CurveUps: Supplemental material

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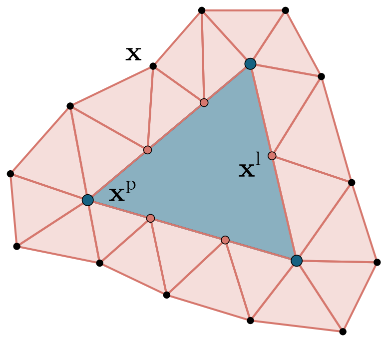

Figure 1: Notation for vertices and associated forces. A pin of a tile is depicted in green, and the simulation mesh of the elastic membrane in red. Pin vertices denoted $\mathbf{x}^{\mathrm{p}}$ are depicted in green (there are 3 of them). Membrane triangulation vertices belonging to the pin $\mathbf{x}^{\mathrm{l}}$ are depicted in red or green (there are 8 of them). All the depicted vertices are membrane triangulation vertices $\mathbf{x}$ (there are 21 of them).

1 Derivatives of the elastic membrane forces

We aim to compute the forces generated by the elastic membrane per pin vertex and their derivatives with respect to the model parameters $\mathbf{P}$. The total number of pin vertices is $6M$, where M is the number of tiles, since each tile has two pins, and each pin has three vertices. On the other hand, we need to use a high enough resolution of the membrane triangulation. This results in a higher number of the membrane vertices fixed along the pin boundaries. We denote the pin vertices and associated forces by the upper index 'p' and the membrane triangulation vertices belonging to the pins and associated forces by the upper index 'l'. We leave the vertices of the membrane triangulation and their associated forces without upper index. See Figure **??** for notation.

1.1 Pin force definition

The total force acting on the i-th pin vertex is

$$\mathbf{f}_i^{\mathrm{p}} = \sum_{j \in \mathrm{pin}(i)} w_{ij} \mathbf{f}_j^{\mathrm{l}},$$

where w_{ij} is the barycentric coordinate (based on the three pin vertices) of the membrane vertex $\mathbf{x}_j^{\mathrm{l}}$ with respect to the pin vertex $\mathbf{x}_i^{\mathrm{p}}$, $\mathbf{f}_j^{\mathrm{l}}$ is the force acting on the membrane vertex, and $\mathrm{pin}(i)$ denotes the set of indices of the membrane vertices attached to the pin containing pin vertex i. We can write down the barycentric coordinates as follows:

$$\mathbf{x}_j^{\mathrm{l}} = \sum_{k=1}^{3} w_{kj} \mathbf{x}_k^{\mathrm{p}}.$$

We also note that in flat configuration the membrane is uniformly stretched by τ in all directions:

$$\mathbf{X}_j^{\mathrm{l}} = \sum_{k=1}^{3} \frac{w_{kj}}{\tau} \mathbf{X}_k^{\mathrm{p}}.$$

Using Constant Strain Triangles we calculate the internal forces as follows:

$$\mathbf{f}_j = -\sum_{e \in \mathcal{F}_j} \frac{\partial \Psi^e}{\partial \mathbf{x}_j} V_e = -\sum_{e \in \mathcal{F}_j} \left(\frac{\partial \Psi^e}{\partial \lambda_1^e} \frac{\partial \lambda_1^e}{\partial \mathbf{x}_i} + \frac{\partial \Psi^e}{\partial \lambda_2^e} \frac{\partial \lambda_2^e}{\partial \mathbf{x}_i} \right) h A_e, \tag{1}$$

where $\mathcal{F}_i$ is the set of faces incident to vertex i, h is the thickness of the membrane in rest configuration, A_e is the area of element e in rest configuration, $V_e = hA_e$ is the volume of element e in rest configuration.

