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Modelled distributions of Triebel–Lizorkin type

by

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Abstract. In order to provide a local description of a regular function in a small neighbourhood of a point x, it is sufficient by Taylor's theorem to know the value of the function as well as all of its derivatives up to the required order at the point x itself. In other words, one could say that a regular function is locally modelled by the set of polynomials. The theory of regularity structures due to Hairer generalizes this observation and provides an abstract setup, which in the application to singular SPDE extends the set of polynomials by functionals constructed from, e.g., white noise. In this context, the notion of Taylor polynomials is lifted to the notion of so-called modelled distributions. The celebrated reconstruction theorem, which in turn was inspired by Gubinelli's *sewing lemma*, is of paramount importance for the theory. It enables one to reconstruct a modelled distribution as a true distribution on $\mathbb{R}^d$ which is locally approximated by this extended set of models or "monomials". In the original work of Hairer, the error is measured by means of Hölder norms. This was then generalized to the whole scale of Besov spaces by Hairer and Labbé. It is the aim of this work to adapt the analytic part of the theory of regularity structures to the scale of Triebel–Lizorkin spaces.

1. Introduction. One common theme in the theory of function spaces is how to measure the "regularity" or "smoothness" of a function. Unsurprisingly, an appropriate answer often depends on the given context or application. The most basic notion of regularity manifests itself in the classical spaces of differentiable functions with integer regularity index. By Taylor's theorem, a function is differentiable up to some fixed order if the small-scale fluctuations of the function are given, up to some error term, by a polynomial with degree of required order. In this sense, "regular" functions are exactly those which locally look like polynomials.

Now, in the context of differential equations there are lots of model problems where this point of view is simply not appropriate. This is for example

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the case for controlled ODEs with rough driving noise, or for singular stochastic PDEs with white noise as the driving force. In the latter example one would rather expect the solution to locally look like functionals which are build from the driving noise. In the last years, two solution theories were developed to formalize this observation and give for a first time a direct meaning to interesting SPDEs like the KPZ equation, the 2D parabolic Anderson model and generalizations thereof, or the 3D stochastic quantization equation for the $(\Phi)^4_3$ euclidean quantum field.

On the one hand, there is the theory of paracontrolled distributions due to Gubinelli, Imkeller and Perkowski [4], where ideas from paradifferential calculus and the theory of controlled rough paths are combined in order to give a rigorous treatment of ill-posed stochastic PDEs. A different but related approach was developed by Hairer [5]. In his theory of regularity structures, one refers to the set of polynomials as a "model" for (classical) differentiable functions. As already alluded to above, the polynomial model is not the right formalization of "regularity" for many interesting SPDEs. Instead, the theory of regularity structures extends the set of "models" beyond the set of polynomials. As a consequence, the notion of Taylor polynomials is generalized and the corresponding "function space" is given by so-called modelled distributions. It is a key result of the theory that all modelled distributions can be reconstructed as genuine distributions which then locally look like the given fixed set of models. This result is referred to as the *reconstruction theorem*.

In the original work on regularity structures [5], the space of modelled distributions was set up in direct analogy to Hölder spaces. The theory in particular allows for potential blow-up on the $t = 0$ hyperplane in order to treat a large class of initial data. Boundary conditions entered the analysis in the form of initial data, since the work in [5] concentrated specifically on spatially periodic problems. We refer the interested reader to the recent work of Gerencsér and Hairer [3], who extended the original framework to also allow for singularities near the boundary of domains in the space variable. This generalization enables studying singular SPDE with certain boundary conditions, e.g. the KPZ equation with Dirichlet and Neumann conditions or the generalized 2D parabolic Anderson model with Dirichlet conditions.

A first step in the direction of the full Besov scale appeared in the work of Hairer and Labbé [6] on multiplicative stochastic heat equations. The motivation for this generalization came from the desire to start the evolution from a Dirac mass. One key observation here is that the Dirac mass has improved regularity when considered as an element of the Besov scale in contrast to Hölder spaces. This then led to a generalization of the original framework to analogues of the Besov spaces of type $B^{\alpha}_{p,\infty}$. We also mention the recent work of Prömel and Teichmann [9] who study modelled distributions of Sobolev–Slobodetskiĭ type. Recall that the classical Sobolev–Slobodetskiĭ spaces are norm equivalent to the Besov spaces $B^{\alpha}_{p,p}$. The whole scale of Besov spaces was then eventually treated in [6].

In this paper, we adapt the analytic part of the theory of regularity structures to a space of modelled distributions which mimics classical Triebel–Lizorkin distributions. In the remainder of this introduction, we first describe the key notions from the theory of regularity structures. Based on that, we will briefly discuss the results on the Besov scale of modelled distributions by Hairer and Labbé [7]. We conclude the introduction with an outline of the paper.

Regularity structures. We recall the basic notions from the theory of regularity structures as introduced by Hairer [5]. We also use this opportunity to clarify notation. First, a *regularity structure* consists of a triple $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ where, following Hairer, we call $\mathcal{A}$ the set of *homogeneities*, $\mathcal{T}$ the *model space* and $\mathcal{G}$ the *structure group*. The set $\mathcal{A}$ is a subset of $\mathbb{R}$ which is bounded from below and locally finite. The model space $\mathcal{T}$ is a graded vector space of type $\bigoplus_{\zeta\in\mathcal{A}} \mathcal{T}_\zeta$, where each $\mathcal{T}_\zeta$ is a Banach space. Finally, $\mathcal{G}$ is a group of linear maps $\Gamma\colon \mathcal{T} \to \mathcal{T}$ such that for all $\zeta \in \mathcal{A}$, all $\tau \in \mathcal{T}_\zeta$ and all $\Gamma \in \mathcal{G}$ we have $\Gamma\tau - \tau \in \bigoplus_{\beta\in\mathcal{A}\cap(-\infty,\zeta)} \mathcal{T}_\beta$.

We will denote by $\mathcal{Q}_\zeta$ the projection from $\mathcal{T}$ onto $\mathcal{T}_\zeta$. Furthermore, we define $|\tau|_\zeta := \|\mathcal{Q}_\zeta\tau\|_{\mathcal{T}_\zeta}$ for all $\tau \in \mathcal{T}$. For notational convenience, let us also set $\mathcal{A}_\gamma := \mathcal{A} \cap (-\infty, \gamma)$ and $\mathcal{T}^-_\gamma := \bigoplus_{\beta\in\mathcal{A}_\gamma} \mathcal{T}_\beta$, for $\gamma \in \mathbb{R}$. We denote by $\mathcal{Q}_{<\gamma}$ the projection of $\mathcal{T}$ onto $\mathcal{T}^-_\gamma$. In what follows, given a fixed regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ and a fixed $\gamma > 0$, the integer $r \in \mathbb{N}$ is assumed to be the smallest positive integer such that $r > |\min \mathcal{A}| \vee |\max \mathcal{A}_\gamma|$. Finally, we also fix a scaling $\mathfrak{s} = (\mathfrak{s}_1, \dots, \mathfrak{s}_d) \in \mathbb{N}^d$ and write $\|x\|_{\mathfrak{s}} = \sup |x_i|^{1/\mathfrak{s}_i}$ for $x \in \mathbb{R}^d$, i.e. we consider the $\mathfrak{s}$-scaled "supremum norm" on $\mathbb{R}^d$. Given $x \in \mathbb{R}^d$ and $R > 0$, we then denote by $Q(x, R)$ the ball centred at x and radius R with respect to this "norm". Furthermore, let $A(x, R)$ denote the annulus $\{z \in \mathbb{R}^d\colon R/2 \le \|z\|_{\mathfrak{s}} \le R\}$.

Apart from the algebraic setup represented by the triple $(\mathcal{A}, \mathcal{T}, \mathcal{G})$, the theory of regularity structures provides the elements of the model space with some analytical structure. This is the content of the key notion of a *model*, which is a pair (Π, Γ) obeying the following requirements. The object Π is a family $(\Pi_x)_{x\in\mathbb{R}^d}$ of continuous linear maps from $\mathcal{T}$ into the space $\mathcal{D}'(\mathbb{R}^d)$ of Schwartz distributions such that, for every $\gamma > 0$,

$$\|\Pi\|_x := \sup_{\eta\in\mathfrak{B}^r} \sup_{\lambda\in(0,1]} \sup_{\zeta\in\mathcal{A}_\gamma} \sup_{\tau\in\mathcal{T}_\zeta} \frac{|\langle \Pi_x\tau, \eta^\lambda_x\rangle|}{|\tau|\lambda^\zeta} \lesssim 1,$$

uniformly over all $x \in \mathbb{R}^d$. Here, we denote by $\mathcal{C}^r(\mathbb{R}^d)$ the space of functions $\eta\colon \mathbb{R}^d \to \mathbb{R}$ that are continuously differentiable for all orders $k \in \mathbb{N}^d$ with scaled degree $|k|_{\mathfrak{s}} := k_1\mathfrak{s}_1 + \cdots + k_d\mathfrak{s}_d \le r$, and $\mathfrak{B}^r$ denotes the space of all functions in $\mathcal{C}^\infty(\mathbb{R}^d)$ which are supported in $Q(0,1)$ and with $\mathcal{C}^r(\mathbb{R}^d)$-norm bounded by 1. In addition, we made use of

$$\eta_x^\lambda(z) := \lambda^{-|\mathfrak{s}|}\eta\big(\lambda^{-\mathfrak{s}_1}(z_1-x_1),\dots,\lambda^{-\mathfrak{s}_d}(z_d-x_d)\big)$$

for all $\eta\colon \mathbb{R}^d \to \mathbb{R}$, all $\lambda \in (0,1]$ and all $x, z \in \mathbb{R}^d$.

The object Γ is a map $\mathbb{R}^d \times \mathbb{R}^d \to \mathcal{G}$ such that

(i) $\Gamma_{x,x} = 1$, $\Gamma_{x,z}\Gamma_{z,y} = \Gamma_{x,y}$ for all $x, y, z \in \mathbb{R}^d$,
(ii) $\Pi_y = \Pi_x\Gamma_{x,y}$ for all $x, y \in \mathbb{R}^d$, and
(iii) the following bound is satisfied:

$$\|\Gamma\|_{x,y} := \sup_{\zeta\in\mathcal{A}_\gamma}\ \sup_{\beta\in\mathcal{A}\cap(-\infty,\zeta]}\ \sup_{\tau\in\mathcal{T}_\zeta} \frac{|\Gamma_{x,y}\tau|_\beta}{|\tau|\,\|x-y\|_{\mathfrak{s}}^{\zeta-\beta}} \lesssim 1,$$

uniformly over all $x \in \mathbb{R}^d$ and all $y \in Q(x,1)$.

Finally, we will write

$$\|\Pi\| := \sup_{x\in\mathbb{R}^d} \|\Pi\|_x, \qquad \|\Gamma\| := \sup_{x,y\in\mathbb{R}^d} \|\Gamma\|_{x,y}.$$

An elementary example of a regularity structure is the *polynomial regularity structure*, which will always be denoted by $(\bar{\mathcal{A}}, \bar{\mathcal{T}}, \bar{\mathcal{G}})$. The associated set of homogeneities is simply $\bar{\mathcal{A}} = \mathbb{N}_0$. For $\zeta \in \bar{\mathcal{A}}$, we let $\mathcal{T}_\zeta$ be the linear span of the monomials $X^k = X_1^{k_1}\cdots X_d^{k_d}$ with scaled degree $|k|_{\mathfrak{s}} = \zeta$, i.e. in particular $\bar{\mathcal{T}} = \mathbb{R}[X_1,\dots,X_d]$. The structure group $\bar{\mathcal{G}} \sim (\mathbb{R}^d,+)$ acts on $Q \in \bar{\mathcal{T}}$ by translations:

$$(\Gamma_h Q)(X) = Q(X + h\mathbf{1}), \qquad h \in \mathbb{R}^d.$$

Finally, the polynomial structure $(\bar{\mathcal{A}}, \bar{\mathcal{T}}, \bar{\mathcal{G}})$ comes with a canonical model given by

$$(\Pi_x X^k)(y) = (y-x)^k, \qquad \Gamma_{x,y} = \Gamma_{x-y}.$$

Besov scale of modelled distributions. Let us now recall from the work of Hairer and Labbé [7] the Besov scale of modelled distributions, which resembles the classical Besov scale. Below, L^p always refers to the Lebesgue space $L^p(\mathbb{R}^d, \mathrm{d}x)$ with x denoting the integration variable.

Definition 1.1. Let $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ be a regularity structure and (Π, Γ) a model. For $1 \le p, q \le \infty$ and $\gamma > 0$, let $\mathcal{B}^\gamma_{p,q}$ be the (Banach) space of all functions $f\colon \mathbb{R}^d \to \mathcal{T}_\gamma^-$ such that

(i) $\displaystyle\sup_{\zeta\in\mathcal{A}_\gamma} \big\||f(x)|_\zeta\big\|_{L^p} < \infty,$

(ii) $\displaystyle\sup_{\zeta\in\mathcal{A}_\gamma} \bigg(\int_{Q(0,1)} \bigg\|\frac{|f(x+h) - \Gamma_{x+h,x}f(x)|_\zeta}{\|h\|_{\mathfrak{s}}^{\gamma-\zeta}}\bigg\|_{L^p}^q \frac{\mathrm{d}h}{\|h\|_{\mathfrak{s}}^{|\mathfrak{s}|}}\bigg)^{1/q} < \infty.$

The associated norm of $f \in \mathcal{B}^\gamma_{p,q}$ is denoted by $\|\!|f|\!\|_{\mathcal{B}^\gamma_{p,q}}$.

One of the key results in [7] was a "countable description" of this space. The idea is to work only with local volume means of a modelled distribution. This leads to a scale of spaces denoted by $\bar{\mathcal{B}}$ (Definition 8.2), which turns out to be equivalent to the scale $\mathcal{B}$ [7, Thm. 2.14]. The idea to work with local volume means in order to construct the reconstruction operator already appeared in the work of Hairer and Labbé [6] on multiplicative stochastic heat equations.

