathsf{Sim}, 2)$ we have only one action $\perp$ available for Player 1 and we have $\delta^{\mathcal{M}}((s, s', \mathsf{Sim}, 2), \perp) = \{(t, s', \mathsf{Sim}, 1) \mid \exists a \in \mathsf{Av}^{\mathsf{Sim}}(s) : t \in \delta^{\mathsf{Sim}}(s, a)\}$.

We write $(G^{\mathsf{Alt}} \otimes G^{\mathsf{Sim}}) \leqslant_{\mathcal{M}} G'$ if and only if $(s_0, s_0') \in [\![\langle\!\langle 1 \rangle\!\rangle(\Box\neg p)]\!]_{G^{\mathcal{M}}}$.

Proposition 4 *Let $G, G', G^{\mathsf{Alt}}, G^{\mathsf{Sim}}$ be games such that $G \leqslant_{\mathcal{A}} G^{\mathsf{Alt}}$ and $G \leqslant_{\mathcal{S}} G^{\mathsf{Sim}}$. Then $(G^{\mathsf{Alt}} \otimes G^{\mathsf{Sim}}) \leqslant_{\mathcal{M}} G'$ implies $G \leqslant_{\mathcal{C}} G'$.*

The key proof idea for the above proposition is as follows: if $G \leqslant_{\mathcal{A}} G^{\mathsf{Alt}}$ and $G \leqslant_{\mathcal{S}} G^{\mathsf{Sim}}$, then in the modified combined-simulation game $G^{\mathcal{M}}$ the adversary (Player 2) is stronger than in the combined-simulation game $G^{\mathcal{C}}$. Hence winning in $G^{\mathcal{M}}$ for the proponent (Player 1) implies winning in $G^{\mathcal{C}}$ and gives the desired result of the proposition.

Sound assume-guarantee method. Given two games G_1 and G_2, checking whether their parallel composition $G_1 \parallel G_2$ is in combined simulation with a game G' can be done explicitly by constructing the synchronized product. The composition, however, may be much larger than the components and thus make the method ineffective in practical cases. We present an alternative method that proves combined simulation in a compositional manner, by abstracting G_2 with some partition Π and then composing it with G_1. The sound assume-guarantee rule follows from Proposition 3 and Proposition 4.

Proposition 5 (Sound assume-guarantee rule) *Given games G_1, G_2, G', and a partition Π of G_2, let $\mathsf{A} = G_1 \parallel Abs^{\Pi}_{\mathcal{A}}(G_2)$ and $\mathsf{S} = G_1 \parallel Abs^{\Pi}_{\mathcal{S}}(G_2)$. If $(\mathsf{A} \otimes \mathsf{S}) \leqslant_{\mathcal{M}} G'$, then $(G_1 \parallel G_2) \leqslant_{\mathcal{C}} G'$, i.e.,*

$$\frac{\mathsf{A} = G_1 \parallel Abs^{\Pi}_{\mathcal{A}}(G_2); \qquad \mathsf{S} = G_1 \parallel Abs^{\Pi}_{\mathcal{S}}(G_2); \qquad (\mathsf{A} \otimes \mathsf{S}) \leqslant_{\mathcal{M}} G'}{(G_1 \parallel G_2) \leqslant_{\mathcal{C}} G'} \tag{6}$$

Remark 4 Note that for the trivial partition Π, where every equivalence relation is a singleton, the modified combined-simulation game coincides with the combined simulation game. We will use this fact to argue about completeness of our CEGAR approach.

If the partition Π is coarse, then the abstractions in the assume-guarantee rule can be smaller than G_2 and also their composition with G_1. As a consequence, combined simulation can be faster as compared to explicitly computing the composition. In Section 6.4 we describe how to effectively compute the partitions Π and refine them using CEGAR approach.

6.3 Counterexamples Analysis

Representation of counterexamples. If the premise $(\mathsf{A} \otimes \mathsf{S}) \leqslant_{\mathcal{M}} G'$ of the assume-guarantee rule (6) is not satisfied, then the adversary (Player 2) has a memoryless winning strategy θ_{abs} in $G^{\mathcal{M}}$, and the memoryless strategy is the *counterexample*. To use the sound assume-guarantee rule (6) in a CEGAR loop, we need analysis of counterexamples. Note that in $G^{\mathcal{M}}$ Player 2 has a reachability objective, and thus a winning strategy θ_{abs} ensures that the target set is always reached from the starting state, and hence no cycle can be formed without reaching the target state once the memoryless winning strategy is fixed. Hence we represent counterexamples as directed-acyclic graphs (given the strategy θ_{abs} we denote the corresponding directed acyclic graph as $\mathsf{DAG}(\theta_{\mathsf{abs}})$), where the leafs are the target states and every non-leaf state has a single successor chosen by the strategy of Player 2 and has all available actions for Player 1.

Abstract, concrete, and spurious counterexamples. Given two-player games G_1 and G_2, let $G = (G_1 \parallel G_2)$ be the parallel composition. Given G and G', let $G^{\mathcal{C}}$ be the combined-simulation game of G and G'. The abstract game $G^{\mathcal{M}}$ is the modified combined-simulation game of $(\mathsf{A} \otimes \mathsf{S})$ and G', where $\mathsf{A} = G_1 \parallel Abs^{\Pi}_{\mathcal{A}}(G_2)$ and $\mathsf{S} = G_1 \parallel Abs^{\Pi}_{\mathcal{S}}(G_2)$. We refer to a counterexample θ_{abs} in $G^{\mathcal{M}}$ as *abstract*, and to a counterexample θ_{con} in $G^{\mathcal{C}}$ as *concrete*. An abstract counterexample θ_{abs} is *feasible* if we can substitute partitions in A and S in a rooted subtree of θ_{abs} with states of G_2 to obtain a concrete counterexample (see [58] for details). An abstract counterexample is *spurious* if it is not feasible.