1.2 Goal derivatives

The derivative we aim to compute is $\frac{\partial \mathbf{f}^{\mathrm{p}}}{\partial \mathbf{P}}$, where $\mathbf{P}$ represents all the CurveUp parameters: vertex positions in the actuated configuration, tile mappings to flat configuration, and pin parameters. We apply the chain rule knowing that $\mathbf{f}^{\mathrm{p}} = \mathbf{f}^{\mathrm{p}}(\mathbf{x}^{\mathrm{p}}, \mathbf{X}^{\mathrm{p}})$:

$$\frac{\partial \mathbf{f}^{\mathrm{p}}}{\partial \mathbf{P}} = \frac{\partial \mathbf{f}^{\mathrm{p}}}{\partial \mathbf{x}^{\mathrm{p}}}\frac{\partial \mathbf{x}^{\mathrm{p}}}{\partial \mathbf{P}} + \frac{\partial \mathbf{f}^{\mathrm{p}}}{\partial \mathbf{X}^{\mathrm{p}}}\frac{\partial \mathbf{X}^{\mathrm{p}}}{\partial \mathbf{P}}, \tag{2}$$

where the derivatives $\frac{\partial \mathbf{x}^{\mathrm{p}}}{\partial \mathbf{P}}$ and $\frac{\partial \mathbf{X}^{\mathrm{p}}}{\partial \mathbf{P}}$ are computed using algorithmic differentiation, and for the rest we present analytic expressions below.

1.3 Pin force derivatives wrt pin positions

1.3.1 Pin force derivatives wrt pin positions in the deformed configuration

$$\frac{\partial \mathbf{f}_i^{\mathrm{p}}}{\partial \mathbf{x}_k^{\mathrm{p}}} = \sum_{j \in \mathrm{pin}(i)} w_{ij} \frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{x}_k^{\mathrm{p}}}.$$

$$\frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{x}_k^{\mathrm{p}}} = \sum_{n \in \mathrm{ring}(j)} \frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{x}_n^{\mathrm{l}}}\frac{\partial \mathbf{x}_n^{\mathrm{l}}}{\partial \mathbf{x}_k^{\mathrm{p}}} = \sum_{n \in \mathrm{ring}(j)} w_{kn} \frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{x}_n^{\mathrm{l}}}.$$

Note that if $\mathbf{x}_n^{\mathrm{l}}$ does not belong to the pin of $\mathbf{x}_k^{\mathrm{p}}$, $\frac{\partial \mathbf{x}_n^{\mathrm{l}}}{\partial \mathbf{x}_k^{\mathrm{p}}} = 0$.

We derive $\frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{x}_n^{\mathrm{l}}}$ in the subsection **??**.

1.3.2 Pin force derivatives wrt pin positions in the rest configuration

$$\frac{\partial \mathbf{f}_i^{\mathrm{p}}}{\partial \mathbf{X}_k^{\mathrm{p}}} = \sum_{j \in \mathrm{pin}(i)} w_{ij} \frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{X}_k^{\mathrm{p}}}.$$

$$\frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{X}_k^{\mathrm{p}}} = \sum_{n \in \mathrm{ring}(j)} \frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{X}_n^{\mathrm{l}}}\frac{\partial \mathbf{X}_n^{\mathrm{l}}}{\partial \mathbf{X}_k^{\mathrm{p}}} = \sum_{n \in \mathrm{ring}(j)} \frac{w_{kn}}{\tau} \frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{X}_n^{\mathrm{l}}}.$$

We derive $\frac{\partial \mathbf{f}_j^{\mathrm{l}}}{\partial \mathbf{X}_n^{\mathrm{l}}}$ in the subsection **??**.