With this "countable description" at hand, Hairer and Labbé can prove the reconstruction theorem for the full scale of Besov type modelled distributions (with non-integer regularity index), and moreover various continuous embeddings which are well known in the classical setting of Besov spaces. They further obtain Schauder type estimates for their spaces of modelled distributions.

The main purpose of the present work is to adapt this theory to spaces of modelled distributions which resemble the scale of Triebel–Lizorkin distributions. We start in Section 2 by providing the necessary results for the classical Triebel–Lizorkin spaces, i.e. a wavelet characterization and a convergence criterion. In a next step, we introduce in Section 3 the corresponding space of modelled distributions. Our definition is in analogy to the classical characterization of Triebel–Lizorkin spaces in terms of volume means of differences (for more on characterizations in terms of differences, see the book of Triebel [10, Sec. 2.5.11]). We then provide results which show that it is again sufficient to work only with a discrete set of volume means (Propositions 3.8 and 3.9).

In Section 4, we make use of this characterization in order to prove embeddings for the spaces considered in this work. We also provide embeddings which involve the Besov scale of modelled distributions. Section 5 is devoted to the proof of the reconstruction theorem, and in Section 6 we obtain Schauder estimates in the Triebel–Lizorkin scale of modelled distributions. We also include a short section on products of modelled distributions (Section 7). Finally, in Section 8 we return to the Besov scale.

Our motivation comes from [6], where the authors use volume means of differences instead of differences themselves (as in [7]) for their space of modelled distributions. In the classical setting of Besov spaces, it is a well-known fact that this actually makes no difference. We show that this is still true in the framework of regularity structures.

2. Some harmonic analysis.

The following definition introduces the precise space of distributions we are going to use. The notation L^q_λ is shorthand for $L^q((0,1],\lambda^{-1}\mathrm{d}\lambda)$. For $\beta\in\mathbb{N}_0$, we denote by $\mathfrak{B}^r_\beta(\mathbb{R}^d)$ the subspace of functions in $\mathfrak{B}^r(\mathbb{R}^d)$ which annihilate polynomials with scaled degree at most β.

Definition 2.1. Let $1 \le p < \infty$, $1 \le q \le \infty$ and $\alpha \in \mathbb{R}$. Furthermore, fix $r \in \mathbb{N}$ such that $r > |\alpha|$. If $\alpha < 0$, we let $F^{\alpha}_{p,q}$ be the (Banach) space of Schwartz distributions ξ on $\mathbb{R}^d$ with

$$\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle\xi,\eta^\lambda_x\rangle|}{\lambda^\alpha}r\Big\|_{L^q_\lambda}\Big\|_{L^p}<\infty. \tag{2.1}$$

If $\alpha \ge 0$, we require that

$$\Big\|\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}|\langle\xi,\eta_x\rangle|\Big\|_{L^p}<\infty,\qquad \Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r_{\lfloor\alpha\rfloor}(\mathbb{R}^d)}\frac{|\langle\xi,\eta^\lambda_x\rangle|}{\lambda^\alpha}\Big\|_{L^q_\lambda}\Big\|_{L^p}<\infty. \tag{2.2}$$

In both cases, we denote by $\|\xi\|_{F^{\alpha}_{p,q}}$ the corresponding norm.

Note that this scale of spaces depends on the chosen scaling $\mathfrak{s}$ through the definition of the spaces $\mathfrak{B}^r(\mathbb{R}^d)$ and $\mathfrak{B}^r_{\lfloor\alpha\rfloor}(\mathbb{R}^d)$. Next, we aim at a characterization of the spaces $F^{\alpha}_{p,q}$ in terms of wavelets. Such a characterization is of course not new (see the book of Triebel [11]), but we still prefer to give a proof since we allow for non-trivial scalings of $\mathbb{R}^d$. For more on wavelets with compact support and certain regularity, we refer to the work of Daubechies [1].

For $r > 0$, there is a compactly supported scaling function $\varphi \in \mathcal{C}^r(\mathbb{R})$ with the following properties:

(i) $\int_{\mathbb{R}^d}\varphi(x)\varphi(x-k)\,\mathrm{d}x = \delta_{k,0}$ for every $k \in \mathbb{Z}$,
(ii) there exists a finite family $(a_k)_{k\in K}$ of constants with $K \subset \mathbb{Z}$ such that $\varphi(x) = \sum_{k\in K} a_k\varphi(2x-k)$, and
(iii) for every $x \in \mathbb{R}^d$ and every polynomial Q of scaled degree at most r, we have $\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}^d} Q(z)\varphi(z-k)\varphi(x-k)\,\mathrm{d}z = Q(x)$.

Now, given such a φ we let V_n be the linear subspace of $L^2(\mathbb{R}^d)$ generated by functions of the form

$$\varphi^n_y(x) := 2^{-n|\mathfrak{s}|/2}\prod_{i=1}^d \varphi^{2^{-n\mathfrak{s}_i}}_{y_i}(x_i),$$

where $y \in \Lambda_n$ and

$$\Lambda_n = \{(y_1,\dots,y_d)\in\mathbb{R}^d\colon y_i = 2^{-n\mathfrak{s}_i}k_i,\ k_i\in\mathbb{Z},\ i=1,\dots,d\}.$$

The properties of φ ensure that $V_n \subset V_{n+1}$. Wavelet theory also guarantees the existence of a finite set $\Psi \subset \mathcal{C}^r(\mathbb{R}^d)$ of compactly supported functions such that

(i) the linear subspace generated by functions of type $\psi^n_y := 2^{-n|\mathfrak{s}|/2}\psi^{2^{-n}}_y$, where $\psi \in \Psi$ and $y \in \Lambda_n$, equals the orthogonal complement of V_n in V_{n+1}, for all $n \ge 0$,
(ii) $\int_{\mathbb{R}^d} x^k\psi(x)\,\mathrm{d}x = 0$ for every $k \in \mathbb{N}^d_0$ with $|k|_{\mathfrak{s}} \le r$ and every $\psi \in \Psi$, and

(iii) for all $n \geq 0$, the set $\{\varphi_y^n : y \in \Lambda_n\} \cup \{\psi_y^m : m \geq n,\ \psi \in \Psi,\ y \in \Lambda_m\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.

Finally, for every $n \geq 0$ and $y \in \Lambda_n$, define the cube

$$Q_y^n = [y_1, y_1 + 2^{-n\mathfrak{s}_1}) \times \cdots \times [y_d, y_d + 2^{-n\mathfrak{s}_d})$$

and let χ_y^n denote the associated characteristic function. Now, the announced wavelet description reads as follows.

Proposition 2.2. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\alpha \in \mathbb{R}$. Choose $r \in \mathbb{N}$ such that $r > |\alpha|$ and let $\xi \in \mathcal{D}'(\mathbb{R}^d)$. Then $\xi \in F_{p,q}^\alpha$ if and only if*

$$(2.3)\qquad \begin{aligned} &\Big(\sum_{y\in\Lambda_0} |\langle \xi, \varphi_y^0\rangle|^p\Big)^{1/p} < \infty, \\ &\sup_{\psi\in\Psi} \Big\| \Big(\sum_{n\geq 0} \sum_{y\in\Lambda_n} \frac{|\langle \xi, \psi_y^n\rangle|^q}{2^{-n(|\mathfrak{s}|/2+\alpha)q}} \chi_y^n(x) \Big)^{1/q} \Big\|_{L^p} < \infty. \end{aligned}$$

In particular, this gives an equivalent norm on $F_{p,q}^\alpha$ in terms of wavelet coefficients.

Proof. The "only if" part follows from the same type of consideration in [7], so we only discuss the "if" assertion. In other words, we assume that the bounds in (2.3) hold, and we have to verify that $\xi \in F_{p,q}^\alpha$. To this end, we distinguish the cases $\alpha \geq 0$ and $\alpha < 0$. Let us begin with the latter case for $q < \infty$.

Following the argument in [7], we show that the series

$$(2.4)\qquad \sum_{y\in\Lambda_0} \langle \varphi_y^0, \eta_x^\lambda\rangle \langle \xi, \varphi_y^0\rangle + \sum_{\psi\in\Psi} \sum_{n\geq 0} \sum_{y\in\Lambda_n} \langle \psi_y^n, \eta_x^\lambda\rangle \langle \xi, \psi_y^n\rangle$$

converges absolutely, and by bounding the two contributions separately, we will eventually obtain the required bound for $\langle \xi, \eta_x^\lambda\rangle$. The first term involving φ is treated exactly as in [7], which also explains the occurrence of the first bound in (2.3). Therefore, we only discuss the second term in detail. By using the triangle inequality for the sum over $\psi \in \Psi$ this amounts to bounding

$$(2.5)\qquad \Big\| \Big(\sum_{n_0=0}^\infty \Big\| \sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)} \frac{\sum_{n\geq 0} \sum_{y\in\Lambda_n} |\langle \psi_y^n, \eta_x^\lambda\rangle|\,|\langle \xi, \psi_y^n\rangle|}{\lambda^\alpha} \Big\|_{L^q_{\lambda,n_0}}^q \Big)^{1/q} \Big\|_{L^p}$$

where L^q_{λ,n_0} denotes $L^q((2^{-(n_0+1)}, 2^{-n_0}], \lambda^{-1}\,d\lambda)$. In addition, let M denote the smallest integer greater than or equal to the maximum sizes of the supports of φ and $\psi \in \Psi$. For fixed $n_0 \in \mathbb{N}_0$, we divide the sum over $n \geq 0$ into sums over $n < n_0$ and over $n \geq n_0$. We note that (cf. [7])

$$(2.6)\qquad |\langle \psi_y^n, \eta_x^\lambda\rangle| \lesssim 2^{n|\mathfrak{s}|/2} \chi_{Q(x,\lambda+M2^{-n})}(y),$$

uniformly over all $\psi \in \Psi$, all $x \in \mathbb{R}^d$, all $n < n_0$, all $y \in \Lambda_n$, all $\lambda \in (2^{-(n_0+1)}, 2^{-n_0}]$ and all $\eta \in \mathfrak{B}^r(\mathbb{R}^d)$. Furthermore,

$$|\langle \psi_y^n, \eta_x^\lambda \rangle| \lesssim 2^{-(n-n_0)(|\mathfrak{s}|/2+r)} 2^{n_0|\mathfrak{s}|/2} \chi_{Q(x,\lambda+M2^{-n})}(y), \tag{2.7}$$

uniformly over all $\psi \in \Psi$, all $x \in \mathbb{R}^d$, all $n \geq n_0$, all $y \in \Lambda_n$, all $\lambda \in (2^{-(n_0+1)}, 2^{-n_0}]$ and all $\eta \in \mathfrak{B}^r(\mathbb{R}^d)$.

Now, by (2.6) we obtain

$$\begin{aligned}\Big\| \sup_{\eta \in \mathfrak{B}^r(\mathbb{R}^d)} \sum_{n<n_0} \sum_{y \in \Lambda_n} \frac{|\langle \psi_y^n, \eta_x^\lambda \rangle| \, |\langle \xi, \psi_y^n \rangle|}{\lambda^\alpha} \Big\|_{L^q_{\lambda,n_0}}^q \\ \lesssim \sum_{n<n_0} 2^{\alpha(n_0-n)} \Big| \sum_{\substack{y \in \Lambda_n \\ \|x-y\|_{\mathfrak{s}} \leq 2^{-n}(M+1)}} \frac{|\langle \xi, \psi_y^n \rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \Big|^q .\end{aligned}$$

Here, we have used Jensen's inequality for the sum over $n < n_0$. In addition, (2.7) yields

$$\begin{aligned}\Big\| \sup_{\eta \in \mathfrak{B}^r(\mathbb{R}^d)} \sum_{n\geq n_0} \sum_{y \in \Lambda_n} \frac{|\langle \psi_y^n, \eta_x^\lambda \rangle| \, |\langle \xi, \psi_y^n \rangle|}{\lambda^\alpha} \Big\|_{L^q_{\lambda,n_0}}^q \\ \lesssim \sum_{n\geq n_0} 2^{-(n-n_0)(r+\alpha)} \Big| \sum_{\substack{y \in \Lambda_n \\ \|x-y\|_{\mathfrak{s}} \leq 2^{-n_0}(M+1)}} 2^{(n_0-n)|\mathfrak{s}|} \frac{|\langle \xi, \psi_y^n \rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \Big|^q .\end{aligned}$$

This time, we have applied Jensen's inequality to the sum over $n \geq n_0$. The next step is to appropriately estimate the restricted sums over $y \in \Lambda_n$ in the last two bounds.

We begin with the large scales regime $n < n_0$. Recall the notation Q_y^n. Let $\Lambda_n^{M,x}$ denote the smallest subset of Λ_n such that the cube $Q(x, 2^{-n}(M+2))$ is covered with the disjoint union of cubes Q_y^n with $y \in \Lambda_n^{M,x}$. Note that $\Lambda_n^{M,x}$ contains the set of all $y \in \Lambda_n$ such that $\|y - x\|_{\mathfrak{s}} \leq 2^{-n}(M+1)$. Hence, for any $\kappa \in (0, 1)$ the monotonicity of ℓ^s-norms yields

$$\begin{aligned}\Big| \sum_{\substack{y \in \Lambda_n \\ \|x-y\|_{\mathfrak{s}} \leq 2^{-n}(M+1)}} \frac{|\langle \xi, \psi_y^n \rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \Big|^\kappa &\leq \sum_{\substack{y \in \Lambda_n \\ \|x-y\|_{\mathfrak{s}} \leq 2^{-n}(M+1)}} \frac{|\langle \xi, \psi_y^n \rangle|^\kappa}{2^{-n(|\mathfrak{s}|/2+\alpha)\kappa}} \\ &\lesssim \fint_{Q(x,2^{-n}(M+2))} \Big| \sum_{y \in \Lambda_n^{M,x}} \frac{|\langle \xi, \psi_y^n \rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \chi_y^n(z) \Big|^\kappa \mathrm{d}z \\ &\leq \mathcal{M}\Big(\Big| \sum_{y \in \Lambda_n} \frac{|\langle \xi, \psi_y^n \rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \chi_y^n \Big|^\kappa \Big)(x),\end{aligned}$$

where $\mathcal{M}f$ denotes the Hardy–Littlewood maximal function for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Due to the additional factor $2^{(n_0-n)|\mathfrak{s}|}$, an analogous type of argument shows for the small scales regime $n \geq n_0$ that

$$\sum_{\substack{y\in\Lambda_n\\ \|x-y\|_{\mathfrak{s}}\leq 2^{-n_0}(M+1)}} 2^{(n_0-n)|\mathfrak{s}|}\frac{|\langle\xi,\psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}}$$
$$\lesssim 2^{-(n-n_0)\frac{\kappa-1}{\kappa}|\mathfrak{s}|}\Big\{\mathcal{M}\Big(\Big|\sum_{y\in\Lambda_n}\frac{|\langle\xi,\psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}}\chi_y^n\Big|^{\kappa}\Big)(x)\Big\}^{1/\kappa},$$

which again holds true for every $\kappa \in (0,1)$.