Concretization of counterexamples. We follow the approach of [58] to check the feasibility of a counterexample by finding a *concretization function* Conc from states in $G^{\mathcal{M}}$ to a set of states in G_2 that witnesses a concrete strategy θ_{con} from the abstract strategy θ_{abs}. A state in $G^{\mathcal{M}}$ has as one of its components a subset of states $\pi_i \in \Pi$, which is an equivalence class from the abstracted game G_2. Intuitively, for a state $\overline{s}$ of $G^{\mathcal{M}}$ in the counterexample graph $\mathsf{DAG}(\theta_{\mathsf{abs}})$, the concretization function represents the subset of π_i where a concrete winning strategy exists that replays the strategy represented by the subtree of the graph $\mathsf{DAG}(\theta_{\mathsf{abs}})$ rooted at state $\overline{s}$.

Computation of the concretization function. Given an abstract counterexample θ_{abs} and a state $\overline{s}$ in $G^{\mathcal{M}}$, let $\mathsf{Succ}(\overline{s})$ be the set of all successors of $\overline{s}$ in $G^{\mathcal{M}}$ given θ_{abs} is fixed by Player 2. The concretization function Conc is computed inductively on the structure of the abstract counterexample θ_{abs} starting from the leaves. The formal description of the concretization computation is given in Figure 5, where the concretization of a state $\overline{s}$ in the abstract counterexample is computed from its successors in the DAG. We use the notation Av^1, Av^2, and δ^2 to represent the action-available functions of G_1 and G_2, and the transition function of G_2, respectively.

Proposition 6 *[58, Proposition 2] An abstract counterexample θ_{abs} is feasible if and only if the concretization function Conc of the root of the graph $\mathsf{DAG}(\theta_{\mathsf{abs}})$ contains the initial state of the game G_2.*

Illustrative examples. We present intuitive description of two representative cases of concretization from Figure 5: (1) Consider a state $\overline{s} = ((s_1, \pi_2), s', \mathsf{Alt}, 2)$ where the abstract counterexample chooses the successor $\overline{s}' = ((s_1, \pi_2), s', \mathsf{Alt}, a, 1)$ (intuitively this corresponds to choice of action a). The concretization $\mathsf{Conc}(\overline{s}) = \{s \in \pi_2 \mid a \in \mathsf{Av}^2(s) \wedge s \in \mathsf{Conc}(\overline{s}')\}$ is the subset of states in π_2 where the action a is available and s also belongs to the concretization of the successor state $\overline{s}'$. (2) For a state $\overline{s} = ((s_1, \pi_2), s', \mathsf{Alt}, a, a', 1)$, the concretization is the set of states where action a is not available or all successors given action a belong to the concretization of the successors of $\overline{s}$.

$$
\begin{array}{lll}
\overline{s} = ((s_1, \pi_2), s') & : & \mathsf{Conc}(\overline{s}) = \begin{cases} \pi_2 & \overline{s} \text{ is a leaf} \\ \mathsf{Conc}(\overline{s}') & \text{otherwise, where } \mathsf{Succ}(\overline{s}) = \{\overline{s}'\} \end{cases} \\
\overline{s} = ((s_1, \pi_2), s', \mathsf{Sim}, 2) & : & \mathsf{Conc}(\overline{s}) = \{s \in \pi_2 \mid \exists a \in \mathsf{Av}^1(s_1) \cap \mathsf{Av}^2(s) : \delta^2(s,a) \cap \mathsf{Conc}(\overline{s}') \neq \emptyset\} \\
 & & \text{where } \mathsf{Succ}(\overline{s}) = \{\overline{s}'\} \\
\overline{s} = ((s_1, \pi_2), s', \mathsf{Sim}, 1) & : & \mathsf{Conc}(\overline{s}) = \bigcap_{\overline{s}' \in \mathsf{Succ}(\overline{s})} \mathsf{Conc}(\overline{s}') \\
\overline{s} = ((s_1, \pi_2), s', \mathsf{Alt}, 2) & : & \mathsf{Conc}(\overline{s}) = \{s \in \pi_2 \mid a \in \mathsf{Av}^2(s) \wedge s \in \mathsf{Conc}(\overline{s}'),\} \text{ where} \\
 & & \mathsf{Succ}(\overline{s}) = \{\overline{s}'\} \text{ and } \overline{s}' = ((s_1, \pi_2), s', \mathsf{Alt}, 2, a) \\
\overline{s} = ((s_1, \pi_2), s', \mathsf{Alt}, a, 1) & : & \mathsf{Conc}(\overline{s}) = \bigcap_{\overline{s}' \in \mathsf{Succ}(\overline{s})} \mathsf{Conc}(\overline{s}') \\
\overline{s} = ((s_1, \pi_2), s', \mathsf{Alt}, a, a', 2) & : & \mathsf{Conc}(\overline{s}) = \mathsf{Conc}(\overline{s}'), \text{ where } \mathsf{Succ}(\overline{s}) = \{\overline{s}'\} \\
\overline{s} = ((s_1, \pi_2), s', \mathsf{Alt}, a, a', 1) & : & \mathsf{Conc}(\overline{s}) = \{s \in \pi_2 \mid a \notin \mathsf{Av}^2(s) \vee \delta^2(s,a) \subseteq \bigcup_{\overline{s}' \in \mathsf{Succ}(\overline{s})} \mathsf{Conc}(\overline{s}')\}
\end{array}
$$

Fig. 5 Concretization function; $\overline{s}$ is a state in an abstract counterexample.

Example 5 Consider MDPs G_1, G_2, G' in Figure 3 interpreted as games and the abstract games $Abs^{\Pi}_{\mathcal{A}}(\widehat{G}_2)$, $Abs^{\Pi}_{\mathcal{S}}(\widehat{G}_2)$ in Figure 4. Let $\mathsf{A} = \widehat{G}_1 \parallel Abs^{\Pi}_{\mathcal{A}}(\widehat{G}_2)$ and $\mathsf{S} = \widehat{G}_1 \parallel Abs^{\Pi}_{\mathcal{S}}(\widehat{G}_2)$. Figure 6 shows part of an abstract counterexample to the modified combined-simulation game of $(\mathsf{A} \otimes \mathsf{S})$ and G'. In this counterexample the adversary first plays in the simulation gadget and the proponent responds by moving to a state $((s^1_1, \pi_1), s'_1)$ or a state $((s^1_1, \pi_1), s'_4)$ (their successors are not depicted in Figure 6). From the state $((s^1_1, \pi_1), s'_1)$ the adversary has a winning strategy by playing in the alternating-simulation gadget, and from $((s^1_1, \pi_1), s'_4)$ by playing in the simulation gadget. The dashed shows assign the concretization of states in the abstract counterexample. The counterexample is spurious, since the initial state of G_2 does not belong to the concretization of the initial state of the counterexample. $\square$