1.4 Boundary membrane vertex forces wrt all membrane vertex positions

1.4.1 Boundary membrane vertex forces wrt all membrane vertex positions in the deformed configuration

Denote the stiffness matrix of the membrane as

$$K = \begin{pmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{pmatrix},$$

where indices a denote the boundary fixed vertices and b denote free vertices. Then the linearized quasistatic balance equation takes the form

$$\begin{pmatrix} \mathbf{f}_a \\ \mathbf{f}_b \end{pmatrix} = \begin{pmatrix} K_{aa} & K_{ab} \\ K_{ba} & K_{bb} \end{pmatrix} \begin{pmatrix} \mathbf{u}_a \\ \mathbf{u}_b \end{pmatrix},$$

where $\mathbf{f}_b = 0$, and thus $\mathbf{u}_b = -K_{bb}^{-1} K_{ba} \mathbf{u}_a$. Applying it back to the upper rows of the equation:

$$\mathbf{f}_a = K_{aa}\mathbf{u}_a - K_{ab}K_{bb}^{-1}K_{ba}\mathbf{u}_a.$$

Finally, we write down the derivative matrix as the Schur complement of the block K_{bb} of the matrix K:

$$\frac{\partial \mathbf{f}^{\mathrm{l}}}{\partial \mathbf{x}^{\mathrm{l}}} = K_{aa} - K_{ab}K_{bb}^{-1}K_{ba}.$$

1.4.2 Boundary membrane vertex forces wrt all membrane vertex positions in the rest configuration

Similarly to the previous subsection, for the displacements $\mathbf{U}$ in rest configuration and the Jacobian M, we write down teh following equation:

$$\begin{pmatrix}\mathbf{f}_a\\ \mathbf{f}_b\end{pmatrix} = \begin{pmatrix}M_a\\ M_b\end{pmatrix}\mathbf{U} + \begin{pmatrix}K_{ab}\\ K_{bb}\end{pmatrix}\mathbf{u}_b,$$

where $\mathbf{f}_b = 0$ which gives $\mathbf{u}_b = -K_{bb}^{-1}M_b\mathbf{U}$, and

$$\mathbf{f}_a = M_a\mathbf{U} - K_{ab}K_{bb}^{-1}M_b\mathbf{U}.$$

Consequently, we obtain the following expression:

$$\frac{\partial \mathbf{f}^{\mathrm{l}}}{\partial \mathbf{X}^{\mathrm{l}}} = M_a - K_{ab}K_{bb}^{-1}M_b.$$

1.5 All membrane vertex derivatives

1.5.1 All membrane vertex derivatives wrt the deformed configuration

The stiffness matrix is defined as $K = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$. We obtain the derivatives of the terms in Equation ?? with respect to the deformed configuration.

By definition, the deformation gradient $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$. Assuming Constant Strain Triangles, we will use the following shape functions: $\xi = J^{-1}\mathbf{D}$, where

$$J = \begin{vmatrix}1 & \mathbf{X}_1^{(1)} & \mathbf{X}_2^{(1)}\\ 1 & \mathbf{X}_1^{(2)} & \mathbf{X}_2^{(2)}\\ 1 & \mathbf{X}_1^{(3)} & \mathbf{X}_2^{(3)}\end{vmatrix},$$

$$\mathbf{D} = \begin{pmatrix}\mathbf{X}_2^{(2)} - \mathbf{X}_2^{(3)} & \mathbf{X}_1^{(3)} - \mathbf{X}_1^{(2)}\\ \mathbf{X}_2^{(3)} - \mathbf{X}_2^{(1)} & \mathbf{X}_1^{(1)} - \mathbf{X}_1^{(3)}\\ \mathbf{X}_2^{(1)} - \mathbf{X}_2^{(2)} & \mathbf{X}_1^{(2)} - \mathbf{X}_1^{(1)}\end{pmatrix} = \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}_1^{(1)} & \mathbf{X}_2^{(1)}\\ \mathbf{X}_1^{(2)} & \mathbf{X}_2^{(2)}\\ \mathbf{X}_1^{(3)} & \mathbf{X}_2^{(3)}\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

Then $\mathbf{F} = \mathbf{t}\xi$, where $\mathbf{t} = \begin{pmatrix}\mathbf{x}^{(1)} & \mathbf{x}^{(2)} & \mathbf{x}^{(3)}\end{pmatrix}$.