To summarize, we see that the term in (2.5) is bounded by

$$\Big\|\Big(\sum_{n_0=0}^{\infty}\sum_{n=0}^{\infty}\theta(n-n_0)\Big\{\mathcal{M}\Big(\Big|\sum_{y\in\Lambda_n}\frac{|\langle\xi,\psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}}\chi_y^n\Big|^{\kappa}\Big)(x)\Big\}^{q/\kappa}\Big)^{1/q}\Big\|_{L^p}, \tag{2.8}$$

where the sequence $(\theta(z))_{z\in\mathbb{Z}}$ is defined by

$$\theta(z) = \begin{cases} 2^{-z(r+\alpha+\frac{\kappa-1}{\kappa}q|\mathfrak{s}|)}, & z \geq 0,\\ 2^{-z\alpha}, & z < 0.\end{cases}$$

If we choose $0 < \kappa < 1$ such that $r + \alpha + \frac{\kappa-1}{\kappa}q|\mathfrak{s}| > 0$, then it follows that $\|\theta\|_{\ell^1(\mathbb{Z})} < \infty$. In particular, due to Young's inequality for convolutions the quantity (2.8) can be estimated by

$$\Big\|\Big(\sum_{n_0=0}^{\infty}\Big\{\mathcal{M}\Big(\Big|\sum_{y\in\Lambda_{n_0}}\frac{|\langle\xi,\psi_y^{n_0}\rangle|}{2^{-n_0(|\mathfrak{s}|/2+\alpha)}}\chi_y^{n_0}\Big|^{\kappa}\Big)(x)\Big\}^{q/\kappa}\Big)^{1/q}\Big\|_{L^p}.$$

Next, since $\kappa < p \wedge q$ we can also apply the Fefferman–Stein maximal inequality (cf. [2], a vector-valued version of the Hardy–Littlewood maximal inequality) to bound this by

$$\Big\|\Big(\sum_{n\geq 0}\sum_{y\in\Lambda_n}\frac{|\langle\xi,\psi_y^n\rangle|^q}{2^{-n(|\mathfrak{s}|/2+\alpha)q}}\chi_y^n(x)\Big)^{1/q}\Big\|_{L^p},$$

which is finite by assumption, i.e. $\xi \in F^{\alpha}_{p,q}$ as asserted.

Let us discuss briefly the case $q = \infty$. We still also assume that $\alpha < 0$. Then a similar argument to the above shows that

$$\Big\|\sup_{\lambda\in(0,1]}\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle\xi,\eta_x^\lambda\rangle|}{\lambda^\alpha}\Big\|_{L^p}$$
$$\lesssim \Big(\sum_{y\in\Lambda_0}|\langle\xi,\varphi_y^0\rangle|^p\Big)^{1/p} + \sup_{\psi\in\Psi}\Big\|\sup_{n\in\mathbb{N}_0}\Big\{\mathcal{M}\Big(\Big|\sum_{y\in\Lambda_n}\frac{|\langle\xi,\psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}}\chi_y^n\Big|^{\kappa}\Big)(x)\Big\}^{1/\kappa}\Big\|_{L^p},$$

where $\kappa \in (0,1)$. With an application of the vector-valued maximal inequality in the version of [11, (9.51)] we obtain

$$\Big\| \sup_{\lambda\in(0,1]} \sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)} \frac{|\langle \xi, \eta_x^\lambda\rangle|}{\lambda^\alpha} \Big\|_{L^p} \lesssim \Big(\sum_{y\in\Lambda_0} |\langle \xi, \varphi_y^0\rangle|^p \Big)^{1/p} + \sup_{\psi\in\Psi} \Big\| \sup_{n\in\mathbb{N}_0} \sum_{y\in\Lambda_n} \frac{|\langle \xi, \psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \chi_y^n(x) \Big\|_{L^p},$$

which concludes the proof for $q=\infty$.

Now, we move on to the case $\alpha \geq 0$. Recall from the definition of $F^\alpha_{p,q}$ that this amounts to bounding the two quantities in (2.2). For the latter, the above argument works verbatim except that one has to replace (2.6) with

$$|\langle \psi_y^n, \eta_x^\lambda\rangle| \lesssim 2^{n(|\mathfrak{s}|/2+\lfloor\alpha\rfloor+1)} \lambda^{\lfloor\alpha\rfloor+1} \chi_{Q(x,\lambda+M2^{-n})}(y), \tag{2.9}$$

which holds uniformly over all $\psi\in\Psi$, all $x\in\mathbb{R}^d$, all $n<n_0$, all $y\in\Lambda_n$, all $\lambda\in(2^{-(n_0+1)}, 2^{-n_0}]$ and all $\eta\in\mathfrak{B}^r_{\lfloor\alpha\rfloor}(\mathbb{R}^d)$. Therefore, we only deal with the first quantity in (2.2), again resorting to the wavelet decomposition from (2.4)—this time with $\lambda=1$. The sum involving φ can be bounded by the arguments of [7]. Hence, we briefly discuss the sum involving $\psi\in\Psi$.

Note that this time there is no integration in $\lambda\in(0,1]$. In other words, we can use the arguments presented above, formally substituting $\lambda=1$, i.e. $n_0=0$. In particular, it is readily seen that

$$\begin{aligned} &\Big\| \sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)} \sum_{n\geq 0} \sum_{y\in\Lambda_n} |\langle \xi, \psi_y^n\rangle|\, |\langle \psi_y^n, \eta_x\rangle| \Big\|_{L^p} \\ &\quad\lesssim \Big\| \sum_{n\geq 0} 2^{-n(r+\alpha+\frac{\kappa-1}{\kappa}|\mathfrak{s}|)} \Big\{ \mathcal{M}\Big(\Big| \sum_{y\in\Lambda_n} \frac{|\langle \xi, \psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \chi_y^n \Big|^\kappa \Big)(x) \Big\}^{1/\kappa} \Big\|_{L^p} \\ &\quad\lesssim \Big\| \Big(\sum_{n\geq 0} \Big\{ \mathcal{M}\Big(\Big| \sum_{y\in\Lambda_n} \frac{|\langle \xi, \psi_y^n\rangle|}{2^{-n(|\mathfrak{s}|/2+\alpha)}} \chi_y^n \Big|^\kappa \Big)(x) \Big\}^{q/\kappa} \Big)^{1/q} \Big\|_{L^p} \end{aligned}$$

(with the obvious modification for $q=\infty$). In the last step, we made use of Hölder's inequality, and that one can choose $0<\kappa<1$ such that $r+\alpha+\frac{\kappa-1}{\kappa}|\mathfrak{s}|>0$. Hence, we can conclude the proof of the "if" assertion by another application of the Fefferman–Stein maximal inequality. ■

Consider now a sequence $(\xi_n)_{n\in\mathbb{N}_0}$ given by

$$\xi_n = \sum_{y\in\Lambda_n} A_y^n \varphi_y^n,$$

and define

$$\delta A_y^n = \langle \xi_{n+1} - \xi_n, \varphi_y^n\rangle.$$

For the proof of the reconstruction theorem, the following criterion for convergence of the sequence $(\xi_n)_{n\in\mathbb{N}}$ in $F^\alpha_{p,q}$ will be of importance. Of course, this criterion is the counterpart to the corresponding assertion in [7, Prop. 3.7].

PROPOSITION 2.3. *Let $\gamma > 0$, $\alpha < 0$, $1 \le p < \infty$ and $1 \le q \le \infty$. Furthermore, assume that*

$$\Big\| \sup_{n\ge 0} \sum_{y\in\Lambda_n} \frac{|A^n_y|}{2^{-n(\alpha+|\mathfrak{s}|/2)}} \chi^n_y(x) \Big\|_{L^p} < \infty, \tag{2.10}$$

$$\Big\| \Big(\sum_{n\ge 0} \Big| \sum_{y\in\Lambda_n} \frac{|\delta A^n_y|}{2^{-n(\gamma+|\mathfrak{s}|/2)}} \chi^n_y(x) \Big|^q \Big)^{1/q} \Big\|_{L^p} < \infty. \tag{2.11}$$

Then, for every $\bar\alpha < \alpha$, the sequence $(\xi_n)_{n\in\mathbb{N}}$ converges in $F^{\bar\alpha}_{p,q}$. Moreover, if $q = \infty$ the limit distribution is actually an element of $F^\alpha_{p,q}$.

Proof. We follow the strategy of Hairer and Labbé [7], adapting some arguments as in the wavelet characterization of $F^\alpha_{p,q}$. In particular, we will again make use of the Fefferman–Stein maximal inequality. First, we decompose

$$\xi_{n+1} - \xi_n = g_n + \delta\xi_n \in V_n \oplus V_n^\perp = V_{n+1},$$

where

$$\begin{aligned} g_n &= \sum_{z\in\Lambda_n} \delta A^n_z \varphi^n_z, \\ \delta\xi_n &= \sum_{z\in\Lambda_n} \Big(\sum_{u\in\Lambda_{n+1}} A^{n+1}_u \langle \varphi^{n+1}_u, \psi^n_z\rangle \Big) \psi^n_z. \end{aligned}$$

Now, fix positive integers $k \le K$. We aim to bound the wavelet norm of $\sum_{n=k}^K g_n$ and $\sum_{n=k}^K \delta\xi_n$ in $F^{\bar\alpha}_{p,q}$, for every $\bar\alpha < \alpha$. Let us first treat the contributions from the $\delta\xi_n$ (with the obvious modification for $q = \infty$). To this end, note first that

$$\begin{aligned} &\Big\| \Big(\sum_{m\ge 0} \sum_{y\in\Lambda_m} \frac{|\langle \sum_{n=k}^K \delta\xi_n, \psi^m_y\rangle|^q}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)q}} \chi^m_y(x) \Big)^{1/q} \Big\|_{L^p} \\ &\qquad = \Big\| \Big(\sum_{k\le n\le K} \sum_{y\in\Lambda_n} \frac{|\sum_{z\in\Lambda_{n+1}} A^{n+1}_z \langle \varphi^{n+1}_z, \psi^n_y\rangle|^q}{2^{-n(\bar\alpha+|\mathfrak{s}|/2)q}} \chi^n_y(x) \Big)^{1/q} \Big\|_{L^p}. \end{aligned}$$

Then, we use the fact that $|\langle \varphi^{n+1}_z, \psi^n_y\rangle| \lesssim 1$ uniformly over the relevant parameters. Actually, the left hand side vanishes whenever $\|y-z\|_\mathfrak{s} > (2M)2^{-n}$.

Hence,

$$\Big\| \Big(\sum_{k\le n\le K} \sum_{y\in\Lambda_n} \frac{|\sum_{z\in\Lambda_{n+1}} A_z^{n+1} \langle \varphi_z^{n+1}, \psi_y^n\rangle|^q}{2^{-n(\bar\alpha+|\mathfrak{s}|/2)q}} \chi_y^n(x) \Big)^{1/q} \Big\|_{L^p}$$
$$\lesssim \Big\| \Big(\sum_{k\le n\le K} \sum_{y\in\Lambda_n} \Big| \sum_{\substack{z\in\Lambda_{n+1}\\ \|z-x\|_{\mathfrak{s}}\le C2^{-n}}} \frac{|A_z^{n+1}|}{2^{-(n+1)(\bar\alpha+|\mathfrak{s}|/2)}} \Big|^q \chi_y^n(x) \Big)^{1/q} \Big\|_{L^p},$$

with $C = 2M+1$. Now, we can proceed similarly to the proof of the wavelet characterization to obtain (with $0<\kappa<1$)

$$\Big\| \Big(\sum_{k\le n\le K} \sum_{y\in\Lambda_n} \Big| \sum_{\substack{z\in\Lambda_{n+1}\\ \|z-x\|_{\mathfrak{s}}\le C2^{-n}}} \frac{|A_z^{n+1}|}{2^{-(n+1)(\bar\alpha+|\mathfrak{s}|/2)}} \Big|^q \chi_y^n(x) \Big)^{1/q} \Big\|_{L^p}$$
$$\lesssim \Big\| \Big(\sum_{k\le n\le K} \Big\{ \mathcal{M}\Big(\Big| \sum_{z\in\Lambda_{n+1}} \frac{|A_z^{n+1}|}{2^{-(n+1)(\bar\alpha+|\mathfrak{s}|/2)}} \chi_z^n \Big|^\kappa \Big)(x) \Big\}^{q/\kappa} \Big)^{1/q} \Big\|_{L^p}$$
$$\lesssim \Big\| \Big(\sum_{k\le n\le K} \Big| \sum_{z\in\Lambda_{n+1}} \frac{|A_z^{n+1}|}{2^{-(n+1)(\bar\alpha+|\mathfrak{s}|/2)}} \chi_z^n(x) \Big|^q \Big)^{1/q} \Big\|_{L^p},$$

where the last step follows of course from the Fefferman–Stein maximal inequality. From this, we immediately infer that

$$\Big\| \Big(\sum_{m\ge 0} \sum_{y\in\Lambda_m} \frac{|\langle \sum_{n=k}^K \delta\xi_n, \psi_y^m\rangle|^q}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)q}} \chi_y^m(x) \Big)^{1/q} \Big\|_{L^p}$$
$$\lesssim 2^{-k(\alpha-\bar\alpha)} \Big\| \sup_{n\ge 0} \sum_{y\in\Lambda_n} \frac{|A_y^n|}{2^{-n(\alpha+|\mathfrak{s}|/2)}} \chi_y^n(x) \Big\|_{L^p}.$$