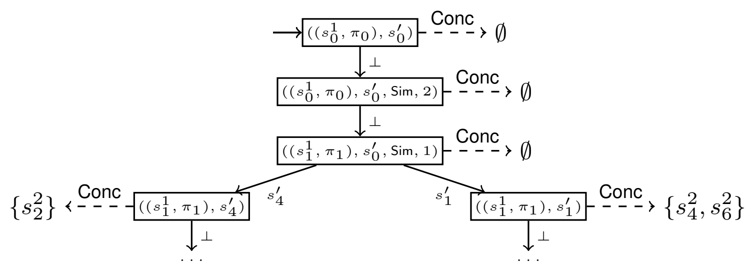


Fig. 6 Abstract counterexample to the modified combined-simulation game of $(\mathsf{A} \otimes \mathsf{S})$ and G', where $\mathsf{A} = \widehat{G}_1 \parallel Abs^{\Pi}_{\mathcal{A}}(\widehat{G}_2)$ and $\mathsf{S} = \widehat{G}_1 \parallel Abs^{\Pi}_{\mathcal{S}}(\widehat{G}_2)$.

Algorithm 1 Assume-guarantee CEGAR for $\leqslant_{\mathcal{C}}$.

```
Input: Two-player games $G_1, G_2, G'$.
Output: yes if $G_1 \parallel G_2 \leqslant_{\mathcal{C}} G'$, otherwise no
  $\Pi \leftarrow$ coarsest partitioning of $G_2$
  loop
      $\mathsf{A} \leftarrow G_1 \parallel Abs^{\Pi}_{\mathcal{A}}(G_2)$;    $\mathsf{S} \leftarrow G_1 \parallel Abs^{\Pi}_{\mathcal{S}}(G_2)$
      $G^{\mathcal{M}} \leftarrow$ modified combined simulation game of $(\mathsf{A} \otimes \mathsf{S})$ and $G'$
      if Player 1 wins in $G^{\mathcal{M}}$ then    return yes
      else
          Cex $\leftarrow$abstract counterexample in $G^{\mathcal{M}}$
          if Feasible(Cex) then    return no
          else    $\Pi \leftarrow$ Refine(Cex, $\Pi$)
```

6.4 CEGAR

The counterexample analysis presented in the previous section allows us to automatically refine abstractions using the CEGAR paradigm [45]. The pseudo-code of the CEGAR algorithm for the assume-guarantee combined simulation is shown in Algorithm 1. The algorithm takes G_1, G_2, G' as arguments and answers whether $(G_1 \parallel G_2) \leqslant_{\mathcal{C}} G'$ holds. Initially, the algorithms computes the coarsest partition Π of G_2. Then, it executes the CEGAR loop: in every iteration the algorithm constructs A (resp. S) as the parallel composition of G_1 and the alternating-simulation abstraction (resp. simulation abstraction) of G_2. Let $G^{\mathcal{M}}$ be the modified combined-simulation game of $(\mathsf{A} \otimes \mathsf{S})$ and G'. If Player 1 has a winning strategy in $G^{\mathcal{M}}$ then the algorithm returns YES; otherwise it finds an abstract counterexample Cex in $G^{\mathcal{M}}$. In case the counterexample is feasible, then it corresponds to a concrete counterexample, and the algorithm returns NO. If Cex is spurious, the algorithm calls a refinement procedure that uses the concretization of Cex to return a partition Π' finer than partition Π. Our technique can be extended to handle multiple components in a similar way as presented in [65, Section 5].

Refinement procedure. Given a partition Π and a spurious counterexample Cex together with its concretization function Conc we describe how to compute the refined partition Π'. Consider a partition $\pi \in \Pi$ and let $\overline{S}_\pi = \{\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_m\}$ denote the states of the abstract counterexample Cex that contain π as its component. Every state $\overline{s}_i$ splits π into at most two sets $\mathsf{Conc}(\overline{s}_i)$ and $\pi \setminus \mathsf{Conc}(\overline{s}_i)$, and let this partition be denoted as T_i. We define a partition $\mathcal{P}_\pi$ as the largest equivalence relation on π that is finer than any of the equivalence relation T_i for all $1 \leq i \leq m$. Formally, $\mathcal{P}_\pi = \{\overline{\pi}_1, \overline{\pi}_2, \ldots, \overline{\pi}_k\}$ is a partition of π such that for all

$1 \leq j \leq k$ and $1 \leq i \leq m$ we have $\overline{\pi}_j \subseteq \mathsf{Conc}(\overline{s}_i)$ or $\overline{\pi}_j \subseteq \pi \setminus \mathsf{Conc}(\overline{s}_i)$. The new partition Π' is then defined as the union over $\mathcal{P}_\pi$ for all $\pi \in \Pi$.

Example 6 We continue with our running example. In Example 5 we showed that the abstractions of $\widehat{G}_2$ by the coarsest partition Π lead to a spurious counterexample depicted in Figure 6. Consider the partition $\pi_1 = \{s_2^2, s_4^2, s_6^2\}$. There are three states in the counterexample that have π_1 as its component and the concretization function assigns to them three subsets of states: $\emptyset, \{s_2^2\}, \{s_4^2, s_6^2\}$. After the refinement partition π_1 is split into two partitions $\pi_1' = \{s_2^2\}$ and $\pi_1'' = \{s_4^2, s_6^2\}$. $\square$

Proposition 7 *Given a partition Π and a spurious counterexample* Cex, *the partition Π' obtained as refinement of Π is finer than Π.*

Sound and completeness of our CEGAR approach. Since we consider finite games, the refinement procedure only executes for finitely many steps. In every iteration of the CEGAR algorithm, either the algorithm returns a correct answer (by soundness), or a finer partition is obtained. Thus either we end up with a correct answer, or the trivial partition, and hence by Remark 4 the completeness of our approach follows. Thus our CEGAR approach is both sound and complete.