$$\mathbf{F} = F_{ij} = t_{ip}\xi_{pj}.$$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{t}} = \frac{F_{ij}}{\partial t_{kl}} = \delta_{ik}\xi_{lj}, \qquad \frac{\partial \mathbf{F}^T}{\partial \mathbf{t}} = \frac{F_{ij}^T}{\partial t_{kl}} = \delta_{jk}\xi_{li}.$$

$$\frac{\partial \mathbf{C}}{\partial \mathbf{t}} = \frac{\partial \mathbf{F}^T}{\partial \mathbf{t}}\mathbf{F} + \mathbf{F}^T\frac{\partial \mathbf{F}}{\partial \mathbf{t}}.$$

$$\frac{\partial \mathbf{C}}{\partial \mathbf{t}} = \delta_{qk}\xi_{li}t_{qp}\xi_{pj} + t_{qp}\xi_{pi}\delta_{qk}\xi_{lj}.$$

$$\frac{\partial \mathbf{C}}{\partial \mathbf{t}} = C'_{ijkl} = t_{kp}\xi_{pj}\xi_{li} + t_{kp}\xi_{pi}\xi_{lj}.$$

We derive:

$$\frac{\partial \lambda_1}{\partial \mathbf{t}} = N_1^T\frac{\partial \mathbf{C}}{\partial \mathbf{t}}N_1.$$

$$\frac{\partial \lambda_1}{\partial \mathbf{x}^{(a)}} = N_1^T\frac{\partial \mathbf{C}}{\partial \mathbf{t}}\bigg|_{l=a}N_1.$$

The second derivative of the Cauchy Green tensor is calculated as follows:

$$\frac{\partial^2 \mathbf{C}}{\partial \mathbf{t}^2} = \frac{\partial C'_{ijkl}}{\partial t_{mn}} = \delta_{km}\big(\xi_{nj}\xi_{li} + \xi_{ni}\xi_{lj}\big).$$

$$\frac{\partial^2 \mathbf{C}}{\partial \mathbf{x}^{(b)}\partial \mathbf{x}^{(a)}} = \frac{\partial C'_{ijkl}}{\partial t_{mn}}\bigg|_{l=a;n=b}.$$

We derive for $\mathbf{x}_i = \mathbf{x}^{(a)}$ and $\mathbf{x}_j = \mathbf{x}^{(b)}$:

$$\frac{\partial^2 \lambda_1}{\partial \mathbf{x}_j\partial \mathbf{x}_i} = \mathbf{N}_1^T\frac{\partial^2 \mathbf{C}}{\partial \mathbf{x}_j\partial \mathbf{x}_i}\mathbf{N}_1 + \frac{1}{\lambda_1 - \lambda_2}\Big(\mathbf{N}_2^T\frac{\partial \mathbf{C}}{\partial \mathbf{x}_j}\mathbf{N}_1\mathbf{N}_2^T\frac{\partial \mathbf{C}}{\partial \mathbf{x}_i}\mathbf{N}_1 + \mathbf{N}_1^T\frac{\partial \mathbf{C}}{\partial \mathbf{x}_i}\mathbf{N}_2\mathbf{N}_1^T\frac{\partial \mathbf{C}}{\partial \mathbf{x}_j}\mathbf{N}_2\Big). \tag{3}$$

$$\frac{\partial^2 \lambda_1}{\partial \mathbf{x}_j \partial \mathbf{x}_i} = \mathbf{N}_1^T \frac{\partial^2 \mathbf{C}}{\partial \mathbf{x}_j \partial \mathbf{x}_i} \mathbf{N}_1 + \frac{2}{\lambda_1 - \lambda_2} \mathbf{N}_2^T \frac{\partial \mathbf{C}}{\partial \mathbf{x}_j} \mathbf{N}_1 \mathbf{N}_2^T \frac{\partial \mathbf{C}}{\partial \mathbf{x}_i} \mathbf{N}_1. \tag{4}$$

It is easy to see that

$$\frac{\partial \Psi}{\partial \lambda_1} = \kappa \left(1 - \frac{1}{\lambda_1^2 \lambda_2} \right).$$