In the second step we treat the contributions from the g_n. We only discuss the bound related to the second quantity in (2.3); the bound related to the first quantity in (2.3) follows from similar considerations. Moreover, the case $q=\infty$ again follows from an obvious modification of the argument for $q<\infty$, so we will only treat the latter in detail. First of all, we have

$$\Big\| \Big(\sum_{m\ge 0} \sum_{y\in\Lambda_m} \frac{|\langle \sum_{n=k}^K g_n, \psi_y^m\rangle|^q}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)q}} \chi_y^m(x) \Big)^{1/q} \Big\|_{L^p}$$
$$\le \Big\| \Big(\sum_{m\ge 0} \sum_{y\in\Lambda_m} \Big(\sum_{(m+1)\vee k\le n\le K} \sum_{z\in\Lambda_n} \frac{|\delta A_z^n|\,|\langle \varphi_z^n, \psi_y^m\rangle|}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)}} \Big)^q \chi_y^m(x) \Big)^{1/q} \Big\|_{L^p}.$$

Then, observe that $|\langle \varphi_z^n, \psi_y^m\rangle| \lesssim 2^{-(n-m)|\mathfrak{s}|/2}$, uniformly over all the relevant parameters—in particular for all $m<n$. This time, the left hand side van-

ishes as soon as $\|z-y\|_\mathfrak{s} > (2M)2^{-m}$. In addition, let us choose $0<\kappa<1$ such that $\bar\gamma := \gamma + \frac{\kappa-1}{\kappa}|\mathfrak{s}| > 0$. We also put $\alpha' := \bar\alpha + \frac{\kappa-1}{\kappa}|\mathfrak{s}| < 0$. Then we obtain the bound

$$\begin{aligned}
&\Big(\sum_{(m+1)\vee k\le n\le K}\sum_{z\in\Lambda_n}\frac{|\delta A^n_z|\,|\langle\varphi^n_z,\psi^m_y\rangle|}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)}}\Big)^q\chi^m_y(x)\\
&\lesssim 2^{\bar\alpha qm}\Big(\sum_{(m+1)\vee k\le n\le K}2^{-n\gamma}\sum_{\substack{z\in\Lambda_n\\ \|z-y\|_\mathfrak{s}\le(2M)2^{-m}}}2^{(m-n)|\mathfrak{s}|}\frac{|\delta A^n_z|}{2^{-n(\gamma+|\mathfrak{s}|/2)}}\Big)^q\chi^m_y(x)\\
&\lesssim 2^{\bar\alpha qm}\Big(\sum_{(m+1)\vee k\le n\le K}2^{-n\gamma}\Big(\sum_{\substack{z\in\Lambda_n\\ \|z-y\|_\mathfrak{s}\le(2M)2^{-m}}}2^{(m-n)|\mathfrak{s}|\kappa}\frac{|\delta A^n_z|^\kappa}{2^{-n(\gamma+|\mathfrak{s}|/2)\kappa}}\Big)^{1/\kappa}\Big)^q\chi^m_y(x)\\
&\lesssim 2^{\alpha' qm}\sum_{(m+1)\vee k\le n\le K}2^{-n\bar\gamma}\Big(\sum_{\substack{z\in\Lambda_n\\ \|z-x\|_\mathfrak{s}\le C2^{-m}}}2^{(m-n)|\mathfrak{s}|}\frac{|\delta A^n_z|^\kappa}{2^{-n(\gamma+|\mathfrak{s}|/2)\kappa}}\Big)^{q/\kappa}\chi^m_y(x),
\end{aligned}$$

where the second-to-last line is a consequence of the monotonicity of ℓ^s-norms and the last line is an application of Jensen's inequality. From this, we can infer that

$$\begin{aligned}
&\Big\|\Big(\sum_{m\ge0}\sum_{y\in\Lambda_m}\frac{|\langle\sum_{n=k}^K g_n,\psi^m_y\rangle|^q}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)q}}\chi^m_y(x)\Big)^{1/q}\Big\|_{L^p}\\
&\lesssim 2^{-k\bar\gamma}\Big\|\Big(\sum_{m\ge0}\sum_{n\ge m}2^{\alpha' qm}\Big|\fint_{Q(x,C'2^{-m})}\Big|\sum_{z\in\Lambda_n}\frac{|\delta A^n_z|}{2^{-n(\gamma+|\mathfrak{s}|/2)}}\chi^n_z(u)\Big|^\kappa\mathrm{d}u\Big|^{q/\kappa}\Big)^{1/q}\Big\|_{L^p}\\
&\lesssim 2^{-k\bar\gamma}\Big\|\Big(\sum_{n\ge0}\Big\{\mathcal{M}\Big(\Big|\sum_{z\in\Lambda_n}\frac{|\delta A^n_z|}{2^{-n(\gamma+|\mathfrak{s}|/2)}}\chi^n_z\Big|^\kappa\Big)(x)\Big\}^{q/\kappa}\Big)^{1/q}\Big\|_{L^p}.
\end{aligned}$$

Here, we have in particular made use of $\sum_{m\ge0}2^{\alpha' qm}\sim1$ due to $\alpha'<0$. As it is by now routine, we can proceed from this point on with the aid of the Fefferman–Stein maximal inequality to deduce that

$$\Big\|\Big(\sum_{m\ge0}\sum_{y\in\Lambda_m}\frac{|\langle\sum_{n=k}^K g_n,\psi^m_y\rangle|^q}{2^{-m(\bar\alpha+|\mathfrak{s}|/2)q}}\chi^m_y(x)\Big)^{1/q}\Big\|_{L^p}
\lesssim 2^{-k\bar\gamma}\Big\|\Big(\sum_{n\ge0}\Big|\sum_{y\in\Lambda_n}\frac{|\delta A^n_y|}{2^{-n(\gamma+|\mathfrak{s}|/2)}}\chi^n_y(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}.$$

All in all, we obtain the desired convergence by making use of our assumptions (2.10) and (2.11). Furthermore, a second look at our argument for the contributions due to the $\delta\xi_n$ reveals that also

$$\Big\|\sup_{m\geq 0}\sum_{y\in\Lambda_m}\frac{|\langle\sum_{n=k}^{K}\delta\xi_n,\psi_y^m\rangle|}{2^{-m(\alpha+|\mathfrak{s}|/2)}}\chi_y^m(x)\Big\|_{L^p}\lesssim\Big\|\sup_{n\geq 0}\sum_{y\in\Lambda_n}\frac{|A_y^n|}{2^{-n(\alpha+|\mathfrak{s}|/2)}}\chi_y^n(x)\Big\|_{L^p},$$

uniformly over all $k\geq 0$ and all $K\geq k$. As the argument for the contributions due to the g_n works verbatim for $\bar{\alpha}=\alpha$, we deduce that in the case of $q=\infty$ the limit distribution actually lives in $F^{\alpha}_{p,q}$. ■

3. Spaces of modelled distributions. In this section, we introduce the scale of spaces of modelled distributions which shall resemble the scale $F^{\alpha}_{p,q}$. We want to follow as close as possible the characterization of Triebel–Lizorkin spaces by volume means of differences (see Triebel [10, Sec. 2.5.11]).

DEFINITION 3.1. Let $(\mathcal{A},\mathcal{T},\mathcal{G})$ be a regularity structure and (Π,Γ) a model. For $1\leq p<\infty$, $1\leq q\leq\infty$ and $\gamma>0$, let $\mathcal{F}^{\gamma}_{p,q}$ be the (Banach) space of all functions $f\colon\mathbb{R}^d\to\mathcal{T}^{-}_{\gamma}$ such that

(i) $\sup_{\zeta\in\mathcal{A}_\gamma}\big\||f(x)|_\zeta\big\|_{L^p}<\infty$,

(ii) $\sup_{\zeta\in\mathcal{A}_\gamma}\Big\|\Big\|\fint_{Q(0,4\lambda)}\frac{|f(x+h)-\Gamma_{x+h,x}f(x)|_\zeta}{\lambda^{\gamma-\zeta}}\,\mathrm{d}h\Big\|_{L^q_\lambda}\Big\|_{L^p}<\infty.$

We will often make use of the difference operator

$$\Delta_{\Gamma,h}f(x)=f(x+h)-\Gamma_{x+h,x}f(x).$$

Finally, the associated norm of $f\in\mathcal{F}^{\gamma}_{p,q}$ is denoted by $|||f|||_{\mathcal{F}^{\gamma}_{p,q}}$.

A wavelet characterization for the space $F^{\alpha}_{p,q}$ of distributions is useful in particular because norms of sequence spaces are typically more easily analyzed than norms of function spaces, say L^p-norms. For the same reason, one should look for a suitable "discretized" characterization for the Triebel–Lizorkin scale $\mathcal{F}^{\gamma}_{p,q}$ of modelled distributions. This was actually one of the key technical achievements in the work of Hairer and Labbé [7] for the Besov scale of modelled distributions. Hence, it is not much of a surprise that we will follow their ideas and constructions, adapting them to our setting.

DEFINITION 3.2. Let $(\mathcal{A},\mathcal{T},\mathcal{G})$ be a regularity structure and (Π,Γ) a model. Furthermore, let $\mathcal{E}_n=Q(0,2^{-n})\cap\Lambda_n\backslash\{0\}$. For $1\leq p<\infty$, $1\leq q\leq\infty$ and $\gamma>0$, we denote by $\bar{\mathcal{F}}^{\gamma}_{p,q}$ the (Banach) space of all sequences of maps

$$\bar{f}^{(n)}\colon\Lambda_n\to\mathcal{T}^{-}_{\gamma},\qquad n\geq 0,$$

such that, uniformly over all $\zeta\in\mathcal{A}_\gamma$,

(i) $\Big(\sum_{y\in\Lambda_0}|\bar{f}^{(0)}(y)|^p_\zeta\Big)^{1/p}<\infty$,

(ii) $$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}\frac{|\bar f^{(n)}(y+h)-\Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}<\infty,$$

(iii) $$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\frac{|\bar f^{(n)}(y)-\bar f^{(n+1)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}<\infty.$$

The associated norm for $\bar f\in\bar{\mathcal{F}}^\gamma_{p,q}$ will be denoted by $|\!|\!|\bar f|\!|\!|_{\bar{\mathcal{F}}^\gamma_{p,q}}$, and we will refer to the three bounds as *local bound*, *translation bound* and *consistency bound* respectively.

For $M\in\mathbb{N}$ we define $\mathcal{E}_n^{M,0}=Q(0,M\cdot 2^{-n})\cap\Lambda_n$ and $\mathcal{E}_n^M=\mathcal{E}_n^{M,0}\setminus\{0\}$. We then have the following additional bounds which will be of great use.

REMARK 3.3. For every $\bar f\in\bar{\mathcal{F}}^\gamma_{p,q}$ and every $M\in\mathbb{N}$, we have

$$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_{n+1}^{M,0}}\frac{|\bar f^{(n)}(y)-\Gamma_{y,y+h}\bar f^{(n+1)}(y+h)|_\zeta}{2^{-n(\gamma-\zeta)}}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}<\infty.$$

In fact, this quantity is bounded by $|\!|\!|\bar f|\!|\!|_{\bar{\mathcal{F}}^\gamma_{p,q}}$ up to some proportionality constant. For a proof, one can use induction on $M\in\mathbb{N}$ and reduce everything to the consistency and translation bounds.

REMARK 3.4. Let $\bar f\in\bar{\mathcal{F}}^\gamma_{p,q}$ and $M\in\mathbb{N}$. Then

$$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n^M}\frac{|\bar f^{(n)}(y+h)-\Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}\lesssim|\!|\!|\bar f|\!|\!|_{\bar{\mathcal{F}}^\gamma_{p,q}}.$$

This follows immediately from the translation bound, e.g. by induction on $M\in\mathbb{N}$.

REMARK 3.5. Let $1\leq q\leq\bar q\leq\infty$, $1\leq p<\infty$ and $\gamma>0$. Then the following elementary (continuous) embeddings hold true (cf. [10, Sec. 2.3.2]):

(i) $\bar{\mathcal{F}}^\gamma_{p,q}\subset\bar{\mathcal{F}}^\gamma_{p,\bar q}$,

(ii) $\bar{\mathcal{B}}^\gamma_{p,q\wedge p}\subset\bar{\mathcal{F}}^\gamma_{p,q}\subset\bar{\mathcal{B}}^\gamma_{p,q\vee p}$.

For the scale $\bar{\mathcal{B}}$, we refer to Definition 8.2. In this context we also use, for $n\geq 0$ and maps $u\colon\Lambda_n\to\mathbb{R}$, the notation

$$\|u\|_{\ell^p_n}:=\Big(\sum_{y\in\Lambda_n}2^{-n|\mathfrak{s}|}|u(y)|^p\Big)^{1/p}.$$

Indeed, (i) follows directly from $\ell^q\subset\ell^{\bar q}$. For (ii), the local bounds are immediate. The arguments for the translation and consistency bounds are almost identical; we only discuss the former. For this, fix $n\geq 0$, $h\in\mathcal{E}_n$ and $\zeta\in\mathcal{A}_\gamma$.