7 Experimental Results

We implemented our CEGAR approach for combined simulation in Java, and experimented with our tool on a number of MDPs and two-player games examples. Our algorithms use explicit representation of MDPs. We use PRISM [67] model checker to specify the examples and generate input files for our tool.

Observable actions. To be compatible with the existing benchmarks (e.g. [65]) in our tool actions are observable instead of atomic propositions. Our algorithms are easily adapted to this setting. We also allow the user to specify silent actions for components, which are not required to be matched by the specification G'.

Improved (modified) combined-simulation game. We leverage the fact that MDPs are interpreted as turn-based games to simplify the (modified) combined-simulation game. When comparing two Player-1 states, the last two steps in the alternating-simulation gadget can be omitted, since the players have unique successors given the actions chosen in the first two steps. Similarly, for two probabilistic states, the first two steps in the alternating-simulation gadget can be skipped. We check the (modified) combined-simulation games using the standard attractor algorithm to solve games with safety (as well as reachability) objectives [2, 81].

Improved partition refinement procedure. In the implementation we adopt the approach of [58] for refinement. Given a state $\overline{s}$ of the abstract counterexample with partition π as its component, the equivalence relation may split the set $\pi \setminus \mathsf{Conc}(\overline{s})$ into multiple equivalence classes. Intuitively, this ensures that similar-shaped spurious counterexamples do not reappear in the following iterations. This approach is more efficient than the naive one, and also implemented in our tool.

MDP examples with safety specifications. We used our tool on all the MDP examples and specifications from [65]. The specifications describe safety properties of the systems.

– **Client-Server protocol** (CS_1 *and* CS_n)

		AGCS			AGSS			MONCS		MONSS	
Ex.	$\lvert G_1\rvert, \lvert G_2\rvert, \lvert G'\rvert$	*Time*	*Mem*	$I, \lvert \Pi\rvert$	*Time*	*Mem*	$I, \lvert \Pi\rvert$	*Time*	*Mem*	*Time*	*Mem*
$\mathsf{CS}_1(5)$	36 / 405 / 16	1.13s	112MB	49 / 85	6.11s	213MB	32 / 33	**0.04s**	**34MB**	0.18s	95MB
$\mathsf{CS}_1(6)$	49 / 1215/ 19	2.52s	220MB	65 / 123	11.41s	243MB	40 / 41	**0.04s**	**51MB**	0.31s	99MB
$\mathsf{CS}_1(7)$	64 / 3645/ 22	5.41s	408MB	84 / 156	31.16s	867MB	56 / 57	**0.05s**	**82MB**	0.77s	113MB
$\mathsf{CS}_n(3)$	125 / 16 / 54	0.65s	102MB	9 / 24	33.43s	258MB	11 / 12	**0.09s**	**35MB**	11.29s	115MB
$\mathsf{CS}_n(4)$	625 / 25 / 189	6.22s	495MB	15 / 42	TO	-	-	**0.4s**	**106MB**	1349.6s	577MB
$\mathsf{CS}_n(5)$	3k / 36 / 648	117.06s	2818MB	24 / 60	TO	-	-	**2.56s**	**345MB**	TO	-
$\mathsf{MER}(3)$	278 / 1728 / 11	**1.42s**	**143MB**	8 / 14	2.74s	189MB	6 / 7	1.96s	228MB	128.1s	548MB
$\mathsf{MER}(4)$	465 / 21k / 14	**4.63s**	**464MB**	13 / 22	10.81s	870MB	10 / 11	11.02s	1204MB	TO	-
$\mathsf{MER}(5)$	700 / 250k / 17	**29.23s**	**1603MB**	20 / 32	67s	2879MB	15 / 16	-	MO	MO	-
$\mathsf{SN}(1)$	43 / 32 / 18	0.13s	38MB	3 / 6	0.28s	88MB	2 / 3	**0.04s**	**29MB**	3.51s	135MB
$\mathsf{SN}(2)$	796 / 32 / 54	0.9s	117MB	3 / 6	66.09s	258MB	2 / 3	**0.38s**	**103MB**	3580.83s	1022MB
$\mathsf{SN}(3)$	7k / 32 / 162	4.99s	**408MB**	3 / 6	TO	-	-	4.99s	612MB	TO	-
$\mathsf{SN}(4)$	52k / 32 / 486	**34.09s**	**2448MB**	3 / 6	TO	-	-	44.47s	3409MB	TO	-
$\mathsf{LE}(3,4)$	4 / 415 / 269	**0.25s**	**70MB**	7 / 16	0.71s	164MB	7 / 8	0.28s	80MB	3.46s	163MB
$\mathsf{LE}(3,5)$	4 / 814 / 513	**0.35s**	**80MB**	7 / 16	Error	-	-	0.81s	157MB	Error	-
$\mathsf{LE}(4,4)$	6 / 5665 / 2561	**1.27**	**128MB**	7 / 19	TO	-	-	12.74s	1186MB	TO	-
$\mathsf{LE}(5,5)$	8 / 29k / 21k	**6.73s**	**517MB**	7 / 21	TO	-	-	TO	-	TO	-
$\mathsf{LE}(6,4)$	10 / 42k / 40k	**8.98s**	**664MB**	7 / 25	TO	-	-	TO	-	TO	-
$\mathsf{LE}(6,5)$	10 / 169k / 56k	**36.38s**	**2372MB**	7 / 25	TO	-	-	TO	-	TO	-
$\mathsf{PETP}(2)$	68 / 3 / 3	0.04s	31MB	0 / 2	0.04s	87MB	0 / 1	0.04s	**30MB**	0.04s	90MB
$\mathsf{PETP}(3)$	4 / 1730 / 4	**0.19s**	**65MB**	6 / 8	0.29s	153MB	3 / 4	0.24s	72MB	1.07s	170MB
$\mathsf{PETP}(4)$	5 / 54k / 5	**1.58s**	**325MB**	8 / 10	3.12s	727MB	4 / 5	7.04s	960MB	31.52s	1741MB

Table 1 Results for MDPs examples with safety specifications: AGCS stands for our assume-guarantee combined simulation; AGSS stands for assume-guarantee with strong simulation; MONCS stands for our monolithic combined simulation; and MONSS stands for monolithic strong simulation. The number I denotes the number of CEGAR iterations and $|\Pi|$ the size of the abstraction in the last CEGAR iteration. TO and MO stand for a time-out and memory-out, respectively, and Error means that an error occurred during execution. The memory consumption is obtained using the Unix `time` command.