To compute the derivative of the internal force $\frac{\partial \mathbf{f}_i}{\partial \mathbf{x}_j}$:

$$\begin{aligned} \frac{\partial}{\partial \mathbf{x}_j} \left(\frac{\partial \Psi}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial \mathbf{x}_i} \right) &= \left(\frac{\partial}{\partial \mathbf{x}_j} \frac{\partial \Psi}{\partial \lambda_1} \right)^T \frac{\partial \lambda_1}{\partial \mathbf{x}_i} + \frac{\partial \Psi}{\partial \lambda_1} \left(\frac{\partial}{\partial \mathbf{x}_j} \frac{\partial \lambda_1}{\partial \mathbf{x}_i} \right) \\ &= \kappa \left(\frac{2}{\lambda_1^3 \lambda_2} \frac{\partial \lambda_1}{\partial \mathbf{x}_j} + \frac{1}{\lambda_1^2 \lambda_2^2} \frac{\partial \lambda_2}{\partial \mathbf{x}_j} \right)^T \frac{\partial \lambda_1}{\partial \mathbf{x}_i} + \frac{\partial \Psi}{\partial \lambda_1} \frac{\partial^2 \lambda_1}{\partial \mathbf{x}_j \partial \mathbf{x}_i} \end{aligned} \tag{5}$$

We can use the same formulas to compute λ_2 by swapping the indices and compute the derivatives of the both terms in Equation **??**.

1.5.2 All membrane vertex derivatives wrt the rest configuration

The Jacobian is defined as $M = \frac{\partial \mathbf{f}}{\partial \mathbf{X}}$. We obtain the derivatives of the terms in Equation **??** with respect to the rest configuration.

$$\frac{\partial \mathbf{f}_j}{\partial \mathbf{X}_n} = - \sum_{e \in \mathcal{F}_j} h \left(\frac{\partial^2 \Psi^e}{\partial \mathbf{X}_n \partial \mathbf{x}_j} A_e + \frac{\partial \Psi^e}{\partial \mathbf{x}_j} \frac{\partial A_e}{\partial \mathbf{X}_n} \right).$$

The rest surface area A_e of the element e with vertex coordinates $\mathbf{X}^{(1)}$, $\mathbf{X}^{(2)}$, $\mathbf{X}^{(3)}$ can be expressed as:

$$A_e = \frac{1}{2} \left[\left(\mathbf{X}_1^{(2)} - \mathbf{X}_1^{(1)} \right) \left(\mathbf{X}_2^{(3)} - \mathbf{X}_2^{(1)} \right) - \left(\mathbf{X}_2^{(2)} - \mathbf{X}_2^{(1)} \right) \left(\mathbf{X}_1^{(3)} - \mathbf{X}_1^{(1)} \right) \right].$$

Lets say $\mathbf{X}_n^1 = \mathbf{X}_a^{(b)}$, then

$$\frac{\partial A_e}{\partial \mathbf{X}_n^1} = \frac{\partial A_e}{\partial \mathbf{X}_a^{(b)}} = \frac{(-1)^{a+1}}{2} \left(\mathbf{X}_{a+1}^{(b+1)} - \mathbf{X}_{a+1}^{(b+2)} \right).$$

Note that when we sum indexes we always assume modulo maximal index value (either 2 or 3).
Finally, we obtain the following derivative:

$$\frac{\partial^2 \Psi^e}{\partial \mathbf{X}_n \partial \mathbf{x}_j}.$$

We follow the same procedure as in the previous section. Lets consider derivatives with respect to the rest configuration. We denote all triangle vertices in the rest configuration as $\mathbf{T} = \begin{pmatrix} \mathbf{X}^{(1)} & \mathbf{X}^{(2)} & \mathbf{X}^{(3)} \end{pmatrix}$.