Due to the continuous embedding $\ell^1 \subset \ell^p$ we obtain the bound

$$\Big\| \frac{|\bar f^{(n)}(y+h) - \Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \Big\|_{\ell^p_n}^{q\vee p} \leq \Big(\int_{\mathbb{R}^d} \Big(\sum_{y\in\Lambda_n} \frac{|\bar f^{(n)}(y+h) - \Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \chi^n_y(x) \Big)^{(q\vee p)\frac{p}{q\vee p}} \mathrm{d}x \Big)^{\frac{q\vee p}{p}}.$$

Since $p/(q\vee p) \leq 1$, the space $L^{p/(q\vee p)}(\mathbb{R}^d)$ admits an inverse triangle inequality for *non-negative* functions. Combining this with the former bound immediately yields

$$\Big(\sum_{n\geq 0} \sum_{h\in\mathcal{E}_n} \Big\| \frac{|\bar f^{(n)}(y+h) - \Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \Big\|_{\ell^p_n}^{q\vee p} \Big)^{\frac{1}{q\vee p}} \leq \Big\| \Big(\sum_{n\geq 0} \sum_{h\in\mathcal{E}_n} \Big| \sum_{y\in\Lambda_n} \frac{|\bar f^{(n)}(y+h) - \Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \chi^n_y(x) \Big|^{q\vee p} \Big)^{\frac{1}{q\vee p}} \Big\|_{L^p}.$$

At this point, it suffices to exploit the continuous embedding $\ell^q \subset \ell^{q\vee p}$ and the fact that the sets $\mathcal{E}_n$ contain finitely many points, with cardinality bounded independently of $n \geq 0$. This proves the embedding $\bar{\mathcal{F}}^\gamma_{p,q} \subset \bar{\mathcal{B}}^\gamma_{p,q\vee p}$. The remaining one is an immediate consequence of the triangle inequality in $L^{p/(q\wedge p)}(\mathbb{R}^d)$.

Apart from these elementary embeddings, we also expect to have $\mathcal{F}^\gamma_{p,q} \subset \mathcal{F}^{\gamma'}_{p,q}$ whenever $\gamma' < \gamma$. In general, this may not be true if $q < \infty$ since the theory imposes Hölder type bounds on models (Π, Γ). Therefore, we make the following standing assumption.

ASSUMPTION 3.6. *For a given set $\mathcal{A}$ of homogeneities, we assume that $\gamma \notin \mathcal{A}$.*

REMARK 3.7. A noteworthy consequence of the embedding $\bar{\mathcal{F}}^\gamma_{p,q} \subset \bar{\mathcal{B}}^\gamma_{p,q\vee p}$ is that we can immediately propagate the local bound to smaller scales, i.e.

$$\sup_{n\geq 0} \big\| |\bar f^{(n)}(y)|_\zeta \big\|_{\ell^p_n} \lesssim \vert\kern-0.25ex\vert\kern-0.25ex\vert \bar f \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\bar{\mathcal{F}}^\gamma_{p,q}},$$

uniformly over all $\bar f \in \bar{\mathcal{F}}^\gamma_{p,q}$ and all $\zeta \in \mathcal{A}_\gamma$ [7, Lemma 2.13]. On the other hand, adapting the arguments in [7, proof of Lemma 2.13] and working directly with the scale $\bar{\mathcal{F}}^\gamma_{p,q}$ shows that we actually have the improved version

$$\Big\| \sup_{n\geq 0} \sum_{y\in\Lambda_n} \sum_{h\in\mathcal{E}_n} |\bar f^{(n)}(y+h)|_\zeta \chi^n_y(x) \Big\|_{L^p} \lesssim \vert\kern-0.25ex\vert\kern-0.25ex\vert \bar f \vert\kern-0.25ex\vert\kern-0.25ex\vert_{\bar{\mathcal{F}}^\gamma_{p,q}}, \tag{3.1}$$

uniformly over all $\bar f \in \bar{\mathcal{F}}^\gamma_{p,q}$ and all $\zeta \in \mathcal{A}_\gamma$.

Proof of (3.1). First of all, due to Assumption 3.6 and [7, proof of Lemma 2.13] it suffices to prove the desired bound for $\zeta = \max \mathcal{A}_\gamma$. We fix $n \geq 0$, $y' \in \Lambda_n$ and $x \in Q^n_{y'}$. Then

$$\sum_{y\in\Lambda_{n+1}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n+1)}(y+h)|_\zeta \chi^{n+1}_y(x) = \sum_{\substack{y\in\Lambda_{n+1}\\ y\in Q^n_{y'}}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n+1)}(y+h)|_\zeta \chi^{n+1}_y(x).$$

Recall the action of the scaling map

$$\mathcal{S}^\lambda_\mathfrak{s} x' := (\lambda^{-\mathfrak{s}_1} x'_1, \ldots, \lambda^{-\mathfrak{s}_d} x'_d), \qquad x' \in \mathbb{R}^d,\ \lambda > 0.$$

For any given $x' \in \mathbb{R}^d$ and $m \in \mathbb{N}$, we denote by $z^m_{x'} = y' \in \Lambda_m$ the unique lattice point with $x' \in Q^m_{y'}$. Now, we bound

$$\begin{aligned}
\sum_{\substack{y\in\Lambda_{n+1}\\ y\in Q^n_{y'}}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n+1)}(y+h)|_\zeta \chi^{n+1}_y(x)
&\leq \sum_{\substack{y\in\Lambda_{n+1}\\ y\in Q^n_{y'}}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n)}(z^n_{y+\mathcal{S}^{2^{-1}}_\mathfrak{s} h})|_\zeta \chi^{n+1}_y(x)\\
&\quad + \sum_{\substack{y\in\Lambda_{n+1}\\ y\in Q^n_{y'}}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n+1)}(y+h) - \bar{f}^{(n)}(z^n_y)|_\zeta \chi^{n+1}_y(x)\\
&\quad + \sum_{\substack{y\in\Lambda_{n+1}\\ y\in Q^n_{y'}}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n)}(z^n_y) - \bar{f}^{(n)}(z^n_{y+\mathcal{S}^{2^{-1}}_\mathfrak{s} h})|_\zeta \chi^{n+1}_y(x).
\end{aligned}$$

Observe that for $y \in \Lambda_{n+1} \cap Q^n_{y'}$ and $h \in \mathcal{E}_{n+1}$ we have $z^n_y = y'$ and $z^n_{y+\mathcal{S}^{2^{-1}}_\mathfrak{s} h} = y' + \mathcal{S}^{2^{-1}}_\mathfrak{s} h$. Moreover, $h \mapsto y' + \mathcal{S}^{2^{-1}}_\mathfrak{s} h$ defines a bijection $\mathcal{E}_{n+1} \to \{y'\} + \mathcal{E}_n$. Hence,

$$\begin{aligned}
\sum_{\substack{y\in\Lambda_{n+1}\\ y\in Q^n_{y'}}} \sum_{h\in\mathcal{E}_{n+1}} |\bar{f}^{(n+1)}(y+h)|_\zeta \chi^{n+1}_y(x) &\leq \sum_{h\in\mathcal{E}_n} |\bar{f}^{(n)}(y'+h)|_\zeta \chi^n_{y'}(x)\\
&\quad + \sum_{h\in\mathcal{E}^{2,0}_{n+1}} |\Gamma_{y',y'+h} \bar{f}^{(n+1)}(y'+h) - \bar{f}^{(n)}(y')|_\zeta \chi^n_{y'}(x)\\
&\quad + \sum_{h\in\mathcal{E}_n} |\Gamma_{y'+h,y'} \bar{f}^{(n)}(y') - \bar{f}^{(n)}(y'+h)|_\zeta \chi^n_{y'}(x).
\end{aligned}$$

Since this bound holds true uniformly over all $n \geq 0$, all $y' \in \Lambda_n$ and all

$x \in Q^n_{y'}$ we infer with the use of Remark 3.3 that

$$\Big\|\sup_{n\geq 0}\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}|\bar f^{(n)}(y+h)|_\zeta\chi^n_y(x)\Big\|_{L^p}$$
$$\leq \Big\|\sum_{y\in\Lambda_0}\sum_{h\in\mathcal{E}_0}|\bar f^{(0)}(y+h)|_\zeta\chi^0_y(x)\Big\|_{L^p} + C\|(2^{-n(\gamma-\zeta)})_n\|_{\ell^{\frac{q}{q-1}}}\, |||\bar f|||_{\bar{\mathcal{F}}^\gamma_{p,q}}$$

for some absolute constant $C > 0$. ∎

The remainder of this section is devoted to the discussion of how the spaces $\mathcal{F}^\gamma_{p,q}$ and $\bar{\mathcal{F}}^\gamma_{p,q}$ are related to each other. We first show how to obtain from a given $f \in \mathcal{F}^\gamma_{p,q}$ an associated element $\bar f \in \bar{\mathcal{F}}^\gamma_{p,q}$. This is settled in the first part of the statement below. On the other hand, to go the other way around, we would like to ensure that both the resulting maps between $\mathcal{F}^\gamma_{p,q}$ and $\bar{\mathcal{F}}^\gamma_{p,q}$ are consistent. We take care of this in the second part of the statement.

Proposition 3.8. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\gamma > 0$.*

(i) *Given $f \in \mathcal{F}^\gamma_{p,q}$, for all $n \geq 0$ and $y \in \Lambda_n$ define*

$$\bar f^{(n)}(y) = \fint_{Q(y,2^{-n})} \Gamma_{y,z} f(z)\,\mathrm{d}z. \tag{3.2}$$

Then $\bar f \in \bar{\mathcal{F}}^\gamma_{p,q}$ and $|||\bar f|||_{\bar{\mathcal{F}}^\gamma_{p,q}} \lesssim |||f|||_{\mathcal{F}^\gamma_{p,q}}$.

(ii) *Let $f \in \mathcal{F}^\gamma_{p,q}$ and define, for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$,*

$$f_n(x) = \Gamma_{x,x_n}\bar f^{(n)}(x_n), \tag{3.3}$$

where $\bar f^{(n)}$ is defined in (i). *Then $\mathcal{Q}_\zeta f_n \to \mathcal{Q}_\zeta f$ in $L^p(\mathbb{R}^d)$ for all $\zeta \in \mathcal{A}_\gamma$.*

Proof. (i) The local bound holds immediately. For the translation bound, we fix $x \in \mathbb{R}^d$, $n \geq 0$, $h \in \mathcal{E}_n$ and $y = x_n \in \Lambda_n$. Then

$$\begin{aligned}
\frac{|\bar f^{(n)}(y+h) - \Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} &\lesssim \sum_{\beta\geq\zeta}\fint_{Q(y,2^{-n})}\frac{|f(z+h)-\Gamma_{z+h,z}f(z)|_\beta}{2^{-n(\gamma-\beta)}}\,\mathrm{d}z\\
&\lesssim \sum_{\beta\geq\zeta}\fint_{Q(x,2\cdot 2^{-n})}\frac{|f(z+h)-\Gamma_{z+h,z}f(z)|_\beta}{2^{-n(\gamma-\beta)}}\,\mathrm{d}z\\
&\lesssim \sum_{\beta\geq\zeta}\fint_{Q(0,2\cdot 2^{-n})}\frac{|f(x+z+h)-\Gamma_{x+z+h,x}f(x)|_\beta}{2^{-n(\gamma-\beta)}}\,\mathrm{d}z\\
&\quad+\sum_{\beta\geq\zeta}\fint_{Q(0,2\cdot 2^{-n})}\frac{|\Gamma_{x+z+h,x+z}(f(x+z)-\Gamma_{x+z,x}f(x))|_\beta}{2^{-n(\gamma-\beta)}}\,\mathrm{d}z\\
&\lesssim \sum_{\delta\geq\beta}\sum_{\beta\geq\zeta}\fint_{Q(0,3\cdot 2^{-n})}\frac{|f(x+z')-\Gamma_{x+z',x}f(x)|_\delta}{2^{-n(\gamma-\delta)}}\,\mathrm{d}z'.
\end{aligned}$$

This immediately yields the required bound. As for the consistency bound, it is straightforward to verify that

$$|\bar{f}^{(n)}(y) - \bar{f}^{(n+1)}(y)|_\zeta \lesssim \sum_{\beta\geq\zeta} 2^{-n(\beta-\zeta)} \fint_{Q(0,2^{-n})} |\Delta_{\Gamma,h} f(x)|_\beta \,\mathrm{d}h,$$

from which the desired bound follows at once.

(ii) is an immediate consequence of the fact that for a.e. $x \in \mathbb{R}^d$,

$$|f(x) - f_n(x)|_\zeta \lesssim \sum_{\beta\geq\zeta} 2^{-n(\gamma-\zeta)} \fint_{Q(0,2\cdot 2^{-n})} \frac{|\Delta_{\Gamma,h} f(x)|_\beta}{2^{-n(\gamma-\beta)}} \,\mathrm{d}h. \quad \blacksquare$$

In a second step, we show how to recover a modelled distribution from an element of the discrete counterpart $\bar{\mathcal{F}}^\gamma_{p,q}$.

Proposition 3.9. *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\gamma > 0$. Let $\bar{f} \in \bar{\mathcal{F}}^\gamma_{p,q}$ and define, for $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_0$,*

$$f_n(x) = \Gamma_{x,x_n} \bar{f}^{(n)}(x_n).$$

Then, for all $\zeta \in \mathcal{A}_\gamma$, the sequence $(\mathcal{Q}_\zeta f_n)_{n\in\mathbb{N}}$ is Cauchy in $L^p(\mathbb{R}^d)$. Furthermore, its limit $f\colon \mathbb{R}^d \to \mathcal{T}^-_\gamma$ is in $\mathcal{F}^\gamma_{p,q}$ with $|||f|||_{\mathcal{F}^\gamma_{p,q}} \lesssim |||\bar{f}|||_{\bar{\mathcal{F}}^\gamma_{p,q}}$.