Model: The example models a Client-Server protocol with mutual exclusion and probabilistic failures in one (CS_1) or all of the clients (CS_n). The model is parametrized by the number of clients.

Specification: The safety specification is exactly the same as in [65] and characterizes the probabilistic failure model of the clients.

- **Mars Exploration Rover** (MER)

  Model: The example models an arbiter module of NASAs software for Mars Exploration Rovers, which grants shared resources for several users. The number of users is the parameter of the model.

  Specification: The specification is exactly the same as in [65] and imposes a safety requirement on the users' behavior.

- **Sensor networks** (SN)

  Model: The example models a network of sensors that communicate via a bounded buffer with probabilistic behavior in the components. The model is parametrized by the number of sensors.

  Specification: The specification is exactly the same as in [65] and is an abstraction of the observed system behavior.

In addition, we also considered three other classical MDP examples:

- **Leader election protocol** (LE)

  Model: The example is based on a PRISM case study [67] that models the *Leader election protocol* [63], where n agents on a ring randomly pick a number from a pool of K numbers. The agent with the highest number becomes the leader. In case there are multiple agents with the same highest number the election proceeds to the next round. The model is parametrized by the values of n and K, respectively.

Specification: The specification requires that two leaders cannot be elected at the same time.

- **Peterson's algorithm** (PETP)

  Model: The example is based on Peterson's algorithm [72] for mutual exclusion of n threads, where the execution order is controlled by a randomized scheduler. We extend Peterson's algorithm by giving the threads a non-deterministic choice to restart before entering the critical section. The restart operation succeeds with probability $\frac{1}{2}$ and with probability $\frac{1}{2}$ the thread enters the critical section.

  Specification: The specification requires only one process to be in the critical section at any time.

Details of experimental results for safety specifications. Table 1 shows the results for MDP examples we obtained using our assume-guarantee algorithm and the monolithic approach (where the composition is computed explicitly). We also compared our results with the tool presented in [65] that implements both assume-guarantee and monolithic approaches for *strong simulation* [77]. All the results were obtained on a Ubuntu-13.04 64-bit machine running on an Intel Core i5-2540M CPU of 2.60GHz. We imposed a 4.3GB upper bound on Java heap memory and one hour time limit. For MER(6) and PETP(5) PRISM cannot parse the input file (probably it runs out of memory).

Summary of results for safety specifications. For all examples, other than the Client-Server protocol, the assume-guarantee method scales better than the monolithic reasoning; and in all examples our qualitative analysis scales better than the strong simulation approach. Qualitative analysis through combined simulation relies on discrete graph algorithms (attractor computation), while checking strong simulation requires calls to an SMT solver.

		AGCS			MONCS	
Ex.	$\lvert G_1\rvert, \lvert G_2\rvert, \lvert G'\rvert$	$Time$	Mem	$I, \lvert\Pi\rvert$	$Time$	Mem
LE$(6,4)$	10 / 42k / 27	**0.82s**	**172MB**	8 / 41	1.66s	275MB
LE$(6,5)$	10 / 169k / 27	**1.73s**	**351MB**	8 / 25	5.05s	808MB
LE$(6,6)$	10 / 521k / 27	**5.01s**	**946MB**	8 / 41	17.3s	2134MB
LE$(7,4)$	12 / 187k / 31	**2.59s**	**447MB**	8 / 52	5.79s	1029MB
LE$(7,5)$	12 / 948k / 31	**12.49s**	**1748MB**	8 / 27	35.49s	3370MB
LE$(7,6)$	12 / 3.5m / 31	**85.61s**	**4303MB**	8 / 27	MO	-
PETP(2)	17 / 184 / 154	0.35s	93MB	3 / 8	**0.18s**	**73MB**
PETP(3)	25 / 10k / 154	**1.23s**	**170MB**	3 / 8	4.79s	593MB
PETP(4)	33 / 864k / 154	**28.61s**	**2187MB**	2 / 8	MO	-
SZYM(2)	24 / 325 / 204	0.5s	108MB	2 / 8	**0.28s**	**99MB**
SZYM(3)	24 / 5010 / 204	**1.1s**	**140MB**	3 / 8	3.34s	407MB
SZYM(4)	24 / 74k / 204	**3.88s**	**343MB**	2 / 8	48.34s	3246MB
SZYM(5)	24 / 1073k / 204	**27.71s**	**2152MB**	2 / 8	MO	-

Table 2 Results for MDPs examples with liveness specifications.

MDP examples with liveness specifications. We have also experimented with MDPs with liveness specifications. We consider the LE and PETP models from the previous safety experiments, as the liveness properties are natural in these models. We also add the additional Szymanski's mutual-exclusion protocol [79].