$$\frac{\partial \mathbf{F}}{\partial \mathbf{T}} = \mathbf{t} \frac{\partial \xi}{\partial \mathbf{T}} = \mathbf{t} J^{-2} \left(- \frac{\partial J}{\partial \mathbf{T}} \mathbf{D} + J \frac{\partial \mathbf{D}}{\partial \mathbf{T}} \right).$$

$$\frac{\partial J}{\partial T_{kl}} = (-1)^{k+1} \left(\mathbf{X}_{k+1}^{(l+1)} - \mathbf{X}_{k+1}^{(l+2)} \right).$$

$$\frac{\partial D_{ij}}{\partial T_{kl}} = \begin{cases} 1 & \text{if} \quad i = l + k, \quad j = k + 1, \\ -1 & \text{if} \quad i = l - k, \quad j = k + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we can calculate

$$\frac{\partial \mathbf{C}}{\partial \mathbf{T}} = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{T}} \right)^T \mathbf{F} + \mathbf{F}^T \frac{\partial \mathbf{F}}{\partial \mathbf{T}}.$$

$$\frac{\partial C_{ij}}{\partial T_{kl}} = \frac{\partial F_{pi}}{\partial T_{kl}} F_{pj} + F_{pi} \frac{\partial F_{pj}}{\partial T_{kl}}.$$

$$\frac{\partial \lambda_1}{\partial \mathbf{T}} = N_1^T \frac{\partial \mathbf{C}}{\partial \mathbf{T}} N_1.$$

The second derivative:

$$\frac{\partial^2 \mathbf{C}}{\partial \mathbf{T} \partial \mathbf{t}} = \frac{\partial^2 \mathbf{F}^T}{\partial \mathbf{T} \partial \mathbf{t}} \mathbf{F} + \frac{\partial \mathbf{F}^T}{\partial \mathbf{t}} \frac{\partial \mathbf{F}}{\partial \mathbf{T}} + \frac{\partial \mathbf{F}^T}{\partial \mathbf{T}} \frac{\partial \mathbf{F}}{\partial \mathbf{t}} + \mathbf{F}^T \frac{\partial^2 \mathbf{F}}{\partial \mathbf{T} \partial \mathbf{t}}.$$

$$\frac{\partial^2 \mathbf{C}}{\partial \mathbf{T} \partial \mathbf{t}} = \frac{\partial^2 \mathbf{F}^T}{\partial \mathbf{T} \partial \mathbf{t}} \mathbf{F} + \mathbf{F}^T \frac{\partial^2 \mathbf{F}}{\partial \mathbf{T} \partial \mathbf{t}} + 2 \frac{\partial \mathbf{F}^T}{\partial \mathbf{t}} \frac{\partial \mathbf{F}}{\partial \mathbf{T}}.$$

$$\frac{\partial^2 \mathbf{F}}{\partial \mathbf{T} \partial \mathbf{t}} = \frac{\partial \mathbf{t}}{\partial \mathbf{t}} J^{-2} \Big(- \frac{\partial J}{\partial \mathbf{T}} \mathbf{D} + J \frac{\partial \mathbf{D}}{\partial \mathbf{T}} \Big).$$

$$\frac{\partial^2 \lambda_1}{\partial \mathbf{X}_j \partial \mathbf{x}_i} = \mathbf{N}_1^T \frac{\partial^2 \mathbf{C}}{\partial \mathbf{X}_j \partial \mathbf{x}_i} \mathbf{N}_1 + \frac{1}{\lambda_1 - \lambda_2} \Big(\mathbf{N}_2^T \frac{\partial \mathbf{C}}{\partial \mathbf{X}_j} \mathbf{N}_1 \mathbf{N}_2^T \frac{\partial \mathbf{C}}{\partial \mathbf{x}_i} \mathbf{N}_1 + \mathbf{N}_1^T \frac{\partial \mathbf{C}}{\partial \mathbf{x}_i} \mathbf{N}_2 \mathbf{N}_1^T \frac{\partial \mathbf{C}}{\partial \mathbf{X}_j} \mathbf{N}_2 \Big).$$