Proof. To prove the convergence, fix $x \in \mathbb{R}^d$ and $n \geq 0$. We then bound $|f_n(x) - f_{n+1}(x)|_\zeta$ by

$$\begin{aligned} (3.4) \qquad & \big|\Gamma_{x,x_n}\big(\bar{f}^{(n)}(x_n) - \Gamma_{x_n,x_{n+1}} \bar{f}^{(n+1)}(x_{n+1})\big)\big|_\zeta \\ &\qquad \lesssim \sum_{\beta\geq\zeta} 2^{-n(\beta-\zeta)} |\bar{f}^{(n)}(x_n) - \Gamma_{x_n,x_{n+1}} \bar{f}^{(n+1)}(x_{n+1})|_\beta. \end{aligned}$$

Next, we make use of Hölder's inequality and Remark 3.3 to deduce that

$$\Big\| \sum_{n\geq n_0} \frac{|\bar{f}^{(n)}(x_n) - \Gamma_{x_n,x_{n+1}} \bar{f}^{(n+1)}(x_{n+1})|_\beta}{2^{n(\beta-\zeta)}} \Big\|_{L^p} \lesssim 2^{-n_0(\gamma-\zeta)} |||\bar{f}|||_{\bar{\mathcal{F}}^\gamma_{p,q}}.$$

Thus, for all $\zeta \in \mathcal{A}_\gamma$, the sequence $(\mathcal{Q}_\zeta f_n)_{n\in\mathbb{N}}$ is indeed Cauchy in $L^p(\mathbb{R}^d)$. Let us denote its limit by $f\colon \mathbb{R}^d \to \mathcal{T}^-_\gamma$. It remains to show that $f \in \mathcal{F}^\gamma_{p,q}$, and that the asserted bound holds.

The local bound is part of the argument above. For the translation bound, we start with

$$\Big\| \Big\| \fint_{Q(0,4\lambda)} \frac{|\Delta_{\Gamma,h} f(x)|_\zeta}{\lambda^{\gamma-\zeta}} \,\mathrm{d}h \Big\|_{L^q_\lambda} \Big\|_{L^p} \lesssim \Big\| \Big(\sum_{n\geq 0} \Big| \fint_{Q(0,4\cdot 2^{-n})} \frac{|\Delta_{\Gamma,h} f(x)|_\zeta}{2^{-n(\gamma-\zeta)}} \,\mathrm{d}h \Big|^q \Big)^{1/q} \Big\|_{L^p}.$$

Fix $x \in \mathbb{R}^d$, $n \geq 0$ and $h \in Q(0, 4\cdot 2^{-n})$. Following [7], we employ the following decomposition of $\Delta_{\Gamma,h} f(x)$:

$$(3.5) \qquad \big(f(x+h) - f_n(x+h)\big) + \Delta_{\Gamma,h} f_n(x) + \Gamma_{x+h,x}\big(f_n(x) - f(x)\big).$$

We treat each term separately; let us begin with the second term. We find $h' \in \mathcal{E}_n^{M,0}$, $M = 4$, such that $(x+h)_n = x_n + h'$. Observe that

$$\Delta_{\Gamma,h} f_n(x) = \Gamma_{x+h,x_n+h'}\big(\bar{f}^{(n)}(x_n+h') - \Gamma_{x_n+h',x_n}\bar{f}^{(n)}(x_n)\big).$$

If $h' = 0$ the right hand side vanishes, i.e. we obtain

$$\Big|\fint_{Q(0,4\cdot 2^{-n})} \frac{|\Delta_{\Gamma,h} f_n(x)|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h\Big| \lesssim \sum_{\beta\geq\zeta}\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n^M} \frac{|\bar{f}^{(n)}(y+h') - \Gamma_{y+h',y}\bar{f}^{(n)}(y)|_\beta}{2^{-n(\gamma-\beta)}}\chi_y^n(x).$$

Therefore, it remains to make use of Remark 3.4 which yields a bound of required order. For the third term in (3.5), we can estimate

$$\begin{aligned}\fint_{Q(0,4\cdot 2^{-n})} \frac{|\Gamma_{x+h,x}(f_n(x) - f(x))|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h &\lesssim \sum_{\beta\geq\zeta}\sum_{m\geq n}\frac{|f_m(x) - f_{m+1}(x)|_\beta}{2^{-n(\gamma-\beta)}}\\ &\lesssim \sum_{\delta\geq\beta}\sum_{\beta\geq\zeta}\sum_{m\geq n}\sum_{y\in\Lambda_m}\sum_{h\in\mathcal{E}_{m+1}^0}\frac{|\bar{f}^{(m)}(y) - \Gamma_{y,y+h}\bar{f}^{(m+1)}(y+h)|_\delta}{2^{-(n-m)(\gamma-\beta)}2^{-m(\gamma-\delta)}}\chi_y^m(x).\end{aligned}$$

Young's inequality for convolutions shows that

$$\begin{aligned}&\Big\|\Big(\sum_{n\geq 0}\Big|\fint_{Q(0,4\cdot 2^{-n})}\frac{|\Gamma_{x+h,x}(f_n(x) - f(x))|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h\Big|^q\Big)^{1/q}\Big\|_{L^p}\\ &\quad\lesssim \sum_{\delta\geq\zeta}\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_{n+1}^0}\frac{|\bar{f}^{(n)}(y) - \Gamma_{y,y+h}\bar{f}^{(n+1)}(y+h)|_\delta}{2^{-n(\gamma-\delta)}}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}.\end{aligned}$$

Thus, Remark 3.3 finally ensures that a required bound holds.

Let us finally discuss the first term from (3.5). To this end, choose $0 < \kappa < 1$ such that $\gamma' := \gamma - \zeta + |\mathfrak{s}|(\kappa - 1)/\kappa > 0$. Then

$$\begin{aligned}&\fint_{Q(0,4\cdot 2^{-n})}\frac{|f_n(x+h) - f(x+h)|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h\\ &\quad\lesssim \sum_{m\geq n}\fint_{Q(0,4\cdot 2^{-n})}\frac{|f_m(x+h) - f_{m+1}(x+h)|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h\\ &\quad\lesssim \sum_{\beta\geq\zeta}\sum_{m\geq n}\sum_{\substack{y\in\Lambda_m\\ \|y-x\|_\mathfrak{s}\leq C2^{-n}}}\sum_{h\in\mathcal{E}_{m+1}^0}\frac{|\bar{f}^{(m)}(y) - \Gamma_{y,y+h}\bar{f}^{(m+1)}(y+h)|_\beta}{2^{-(n-m)|\mathfrak{s}|}2^{-(n-m)(\gamma-\zeta)}2^{-m(\gamma-\beta)}}.\end{aligned}$$

Each term in the sum over $m \geq n$ can be bounded by

$$2^{(n-m)\gamma'} \Big\{ \mathcal{M}\Big(\Big| \sum_{y\in\Lambda_m} \sum_{h\in\mathcal{E}^0_{m+1}} \frac{|\bar f^{(m)}(y) - \Gamma_{y,y+h}\bar f^{(m+1)}(y+h)|_\beta}{2^{-m(\gamma-\beta)}} \chi^m_y \Big|^\kappa \Big)(x) \Big\}^{1/\kappa}.$$

Now, apply first Young's inequality for convolutions and then the vector-valued maximal inequality; then it remains to resort another time to the bound of Remark 3.3 to conclude the argument. ∎

Remark 3.10. As a consequence of the bounds in Propositions 3.8 and 3.9, the elementary embeddings of Remark 3.5 carry over immediately to the spaces $\mathcal{F}^\gamma_{p,q}$ and $\mathcal{B}^\gamma_{p,q}$. More elaborate embeddings follow in the next section.

4. Embedding theorems. It is the aim of this section to provide embeddings for (and between) the two scales $\mathcal{F}^\gamma_{p,q}$ and $\mathcal{B}^\gamma_{p,q}$. Embeddings for the scale of Besov spaces were already proven in [7, Sec. 4]. Therefore, we focus on statements which involve Triebel–Lizorkin modelled distributions. Recall that in the last section we have already proven the embeddings

$$\mathcal{F}^\gamma_{p,q} \subset \mathcal{F}^\gamma_{p,\bar q}, \qquad \mathcal{B}^\gamma_{p,q\wedge p} \subset \mathcal{F}^\gamma_{p,q} \subset \mathcal{B}^\gamma_{p,q\vee p}.$$

Here, $1 \leq q \leq \bar q \leq \infty$, $1 \leq p < \infty$ and $\gamma > 0$. The main result of this section provides further embeddings (we remind the reader of Assumption 3.6).

Proposition 4.1. *Let $0 < \gamma \leq \bar\gamma$, $1 \leq q, \bar q, r \leq \infty$ and $1 \leq p, \bar p < \infty$. Then*

(i) $\mathcal{F}^{\bar\gamma}_{p,\bar q} \subset \mathcal{F}^\gamma_{p,q}$ *if* $q < \bar q$ *and* $\gamma < \bar\gamma$,
(ii) $\mathcal{F}^{\bar\gamma}_{\bar p,q} \subset \mathcal{F}^\gamma_{p,r}$ *if* $\bar p < p$ *and* $\gamma < \bar\gamma - |\mathfrak{s}|(1/\bar p - 1/p)$,
(iii) $\mathcal{F}^{\bar\gamma}_{\bar p,q} \subset \mathcal{B}^\gamma_{p,\bar p}$ *if* $\bar p < p$ *and* $\gamma < \bar\gamma - |\mathfrak{s}|(1/\bar p - 1/p)$.

Remark 4.2. The first embedding is the pendant of the corresponding embedding for the Besov scale of modelled distributions from [7]. The other two are analogues of well known embeddings for the classical Triebel–Lizorkin scale due to Jawerth [8].

Before we dive into the proof, we provide some helpful results. To obtain the embedding $\mathcal{F}^{\bar\gamma}_{\bar p,q} \subset \mathcal{F}^\gamma_{p,r}$, we will rely on the following bound.

Lemma 4.3. *Let $1 \leq p_1 < p < \infty$, $1 \leq q, r \leq \infty$ and $0 < \delta \leq \delta_1$ be such that $\delta \leq \delta_1 - |\mathfrak{s}|(1/p_1 - 1/p)$. Furthermore, let $u_n\colon \Lambda_n \to \mathbb{R}$, $n \geq 0$, be a sequence of functions. Then*

$$\Big\| \Big\| \sum_{y\in\Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y(x) \Big\|_{\ell^r} \Big\|_{L^p} \lesssim \Big\| \Big\| \sum_{y\in\Lambda_n} \frac{u_n(y)}{2^{-n\delta_1}} \chi^n_y(x) \Big\|_{\ell^q} \Big\|_{L^{p_1}}.$$

Proof. We use an interpolation trick of Jawerth (cf. also Triebel [10, Sec. 2.7.1]). We first remark that we may assume without loss of generality that the right hand side equals 1. Let us also recall some notation: for a measurable set $M \subset \mathbb{R}^d$ we denote by $|M|$ its Lebesgue measure, and if h is a measurable function, its distribution function is $\lambda_h(t) = |\{x \in \mathbb{R}^d : h(x) > t\}|$, where $t > 0$.

We observe first that

$$\sup_{y \in \Lambda_n} \frac{|u_n(y)|}{2^{-n\delta}} \leq 2^{n|\mathfrak{s}|/p} \left\| \frac{u_n(y)}{2^{-n\delta}} \right\|_{\ell^p_n} \leq 2^{n|\mathfrak{s}|/p},$$

uniformly over $n \geq 0$. In particular, there exists $K > 0$ such that

$$\left(\sum_{n=0}^{N} \left| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y(x) \right|^r \right)^{1/r} \leq K^{1/r} 2^{N|\mathfrak{s}|/p}, \tag{4.1}$$

and

$$\left(\sum_{n \geq N+1} \left| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y(x) \right|^r \right)^{1/r} \leq K^{1/r} 2^{N(\delta - \delta_1)} \sup_{n \geq 0} \left| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta_1}} \chi^n_y(x) \right|, \tag{4.2}$$

uniformly over all $N \geq -1$ and all $x \in \mathbb{R}^d$. Next,

$$\left\| \left\| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y(x) \right\|_{\ell^r} \right\|^p_{L^p} \lesssim \sum_{\substack{k \in \mathbb{Z} \\ 2^k \leq 2(2K)^{1/r}}} 2^{kp} \lambda_{\| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y \|_{\ell^r}}(2^k) + \sum_{\substack{k \in \mathbb{Z} \\ 2^k > 2(2K)^{1/r}}} 2^{kp} \lambda_{\| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y \|_{\ell^r}}(2^k).$$

In the following, we will bound each of the two sums on the right hand side separately. For the first sum, due to (4.2) with $N = -1$ we have, for some $c > 0$,

$$\sum_{\substack{k \in \mathbb{Z} \\ 2^k \leq 2(2K)^{1/r}}} 2^{kp} \lambda_{\| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta}} \chi^n_y \|_{\ell^r}}(2^k) \lesssim \sum_{\substack{k \in \mathbb{Z} \\ 2^k \leq c2(2K)^{1/r}}} 2^{kp_1} \lambda_{\sup_{n \geq 0} | \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta_1}} \chi^n_y |}(2^k)$$
$$\lesssim \sum_{k \in \mathbb{Z}} 2^{kp_1} \lambda_{\| \sum_{y \in \Lambda_n} \frac{u_n(y)}{2^{-n\delta_1}} \chi^n_y \|_{\ell^q}}(2^k) \lesssim 1,$$

which is a bound of required order. For the second sum, we first consider for each $k \in \mathbb{Z}$ with $2^k > 2(2K)^{1/r}$ the uniquely determined integer $N(k) \in \mathbb{N}$ that is maximal with respect to $2(2K)^{1/r} 2^{N(k)|\mathfrak{s}|/p} \leq 2^k$. Then due to (4.1) and (4.2),

$$\sum_{\substack{k\in\mathbb{Z}\\ 2^k>2(2K)^{1/r}}} 2^{kp}\lambda_{\|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta}}\chi_y^n\|_{\ell^r}}(2^k)$$

$$\le \sum_{\substack{k\in\mathbb{Z}\\ 2^k>2(2K)^{1/r}}} 2^{kp}\bigg|\bigg\{x\in\mathbb{R}^d\colon 2^{1/r}\Big(\sum_{n\ge N(k)+1}\Big|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta}}\chi_y^n(x)\Big|^r\Big)^{1/r} + 2^{1/r}\Big(\sum_{n\le N(k)}\Big|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta}}\chi_y^n(x)\Big|^r\Big)^{1/r} > 2^k\bigg\}\bigg|$$

$$\le \sum_{\substack{k\in\mathbb{Z}\\ 2^k>2(2K)^{1/r}}} 2^{kp}\bigg|\bigg\{x\in\mathbb{R}^d\colon (2K)^{1/r}2^{N(k)(\delta-\delta_1)}\sup_{n\ge0}\Big|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta_1}}\chi_y^n(x)\Big| + (2K)^{1/r}2^{N(k)|\mathfrak{s}|/p} > 2^k\bigg\}\bigg|$$

$$\le \sum_{\substack{k\in\mathbb{Z}\\ 2^k>2(2K)^{1/r}}} (2^{p/p_1})^{kp_1}\bigg|\bigg\{x\in\mathbb{R}^d\colon (2K)^{1/r}2^{N(k)(\delta-\delta_1)} \times \sup_{n\ge0}\Big|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta_1}}\chi_y^n(x)\Big| > 2^{k-1}\bigg\}\bigg|.$$