		AGCS			MONCS		AGAS			MONAS	
Ex.	$\lvert G_1\rvert, \lvert G_2\rvert, \lvert G'\rvert$	$Time$	Mem	$I, \lvert\Pi\rvert$	$Time$	Mem	$Time$	Mem	$I, \lvert\Pi\rvert$	$Time$	Mem
EC$(32, 6, 16)$	71k / 193 / 129	3.55s	446MB	1 / 7	**1.15s**	281MB	2.34s	391MB	0 / 2	1.03s	**251MB**
EC$(64, 7, 16)$	549k / 385 / 257	70.5s	3704MB	1 / 131	9.07s	1725MB	16.79s	1812MB	0 / 2	**4.83s**	**1467MB**
EC$(64, 8, 16)$	1.1m / 769 / 513	-	MO	-	-	MO	**52.63s**	**3619MB**	0 / 2	-	MO
EC$(64, 8, 32)$	1.1m / 1025 / 513	-	MO	-	-	MO	**54.08s**	**3665MB**	0 / 2	-	MO
PETG(2)	3 / 52 / 3	0.08s	35MB	4 / 6	0.03s	30MB	0.07s	35MB	4 / 6	0.03s	**29MB**
PETG(3)	4 / 1514 / 4	**0.2s**	63MB	6 / 8	0.25s	74MB	0.22s	**62MB**	6 / 8	0.21s	64MB
PETG(4)	5 / 49k / 5	1.75s	316MB	8 / 10	8.16s	1080MB	**1.6s**	**311MB**	8 / 10	6.94s	939MB
VIR1(12)	14 / 4097 / 1	0.91s	159MB	15 / 30	1.69s	255MB	**0.35s**	**114MB**	2 / 4	1.53s	215MB
VIR1(13)	15 / 8193 / 1	1.47s	197MB	16 / 32	4.36s	601MB	**0.6s**	**178MB**	2 / 4	2.8s	402MB
VIR1(14)	16 / 16k / 1	3.09s	326MB	17 / 34	8.22s	992MB	**0.75s**	**241MB**	2 / 4	6.49s	816MB
VIR1(15)	17 / 32k / 1	4.47s	643MB	18 / 36	15.13s	2047MB	**1.05s**	**490MB**	2 / 4	9.67s	1361MB
VIR1(16)	18 / 65k / 1	8.65s	1015MB	19 / 38	41.28s	3785MB	**1.37s**	**839MB**	2 / 4	23.71s	2591MB
VIR1(17)	19 / 131k / 1	18.68s	1803MB	20 / 40	-	MO	**2.12s**	**1653MB**	2 / 4	62.24s	4309MB
VIR1(18)	20 / 262k / 1	38.68s	3079MB	21 / 42	-	MO	**3.35s**	**2878MB**	2 / 4	-	MO
VIR2(12)	13 / 4096 / 1	1.02s	151MB	19 / 34	0.81	154MB	0.68s	**122MB**	9 / 14	**0.57s**	133MB
VIR2(13)	14 / 8192 / 1	1.48s	190MB	20 . 36	1.13s	216MB	1.01s	**183MB**	9 / 14	1.01s	208MB
VIR2(14)	15 / 16k / 1	2.9s	315MB	21 / 38	2.33s	389MB	**1.94s**	**311MB**	9 / 14	2.09s	388MB
VIR2(15)	16 / 32k / 1	5s	631MB	22 / 40	6.29s	964MB	**2.12s**	**489MB**	9 / 14	4.69s	757MB
VIR2(16)	17 / 65k / 1	9.82s	949MB	23 / 42	7.55s	1468MB	**3.96s**	**897MB**	9 / 14	6.09s	1315MB
VIR2(17)	18 / 131k / 1	23.33s	1815MB	24 / 44	23.54s	3012MB	**8.16s**	**1676MB**	9 / 14	15.36s	2542MB
VIR2(18)	19 / 262k / 1	45.89s	3049MB	25 / 46	55.28s	4288MB	**20.3s**	**2875MB**	9 / 14	28.79s	3755MB

Table 3 Results for two-player games examples.

- **Leader election protocol.** The specification for the leader election protocol LE in addition to the safety property, requires a liveness property that a leader is elected with probability one.
- **Peterson's algorithm.** The specification for Peterson's algorithm PETP requires the liveness property that a requesting process is eventually granted access to the critical section with probability one.
- **Szymanski's algorithm** (SZYM)

  Model: The example is based on the Szymanski's algorithm [79] for mutual exclusion of n threads with a randomized scheduler. Threads were extended by a nondeterministic choice to wait, rather than request access to the critical section.

  Specification: As for PETP, the specification requires that a requesting process eventually enters the critical section with probability one.

Table 2 shows the experimental results for systems with liveness specification. We did not compare our results with the tool prototype from [65], which checks strong simulation, since the tool was tailored to safety specifications only. For all the examples, the assume-guarantee method scales better than the monolithic method both in time and memory.

Two-player games examples. We also experimented with our tool on several examples of games, where one of the players controls the choices of the system and the other player represents the environment.

- **Error-correcting device** (EC)

  Model: The example is based on [12] and models an error-correcting device that sends and receives data blocks over a communication channel. Notation EC(n, k, d) means that a data block consists of n bits and it encodes k bits of data; value d is the minimum Hamming distance between two distinct blocks. In the first component Player 2 chooses a message to be sent over the channel and is allowed to flip some bits in the block during the transmission. The second component restricts the number of bits that Player 2 can flip.

  Specification: The specification requires that every message is correctly decoded.
- **Peterson's algorithm** (PETG)

Model: The model is similar to the Peterson's algorithm [72] example for MDPs, with the following differences: (a) the system may choose to restart instead of entering the critical section; (b) instead of a randomized scheduler we consider an adversarial scheduler.

Specification: The specification requires only one process to be in the critical section at any time.

– **Virus attack** (VIR1)

Model: The example models a virus that attacks a computer system with n nodes (based on case study from PRISM [67]). Player 1 represents the virus and is trying to infect as many nodes of the network as possible. Player 2 represents the system and may recover an infected node to an uninfected state. VIR2 is a modified version of VIR1 with two special critical nodes in the network. Whenever both of the nodes are infected, the virus can overtake the system.

Specification: The specification for VIR1 requires that the virus has a strategy to avoid being completely erased, i.e., maintain at least one infected node in the network. The specification for VIR2 is like for VIR1, i.e., the virus can play such that at least one node in the network remains infected, but it additionally requires that even if the system cooperates with the virus, the system is designed in a way that the special nodes will never be infected at the same time.

The results for two-player game examples are shown in Table 3. Along with AGCS and MONCS for assume-guarantee and monolithic combined simulation, we also consider AGAS and MONAS for assume-guarantee and monolithic alternating simulation, as for properties in 1-ATL it suffices to consider only alternating simulation. For all the examples, the assume-guarantee algorithms scale better than the monolithic ones. Combined simulation is finer than alternating simulation and therefore combined simulation may require more CEGAR iterations.

Concluding remarks. In this work we considered compositional analysis of MDPs for qualitative properties and presented a CEGAR approach. Our algorithms are discrete graph algorithms. An interesting direction of future work would be to consider symbolic approaches to the problem.