$$\frac{\partial^2 \lambda_1}{\partial \mathbf{X}_j \partial \mathbf{x}_i} = \mathbf{N}_1^T \frac{\partial^2 \mathbf{C}}{\partial \mathbf{X}_j \partial \mathbf{x}_i} \mathbf{N}_1 + \frac{2}{\lambda_1 - \lambda_2} \mathbf{N}_2^T \frac{\partial \mathbf{C}}{\partial \mathbf{X}_j} \mathbf{N}_1 \mathbf{N}_2^T \frac{\partial \mathbf{C}}{\partial \mathbf{x}_i} \mathbf{N}_1.$$

$$\begin{aligned} \frac{\partial}{\partial \mathbf{X}_j} \left(\frac{\partial \Psi}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial \mathbf{x}_i} \right) &= \left(\frac{\partial}{\partial \mathbf{X}_j} \frac{\partial \Psi}{\partial \lambda_1} \right)^T \frac{\partial \lambda_1}{\partial \mathbf{x}_i} + \frac{\partial \Psi}{\partial \lambda_1} \left(\frac{\partial}{\partial \mathbf{X}_j} \frac{\partial \lambda_1}{\partial \mathbf{x}_i} \right) \\ &= \kappa \left(\frac{2}{\lambda_1^3 \lambda_2} \frac{\partial \lambda_1}{\partial \mathbf{X}_j} + \frac{1}{\lambda_1^2 \lambda_2^2} \frac{\partial \lambda_2}{\partial \mathbf{X}_j} \right)^T \frac{\partial \lambda_1}{\partial \mathbf{x}_i} + \kappa \left(1 - \frac{1}{\lambda_1^2 \lambda_2} \right) \frac{\partial^2 \lambda_1}{\partial \mathbf{X}_j \partial \mathbf{x}_i}. \end{aligned} \tag{6}$$

2 Derivatives of the contact forces

Denote by $\mathbf{f}^{\mathrm{c}}$ the concatenation of contact forces between all actuating pairs of tiles (four per pair), $\mathbf{b}$ is the concatenation of total elastic forces and torques per each tile, and $\mathbf{A}$ a matrix built on tile geometry, such that $\mathbf{A}\mathbf{f}^{\mathrm{c}} = \mathbf{b}$ is true for the balanced configuration (expresses force and torque balance for all tiles). Note that we exclude from the system of equations one tile, since all elastic forces and torques sum up to zero, which would result in zero determinant of $\mathbf{A}$. The resulting system of equations is underdetermined and we find the least squares solution as follows:

$$\mathbf{f}^{\mathrm{c}} = \mathbf{A}^{\mathrm{T}} \big(\mathbf{A}\mathbf{A}^{\mathrm{T}} \big)^{-1} \mathbf{b}.$$

We compute the derivatives of the contact forces with respect to the geometric variables p_i:

$$\frac{\partial \mathbf{f}^{\mathrm{c}}}{\partial p_i} = \frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial p_i} \big(\mathbf{A}\mathbf{A}^{\mathrm{T}} \big)^{-1} \mathbf{b} + \mathbf{A}^{\mathrm{T}} \big(\mathbf{A}\mathbf{A}^{\mathrm{T}} \big)^{-1} \left(\frac{\partial \mathbf{b}}{\partial p_i} - \frac{\partial \big(\mathbf{A}\mathbf{A}^{\mathrm{T}} \big)}{\partial p_i} \big(\mathbf{A}\mathbf{A}^{\mathrm{T}} \big)^{-1} \right),$$

where

$$\frac{\partial \big(\mathbf{A}\mathbf{A}^{\mathrm{T}} \big)}{\partial p_i} = \frac{\partial \mathbf{A}}{\partial p_i} \mathbf{A}^{\mathrm{T}} + \mathbf{A} \frac{\partial \mathbf{A}^{\mathrm{T}}}{\partial p_i},$$

and all derivatives $\frac{\partial \mathbf{A}}{\partial p_i}$ and $\frac{\partial \mathbf{b}}{\partial p_i}$ are computed using algorithmic differentiation.