Since $c'2^k \le 2(2K)^{1/r}2^{N(k)|\mathfrak{s}|/p} \le 2^k$ for some $c'>0$ and since $1\le p_1<p<\infty$ and $\delta\le\delta_1-|\mathfrak{s}|(1/p_1-1/p)$, we may further estimate

$$\bigg|\bigg\{x\in\mathbb{R}^d\colon (2K)^{1/r}2^{N(k)(\delta-\delta_1)}\sup_{n\ge0}\Big|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta_1}}\chi_y^n(x)\Big| > 2^k\bigg\}\bigg|$$
$$\le \bigg|\bigg\{x\in\mathbb{R}^d\colon \sup_{n\ge0}\Big|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta_1}}\chi_y^n(x)\Big| > 2^{N(k)(|\mathfrak{s}|/p-(\delta-\delta_1))}\bigg\}\bigg|$$
$$\le \lambda_{c''\sup_{n\ge0}|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta_1}}\chi_y^n|}((2^{p/p_1})^k).$$

Combining the last two bounds we infer that

$$\sum_{\substack{k\in\mathbb{Z}\\ 2^k>2(2K)^{1/r}}} 2^{kp}\lambda_{\|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta}}\chi_y^n\|_{\ell^r}}(2^k) \le \sum_{k\in\mathbb{Z}}(2^{p/p_1})^{kp_1}\lambda_{c''\sup_{n\ge0}|\sum_{y\in\Lambda_n}\frac{u_n(y)}{2^{-n\delta_1}}\chi_y^n|}((2^{p/p_1})^k)$$

which is again of required order. ■

To prove the embedding $\mathcal{F}^{\bar\gamma}_{\bar p,q}\subset\mathcal{B}^{\gamma}_{p,\bar p}$ we make use of the real interpolation method based on Peetre's K-functional. Let us introduce some notation. For two real Banach spaces X and Y which are both continuously embedded into another real Banach space Z (in our case of interest, $Z = L^p$), we define the functional

$$K(t,z) := \inf_{\substack{z=x+y \\ x\in X,\, y\in Y}} (\|x\|_X + t\|y\|_X).$$

For $0 < \theta < 1$ and $1 \le q \le \infty$, we then denote by $(X,Y)_{\theta,q}$ the subset of all elements $z \in X + Y$ such that

$$\|z\|_{(X,Y)_{\theta,q}} := \left(\int_0^\infty \left|\frac{K(t,z)}{t^\theta}\right|^q \frac{\mathrm{d}t}{t}\right)^{1/q} < \infty.$$

It is well known that the space $(X,Y)_{\theta,q}$ with the norm $\|\cdot\|_{(X,Y)_{\theta,q}}$ is a real Banach space.

Proof of Proposition 4.1. Due to Propositions 3.8 and 3.9 it suffices to establish the asserted embeddings on the level of the sequence spaces $\bar{\mathcal{F}}$ and $\bar{\mathcal{B}}$.

(i) Here, a straightforward modification of the proof of the second case of [7, Thm. 4.1] also works for the Triebel–Lizorkin scale of modelled distributions. We leave the details to the interested reader.

(ii) Put $\bar\zeta = \max \mathcal{A}_{\bar\gamma}$ and first consider the case $\gamma \in (\bar\zeta, \bar\gamma)$. For $\zeta \in \mathcal{A}_\gamma$, we define the functions

$$u_n^\zeta(y) = \sum_{h\in\mathcal{E}_n} |\bar f^{(n)}(y+h) - \Gamma_{y+h,y}\bar f^{(n)}(y)|_\zeta, \qquad n \ge 0,\, y \in \Lambda_n.$$

As the local bound is an immediate consequence of $\ell^{\bar p} \subset \ell^p$, the asserted embedding follows directly from Lemma 4.3. For what follows, it is crucial that this result holds true even if $\gamma = \bar\gamma - |\mathfrak{s}|(1/\bar p - 1/p)$.

Now, let us turn to the case $\gamma \in (\underline{\zeta}, \bar\zeta)$, where $\underline{\zeta} = \max(\mathcal{A}_{\bar\gamma} \setminus \{\bar\zeta\})$. Before proving the required bounds, let us remark that after having established this particular case, one obtains the general statement by recursion over the (finite) set of homogeneities $\mathcal{A}_{\bar\gamma}$.

Now, to verify the embedding for $\gamma \in (\underline{\zeta}, \bar\zeta)$, we follow the argument in [7]. We consider the case where $\bar\gamma - |\mathfrak{s}|(1/\bar p - 1/p) \le \bar\zeta$. Then for every $\varepsilon \ge 0$ such that $\bar\zeta + \varepsilon < \bar\gamma$ we denote by $p_{\bar\zeta^\varepsilon} \in [\bar p, p]$ the unique number with

$$\bar\zeta + \varepsilon = \bar\gamma - |\mathfrak{s}|(1/\bar p - 1/p_{\bar\zeta^\varepsilon}).$$

We also define $p_{\bar\zeta} = p_{\bar\zeta^0}$ and choose $\gamma' < \bar\zeta$ such that $\gamma \le \gamma' - |\mathfrak{s}|(1/p_{\bar\zeta} - 1/p)$. Then it suffices to show that

$$\|\mathcal{Q}_{<\gamma}\bar f\|_{\bar{\mathcal{F}}^{\gamma'}_{p_{\bar\zeta},r}} \lesssim \|\bar f\|_{\bar{\mathcal{F}}^{\bar\gamma}_{\bar p,q}}. \tag{4.3}$$

Indeed, by the assumptions on γ' the result then follows from the previous calculation. To prove (4.3) it suffices to bound

$$\|\!|\mathcal{Q}_{<\gamma}\bar{f}|\!\|_{\bar{\mathcal{F}}^{\gamma'}_{p_{\bar{\zeta}^\varepsilon},r}} \lesssim \|\!|\bar{f}|\!\|_{\bar{\mathcal{F}}^{\bar{\gamma}}_{\bar{p},q}}, \tag{4.4}$$

uniformly over all $\varepsilon > 0$ with $\bar{\zeta}+\varepsilon < \bar{\gamma}$. Indeed, let $\gamma'_\varepsilon = \gamma' - |\mathfrak{s}|(1/p_{\bar{\zeta}^\varepsilon} - 1/p_{\bar{\zeta}})$. Then, for sufficiently small $\varepsilon > 0$ we have $\mathcal{A}_{\gamma'_\varepsilon} = \mathcal{A}_{\gamma'}$ and by an application of Fatou's Lemma as well as the first case we have

$$\|\!|\mathcal{Q}_{<\gamma}\bar{f}|\!\|_{\bar{\mathcal{F}}^{\gamma'}_{p_{\bar{\zeta}},r}} \le \varliminf_{\varepsilon\to 0} \|\!|\mathcal{Q}_{<\gamma}\bar{f}|\!\|_{\bar{\mathcal{F}}^{\gamma'_\varepsilon}_{p_{\bar{\zeta}},r}} \lesssim \varliminf_{\varepsilon\to 0} \|\!|\mathcal{Q}_{<\gamma}\bar{f}|\!\|_{\bar{\mathcal{F}}^{\gamma'}_{p_{\bar{\zeta}^\varepsilon},r}},$$

with a proportionality constant of required order, i.e. the bound in (4.3) follows. Thus, it remains to prove (4.4). To this end, we first make use of

$$\begin{aligned} |\mathcal{Q}_{<\gamma}\bar{f}^{(n)}(y+h) - \Gamma_{y+h,h}\mathcal{Q}_{<\gamma}\bar{f}^{(n)}(y)|_\zeta \\ \le |\bar{f}^{(n)}(y+h) - \Gamma_{y+h,h}\bar{f}^{(n)}(y)|_\zeta + |\Gamma_{y+h,y}\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(n)}(y)|_\zeta, \end{aligned} \tag{4.5}$$

which holds for all $\zeta \in \mathcal{A}_{\gamma'}$. Of course, for the first term on the right hand side we again simply apply Lemma 4.3. For the second quantity we have

$$\begin{aligned} &\Big\|\Big(\sum_{n\ge 0}\sum_{h\in\mathcal{E}_n}\Big|\sum_{y\in\Lambda_n}\frac{|\Gamma_{y+h,y}\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(n)}(y)|_{\bar{\zeta}}}{2^{-n(\gamma'-\zeta)}}\chi^n_y(x)\Big|^r\Big)^{1/r}\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}} \\ &\qquad\lesssim \Big\|\sup_{n\ge 0}\sum_{y\in\Lambda_n}|\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(n)}(y)|_{\bar{\zeta}}\chi^n_y(x)\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}}\|(2^{n(\gamma'-\bar{\zeta})})_n\|_{\ell^r} \\ &\qquad\lesssim \Big\|\sup_{n\ge 0}\sum_{y\in\Lambda_n}|\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(n)}(y)|_{\bar{\zeta}}\chi^n_y(x)\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}}. \end{aligned}$$

Now, as in the proof of Remark 3.7 there exists a constant $K > 0$ such that

$$\begin{aligned} \Big\|\sup_{n\ge 0}\sum_{y\in\Lambda_n}|\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(n)}(y)|_{\bar{\zeta}}\chi^n_y(x)\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}} &\le \Big\|\sum_{y\in\Lambda_0}|\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(0)}(y)|_{\bar{\zeta}}\chi^0_y(x)\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}} \\ &+ K\Big\|\Big(\sum_{n\ge 0}\sum_{h\in\mathcal{E}^0_{n+1}}\Big|\sum_{y\in\Lambda_n}\frac{|\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(n)}(y)|_{\bar{\zeta}}}{2^{-n\varepsilon}}\chi^n_y(x)\Big|^r\Big)^{1/r}\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}}. \end{aligned}$$

Hence, with the help of Remark 3.3 and Lemma 4.3 we obtain

$$\Big\|\sup_{n\ge 0}\sum_{y\in\Lambda_n}|\mathcal{Q}_{\bar{\zeta}}\bar{f}^{(n)}(y)|_{\bar{\zeta}}\chi^n_y(x)\Big\|_{L^{p_{\bar{\zeta}^\varepsilon}}} \lesssim \|\!|\bar{f}|\!\|_{\bar{\mathcal{F}}^{\bar{\gamma}}_{\bar{p},q}},$$

uniformly over the relevant data.

It remains to consider the case $\bar{\gamma} - |\mathfrak{s}|(1/\bar{p} - 1/p) > \bar{\zeta}$, where still $\gamma \in (\underline{\zeta}, \bar{\zeta})$. Here, we find $\varepsilon > 0$ such that $\bar{\zeta} + \varepsilon < \bar{\gamma} - |\mathfrak{s}|(1/\bar{p} - 1/p)$. Then, again making use of (4.5) and slightly modifying the argument shows that

$$\|\!|\mathcal{Q}_{<\gamma}\bar{f}|\!\|_{\bar{\mathcal{F}}^{\gamma}_{p,r}} \lesssim \|\!|\bar{f}|\!\|_{\bar{\mathcal{F}}^{\bar{\gamma}}_{\bar{p},q}}.$$

This concludes the discussion of the second embedding.