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A Technical appendix

We start with an example that shows that also for alternating games combined simulation is finer that the intersection of simulation and alternating-simulation relation.

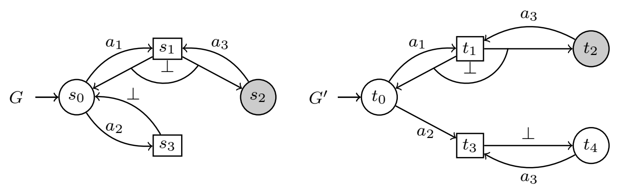

Fig. 7 Games G, G' such that $G \leqslant_{\mathcal{S}} G'$ and $G \leqslant_{\mathcal{A}} G'$, but $G \nleqslant_{\mathcal{C}} G'$.

Example 7 Figure 7 shows two alternating games G, G', where the circular states belong to Player 1 and the rectangular states belong to Player 2, white nodes are labeled by proposition p and gray nodes by proposition q. The largest simulation and alternating-simulation relations between G and G' are: $\mathcal{S}_{\max} = \{(s_0,t_0),(s_1,t_1),(s_2,t_2),(s_3,t_1)\}, \mathcal{A}_{\max} = \{(s_0,t_0),(s_0,t_4),(s_2,t_2),(s_3,t_3),(s_1,t_3),(s_1,t_1)\}$. Formula $\langle\langle 1\rangle\rangle(\Box(p \wedge \langle\langle 1,2\rangle\rangle(\mathsf{true}\,\mathcal{U}\,q)))$ is satisfied in state s_0, but not in state t_0, hence $(s_0,t_0) \notin \mathcal{C}_{\max}$. $\Box$

We now present detailed proofs of Lemma 1 and Theorem 2 in the context of alternating games.

Lemma 8 *Given two alternating games G and G', let $\mathcal{C}_{\max}$ be the combined simulation. For all $(s,s') \in \mathcal{C}_{\max}$ the following assertions hold:*

1. *For all Player 1 strategies σ in G, there exists a Player 1 strategy σ' in G' such that for every play $\omega' \in \mathsf{Plays}(s',\sigma')$ there exists a play $\omega \in \mathsf{Plays}(s,\sigma)$ such that $\omega \leqslant_{\mathcal{C}} \omega'$.*
2. *For all pairs of strategies σ and θ in G, there exists a pair of strategies σ' and θ' in G' such that $\mathsf{Play}(s,\sigma,\theta) \leqslant_{\mathcal{C}} \mathsf{Play}(s',\sigma',\theta')$,*

Proof Assertion 1. As the states of Player 1 and Player 2 are distinguished by the turn atomic proposition, it follows from the fact that $(s,s') \in \mathcal{C}_{\max}$, that either (i) $s \in S_1$ and $s' \in S_1'$ or (ii) $s \in S_2$ and $s' \in S_2'$.

For the first case (i) we consider a winning strategy $\sigma^{\mathcal{C}}$ in $G^{\mathcal{C}}$ such that for all $(s,s') \in \mathcal{C}_{\max}$ and against all strategies $\theta^{\mathcal{C}}$ we have $\mathsf{Play}((s,s'),\sigma^{\mathcal{C}},\theta^{\mathcal{C}}) \in [\![\Box(\neg p)]\!]_{G^{\mathcal{C}}}$. Given the Player 1 strategy σ in G we construct σ' in G' using the strategy $\sigma^{\mathcal{C}}$. Let h be an arbitrary history in $G^{\mathcal{C}}$ that visits only states of type $(S \times S')$ that are in $\mathcal{C}_{\max}$ and ends in (s,s'). Consider a history $w \cdot s$ in G and $w' \cdot s'$ in G'. Let $\sigma(w \cdot s) = a$, we define $\sigma'(w' \cdot s')$ as action $a' = \sigma^{\mathcal{C}}(h \cdot ((s,s'),\mathsf{Alt},2) \cdot ((s,s'),\mathsf{Alt},a,2))$, i.e., action a' corresponds to the choice of the proponents winning strategy $\sigma^{\mathcal{C}}$ in response to the adversarial choice of checking step-wise alternating-simulation followed by action a in G. As both s and s' are Player-1 states we have that $|\delta(s,a)| = 1$ and $|\delta'(s',a')| = 1$. Let (t,t') be the unique state reached in 2 steps from $((s,s'),\mathsf{Alt},a,a',2)$ in $G^{\mathcal{C}}$. Assume towards contradiction that $\mathcal{L}^{\mathcal{C}}((t,t')) = \{p\}$, then there exists a strategy for adversary that reaches a loosing state while the proponent plays a winning strategy $\sigma^{\mathcal{C}}$ and the contradiction follows. For the second case (ii) we have that states s and s' belong to Player 2, and there is a single action available for σ'.

Assertion 2 The proof is similar to the first assertion, and instead of using the step-wise alternating-simulation gadget for strategy construction (of the first item) we use the step-wise simulation gadget from $G^{\mathcal{C}}$ to construct the strategy pairs.

Theorem 7 *For all alternating games G and G' we have* $\mathcal{C}_{\max} = \preccurlyeq_C^* = \preccurlyeq_C$.

Proof First implication. We first prove the implication $\mathcal{C}_{\max} \subseteq \preccurlyeq_C^*$. We will show the following assertions:

– For all states s and s' such that $(s, s') \in \mathcal{C}_{\max}$, we have that every C-ATL* state formula satisfied in s is also satisfied in s'.
– For all plays ω and ω' such that $\omega \leqslant_C \omega'$, we have that every C-ATL* path formula satisfied in ω is also satisfied in ω'.

We will prove the theorem by induction on the structure of the formulas. The interesting cases for the induction step are formulas $\langle\langle 1\rangle\rangle(\varphi)$ and $\langle\langle 1,2\rangle\rangle(\varphi)$, where φ are path formulas.

– Assume $s \models \langle\langle 1\rangle\rangle(\varphi)$ and $(s, s') \in \mathcal{C}_{\max}$. It follows that there exists a strategy $\sigma \in \Sigma$ that ensures the path formula φ from state s against any strategy $\theta \in \Theta$. We want to show that $s' \models \langle\langle 1\rangle\rangle(\varphi)$. By Lemma 8(item 1) we have that there exists a strategy σ' for Player 1 from s' such that for every play $\omega' \in \mathsf{Plays}(s', \sigma')$ there exists a play $\omega \in \mathsf{Plays}(s, \sigma)$ such that $\omega \leqslant_C \omega'$. By inductive hypothesis we have that $s' \models \langle\langle 1\rangle\rangle(\varphi)$.
– Assume $s \models \langle\langle 1,2\rangle\rangle(\varphi)$ and $(s, s') \in \mathcal{C}_{\max}$. It follows that there exist strategies $\sigma \in \Sigma, \theta \in \Theta$ that ensure the path formula φ from state s. By Lemma 8 (item 2) we have that there exist strategies σ' and θ' such that the two plays $\omega' = \mathsf{Play}(s', \sigma', \theta')$ and $\omega = \mathsf{Play}(s, \sigma, \theta)$ satisfy $\omega \leqslant_C \omega'$. By inductive hypothesis we have that $s' \models \langle\langle 1,2\rangle\rangle(\varphi)$.
– Consider a path formula φ. If $\omega \leqslant_C \omega'$, then by inductive hypothesis for every subformula φ' of φ we have that if $\omega \models \varphi'$ then $\omega' \models \varphi'$. It follows that if $\omega \models \varphi$ then $\omega' \models \varphi$.

Second implication. It remains to prove the second implication $\preccurlyeq_C^* \subseteq \preccurlyeq_C \subseteq \mathcal{C}_{\max}$. We prove that from the assumption that $(s, s') \notin \mathcal{C}_{\max}$ we can construct a C-ATL formula φ such that $s \models \varphi$ and $s' \not\models \varphi$. We refer to the formula φ as a distinguishing formula. Assume that given states s and s' we have that $(s, s') \notin \mathcal{C}_{\max}$, then there exists a winning strategy in the corresponding combined-simulation game for the adversary from state (s, s'), i.e., there exists a strategy $\theta^{\mathcal{C}}$ such that against all strategies $\sigma^{\mathcal{C}}$ we have $\mathsf{Play}((s, s'), \sigma^{\mathcal{C}}, \theta^{\mathcal{C}})$ reaches a state labeled by p. As memoryless strategies are sufficient for both players in $G^{\mathcal{C}}$ [55], there also exists a bound $i \in \mathbb{N}$, such that the proponent fails to match the choice of the adversary in at most i turns. We construct the C-ATL formula φ inductively:

Base case: Assume $(s, s') \notin \mathcal{C}_{\max}$ and let 0 be the number of turns the adversary needs to play in order to win. It follows that (s, s') is a winning state for the adversary, i.e., $\mathcal{L}^{\mathcal{C}}((s, s')) = \{p\}$. It follows that $\mathcal{L}(s) \neq \mathcal{L}'(s')$. There are two options: (i) there exists an atomic proposition $q \in \mathsf{AP}$ that is true in s and not true in s' and distinguishes the two states, or (ii) there exists an atomic proposition $q \in \mathsf{AP}$ that is not true in s and true in s', in that case the formula $\neg q$ distinguishes the two states.

Induction step: Assume $(s, s') \notin \mathcal{C}_{\max}$ and let $n + 1$ be the number of turns the adversary needs to play in order to win. As the states of Player 1 and Player 2 are distinguished by the turn atomic proposition, it follows that either (i) $s \in S_1$ and $s' \in S_1'$ or (ii) $s \in S_2$ and $s' \in S_2'$. Otherwise the adversary could win in 0 turns from (s, s').
We first consider case (i), i.e., $(s, s') \in S_1 \times S_1'$. The adversary can choose whether to verify (1) step-wise alternating-simulation (Alt) or (2) step-wise simulation (Sim). After

that he chooses an action a to be played according the adversarial strategy $\theta^{\mathcal{C}}$ in state (s, s'), such that no matter what the proponent plays, the adversary will win in n turns. We consider two cases: (1) the adversary checks for step-wise alternating-simulation relation (Alt), or (2) the adversary checks for step-wise simulation relation (Sim). For case (1) we have that there exists an action a for the adversary such that for all actions a' of the proponent the adversary can win in n turns from the unique successor (t, t') of (s, s') given Alt and a was played by the adversary and a' by the proponent. From the induction hypothesis there exists a C-ATL formula φ_n such that $t \models \varphi_n$ and $t' \not\models \varphi_n$. We define the formula φ_{n+1} that distinguishes states s and s' as $\langle\langle 1 \rangle\rangle(\bigcirc\varphi_n)$. For case (2), where the adversary plays Sim the proof is exactly the same, as step-wise simulation turn from Player 1 states coincides with step-wise alternating-simulation turn.
Next we first consider case (ii), i.e., $(s, s') \in S_2 \times S'_2$. The adversary can choose whether to verify (1) step-wise alternating-simulation (Alt) or(2) step-wise simulation (Sim). We start with first case (1): there is a unique action a available to the adversary from state $((s, s'), \mathsf{Alt}, 2)$ and similarly a unique action a' for the proponent from $((s, s'), a, \mathsf{Alt}, 1)$. The adversary chooses an action t' from the $((s, s'), a, a', \mathsf{Alt}, 2)$ according to the winning strategy and the proponent chooses some action t_i from a set of available successor $(t_1, t_2, \ldots, t_m)$. As the adversary follows a winning strategy $\theta^{\mathcal{C}}$ we have that it wins from all states (t_i, t') for $1 \leq i \leq m$ in at most n turns. From the induction hypothesis there exist C-ATL formulas φ_n^i such that $t_i \models \varphi_n^i$ and $t' \not\models \varphi_n^i$. We define the formula φ_{n+1} that distinguishes states s and s' as $\langle\langle 1 \rangle\rangle(\bigcirc(\bigvee_{1 \leq i \leq m} \varphi_n^i)$. For case (2) where the adversary verifies the step-wise simulation step, the proof is analogous. The formula that distinguishes states s and s' is $\langle\langle 1, 2 \rangle\rangle((\bigcirc \bigvee_{1 \leq i \leq m} \varphi_n^i))$.

The desired result follows. $\square$