(iii) First, choose $\varepsilon > 0$ such that $\gamma \pm \varepsilon$ satisfy Assumption 3.6, $\mathcal{A}_\gamma = \mathcal{A}_{\gamma\pm\varepsilon}$ and $\gamma \pm \varepsilon < \bar{\gamma} - |\mathfrak{s}|(1/\bar{p} - 1/p)$. We already know that $\mathcal{F}^{\bar{\gamma}}_{\bar{p},\infty} \subset \mathcal{F}^{\gamma\pm\varepsilon}_{p,\infty} \subset \mathcal{B}^{\gamma\pm\varepsilon}_{p,\infty}$, so by real interpolation

$$(\mathcal{F}^{\bar{\gamma}}_{\bar{p},\infty}, \mathcal{F}^{\bar{\gamma}}_{\bar{p},\infty})_{1/2,\bar{p}} \subset (\mathcal{B}^{\gamma+\varepsilon}_{p,\infty}, \mathcal{B}^{\gamma-\varepsilon}_{p,\infty})_{1/2,\bar{p}}.$$

Furthermore, it is trivial that $\mathcal{F}^{\bar{\gamma}}_{\bar{p},\infty} \subset (\mathcal{F}^{\bar{\gamma}}_{\bar{p},\infty}, \mathcal{F}^{\bar{\gamma}}_{\bar{p},\infty})_{1/2,\bar{p}}$. Thus, the asserted relation follows at once if we show that

$$(\bar{\mathcal{B}}^{\gamma+\varepsilon}_{p,\infty}, \bar{\mathcal{B}}^{\gamma-\varepsilon}_{p,\infty})_{1/2,\bar{p}} \subset \bar{\mathcal{B}}^{\gamma}_{p,\bar{p}}.$$

The corresponding local bound is trivial. For the translation bound, we follow the argument of Triebel [10, Sec. 2.4.2]. Let $f \in (\bar{\mathcal{B}}^{\gamma+\varepsilon}_{p,\infty}, \bar{\mathcal{B}}^{\gamma-\varepsilon}_{p,\infty})_{1/2,\bar{p}}$. Consider $\bar{f}_1 \in \bar{\mathcal{B}}^{\gamma+\varepsilon}_{p,\infty}$ and $\bar{f}_2 \in \bar{\mathcal{B}}^{\gamma-\varepsilon}_{p,\infty}$ such that $\bar{f} = \bar{f}_1 + \bar{f}_2$. Then

$$\sum_{h\in\mathcal{E}_n} \left\| \frac{|\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \right\|^{\bar{p}}_{\ell^p_n} \lesssim 2^{-n\varepsilon\bar{p}} | \|\|\bar{f}_1\|\|_{\bar{\mathcal{B}}^{\gamma+\varepsilon}_{p,\infty}} + 2^{2n\varepsilon} \|\|\bar{f}_2\|\|_{\bar{\mathcal{B}}^{\gamma-\varepsilon}_{p,\infty}} |^{\bar{p}},$$

uniformly over all $n \geq 0$. By taking the infimum we deduce that

$$\begin{aligned}\biggl(\sum_{n\geq 0} \left\| \frac{|\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \right\|^{\bar{p}}_{\ell^p_n} \biggr)^{1/\bar{p}} &\lesssim \Bigl(\sum_{n\geq 0} |2^{-n\varepsilon} K(2^{2n\varepsilon}, \bar{f})|^{\bar{p}}\Bigr)^{1/\bar{p}} \\ &\lesssim \|\bar{f}\|_{(\bar{\mathcal{B}}^{\gamma+\varepsilon}_{p,\infty}, \bar{\mathcal{B}}^{\gamma-\varepsilon}_{p,\infty})_{1/2,\bar{p}}}. \ \blacksquare\end{aligned}$$

5. The reconstruction operator. This section is devoted to the proof of the reconstruction theorem. We work in a fixed regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ which we always assume to include the polynomial regularity structure $(\bar{\mathcal{A}}, \bar{\mathcal{T}}, \bar{\mathcal{G}})$. Furthermore, models for the regularity structure are always understood to act canonically on the polynomial structure.

A powerful additional ingredient of the theory of regularity structures is an appropriate notion of distance between two modelled distributions. We emphasize that the theory in particular allows for the situation where two modelled distributions are modelled with respect to two *different* models. Of course, we want to translate the corresponding notion from [5] to the scale $\mathcal{F}^{\gamma}_{p,q}$ and therefore we define

$$\begin{aligned}\|\|f;\bar{f}\|\| &= \sup_\zeta \| |f(x) - \bar{f}(x)|_\zeta \|_{L^p} \\ &\quad + \sup_\zeta \left\| \left\| \fint_{Q(0,4\lambda)} \frac{|f(x+h) - \bar{f}(x+h) - \Gamma_{x+h,x} f(x) + \bar{\Gamma}_{x+h,x}\bar{f}(x)|_\zeta}{\lambda^{\gamma-\zeta}} \, \mathrm{d}h \right\|_{L^q_\lambda} \right\|_{L^p}.\end{aligned}$$

The reconstruction theorem then reads as follows.

Theorem 5.1. *Let $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ be a regularity structure and (Π, Γ) a model as above. Let $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Set $\alpha = \min(\mathcal{A}\setminus\mathbb{N}) \wedge \gamma$. If $q < \infty$, choose $\bar{\alpha} < \alpha$. In the case $q = \infty$, let $\bar{\alpha} = \alpha$. Then there exists a* *unique continuous linear map* $\mathcal{R}\colon \mathcal{F}^{\gamma}_{p,q} \to F^{\bar{\alpha}}_{p,q}$ *such that*

$$\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r} \frac{|\langle \mathcal{R}f - \Pi_x f(x), \eta^\lambda_x\rangle|}{\lambda^\gamma}\Big\|_{L^q_\lambda}\Big\|_{L^p} \lesssim \|\!|\!| f |\!|\!\|_{\mathcal{F}^{\gamma}_{p,q}} \|\Pi\|(1+\|\Gamma\|), \tag{5.1}$$

uniformly over all modelled distributions and all models. Furthermore, given another model $(\bar{\Pi}, \bar{\Gamma})$ *and denoting by* $\bar{\mathcal{R}}$ *the associated reconstruction operator, we have*

$$\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r} \frac{|\langle \mathcal{R}f - \bar{\mathcal{R}}\bar{f} - \Pi_x f(x) + \bar{\Pi}_x\bar{f}(x), \eta^\lambda_x\rangle|}{\lambda^\gamma}\Big\|_{L^q_\lambda}\Big\|_{L^p} \tag{5.2}$$
$$\lesssim \|\!|\!| f;\bar{f} |\!|\!\| \, \|\Pi\|(1+\|\Gamma\|) + \|\!|\!| \bar{f} |\!|\!\| \big(\|\Pi - \bar{\Pi}\|(1+\|\Gamma\|) + \|\bar{\Pi}\|\,\|\Gamma - \bar{\Gamma}\|\big).$$

Proof. As in [7], we will build a discrete approximation of the reconstruction operator, namely we start with $f \in \mathcal{F}^{\gamma}_{p,q}$ and build $\bar{f}$ as in Proposition 3.8. Then we define, for $n \in \mathbb{N}$ and $x \in \Lambda_n$,

$$A^n_x = \langle \Pi_x \bar{f}^{(n)}(x), \varphi^n_x\rangle, \qquad \mathcal{R}_n f = \sum_{x\in\Lambda_n} A^n_x \varphi^n_x.$$

Following [7], we divide the proof into three steps. For simplicity, we assume that $q < \infty$; as already discussed previously, the case $q = \infty$ follows in essentially the same way by replacing sums with suprema. The different choice of $\bar{\alpha}$ for $q = \infty$ is a direct consequence of the improved convergence result in Proposition 2.3.

Step 1: We prove that $\mathcal{R}_n f \to \mathcal{R}f$ in $F^{\bar{\alpha}}_{p,q}$ for $\bar{\alpha} < \alpha \wedge 0$. To see this, we only need to prove the assumptions of Proposition 2.3. We start by estimating

$$|A^n_x| \le \sum_{\gamma>\zeta} \|\Pi\| 2^{-n(|\mathfrak{s}|/2+\zeta)} |\bar{f}^{(n)}(x)|_\zeta.$$

Hence, by the bound from Remark 3.7 and the definition of α,

$$\Big\|\sup_{n\ge 0}\sum_{y\in\Lambda_n} \frac{|A^n_y|}{2^{-n(\alpha+|\mathfrak{s}|/2)}}\chi^n_y(x)\Big\|_{L^p} \lesssim \sup_\zeta \Big\|\sup_{n\ge 0}\sum_{y\in\Lambda_n} |\bar{f}^{(n)}(y)|_\zeta \chi^n_y(x)\Big\|_{L^p}$$
$$\lesssim \|\!|\!| \bar{f} |\!|\!\|_{\bar{\mathcal{F}}^{\gamma}_{p,q}} \lesssim \|\!|\!| f |\!|\!\|_{\mathcal{F}^{\gamma}_{p,q}},$$

while

$$|\delta A^n_x| = \Big|\sum_{y\in\Lambda_{n+1}} \langle \Pi_y \bar{f}^{(n+1)}(y) - \Pi_x \bar{f}^{(n)}(x), \varphi^{n+1}_y\rangle \langle \varphi^{n+1}_y, \varphi^n_x\rangle\Big|$$
$$\lesssim \sup_\zeta \sum_{\substack{y\in\Lambda_{n+1}\\ \|y-x\|_\mathfrak{s}\le C2^{-n}}} |\bar{f}^{(n)}(x) - \Gamma_{x,y}\bar{f}^{(n+1)}(y)|_\zeta 2^{-n(|\mathfrak{s}|/2+\zeta)}$$

so that we indeed obtain the second bound of Proposition 2.3:

$$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\frac{|\delta A^n_y|}{2^{-n(\gamma+|\mathfrak{s}|/2)}}\chi^n_y(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}$$
$$\lesssim \Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}^{C,0}_{n+1}}\frac{|\bar{f}^{(n)}(y)-\Gamma_{y,y+h}\bar{f}^{(n+1)}(y+h)|_\zeta}{2^{-n(\gamma-\zeta)}}\chi^n_y(x)\Big|^q\Big)^{1/q}\Big\|_{L^p},$$

which is bounded by the norm of f thanks to Remark 3.3. Hence, $\mathcal{R}_n f$ converges to some $\mathcal{R}f$ in $\mathcal{F}^{\bar{\alpha}}_{p,q}$. From the reconstruction bound which we will prove below, we then recover the convergence of $\mathcal{R}_n f$ in $\mathcal{F}^{\bar{\alpha}}_{p,q}$ for all $\bar{\alpha}<\alpha$, even if $\alpha>0$. The proof is the same as in [7] and we omit it.

Step 2: We prove the reconstruction bound (5.1). To this end, for any $n\in\mathbb{N}$ we write

$$\mathcal{R}f-\Pi_x f(x)=\mathcal{R}_n f-\mathcal{P}_n\Pi_x f(x)+\sum_{m\geq n}\mathcal{R}_{m+1}f-\mathcal{R}_m f-\mathcal{P}^\perp_m\Pi_x f(x),$$

where $\mathcal{P}_n$ is the projection onto V_n and $\mathcal{P}^\perp_n$ is the projection onto the orthogonal complement $V^\perp_n$ of V_n in V_{n+1}. We will treat separately the terms of order λ and the terms of higher order, that is, for any λ we choose n such that $\lambda\in[2^{-n},2^{-n+1})$. Then (cf. [7, Rem. 2.2]) we find immediately that

$$\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\mathcal{R}f-\Pi_x f(x),\eta^\lambda_x\rangle|}{\lambda^\gamma}\Big\|_{L^q_\lambda}\Big\|_{L^p}\simeq\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\mathcal{R}f-\Pi_x f(x),\eta^{2^{-n}}_x\rangle|}{2^{-n\gamma}}\Big\|_{\ell^q}\Big\|_{L^p}$$

and

$$\mathcal{R}_n f-\mathcal{P}_n\Pi_x f(x)=\sum_{y\in\Lambda_n}(A^n_y-\langle\Pi_x f(x),\varphi^n_y\rangle)\varphi^n_y.$$

For terms of order n, we get the following bound uniformly in n and with the constant C depending only on the support of ϕ:

$$\begin{aligned}|\langle\mathcal{R}_n f-\mathcal{P}_n\Pi_x f(x),\eta^{2^{-n}}_x\rangle|&=\Big|\sum_{y\in\Lambda_n}(A^n_y-\langle\Pi_x f(x),\varphi^n_y\rangle)\langle\varphi^n_y,\eta^{2^{-n}}_x\rangle\Big|\\&\lesssim\sum_{\substack{y\in\Lambda_n\\\|y-x\|_\mathfrak{s}\leq C2^{-n}}}\fint_{Q(y,2^{-n})}|\langle\Pi_z(f(z)-\Gamma_{z,x}f(x)),2^{n|\mathfrak{s}|/2}\varphi^n_y\rangle|\,\mathrm{d}z\\&\lesssim\sup_\zeta\fint_{Q(0,2C2^{-n})}\frac{|f(x+h)-\Gamma_{x+h,x}f(x)|_\zeta}{2^{n\zeta}}\,\mathrm{d}h,\end{aligned}$$

where in the last inequality we have used the fact that the sum is actually finite, uniformly in n. In this way, we can find a bound of required order for

$$\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\mathcal{R}_n f-\mathcal{P}_n\Pi_x f(x),\eta^{2^{-n}}_x\rangle|}{2^{-n\gamma}}\Big\|_{\ell^q}\Big\|_{L^p}.$$

Now, we pass to the terms of order greater than n. For this, we use the decomposition $\mathcal{R}_{m+1}f - \mathcal{R}_m f = g_m + \delta f_m$, where $g_m \in V_m$ and $\delta f_m \in V_m^\perp$. We will treat the contributions from $\sum_{m\geq n} g_m$ and $\sum_{m\geq n}(\delta f_m - \mathcal{P}_m^\perp \Pi_x f(x))$ differently. We start with the latter, following [7]:

$$
\begin{aligned}
&|\langle \delta f_m - \mathcal{P}_m^\perp \Pi_x f(x), \eta_x^{2^{-n}}\rangle| \\
&\qquad = \Big| \sum_{\substack{y\in\Lambda_{m+1}\\ z\in\Lambda_m}} (A_y^{m+1} - \langle \Pi_x f(x), \varphi_y^{m+1}\rangle)\langle \varphi_y^{m+1}, \psi_z^m\rangle\langle \psi_z^m, \eta_x^{2^{-n}}\rangle \Big| \\
&\qquad \lesssim \sup_\zeta \frac{2^{-m(r+\zeta)}}{2^{-nr}} \fint_{Q(0,C'2^{-n})} |f(x+h) - \Gamma_{x+h,x} f(x)|_\zeta \,\mathrm{d}h.
\end{aligned}
$$

So, as before we can now easily bound the contribution from

$$
\Big\|\Big\| \sup_{\eta\in\mathfrak{B}^r} \sum_{m\geq n} \frac{|\langle \delta f_m - \mathcal{P}_m^\perp \Pi_x f(x), \eta_x^{2^{-n}}\rangle|}{2^{-n\gamma}} \Big\|_{\ell^q} \Big\|_{L^p}
$$

in terms of the norm of f and the norms of the model, as asserted. Finally, we also estimate the contribution from g_m:

$$
\begin{aligned}
&|\langle g_m, \eta_x^{2^{-n}}\rangle| \\
&\quad = \Big| \sum_{\substack{y\in\Lambda_m\\ z\in\Lambda_{m+1}}} \langle \Pi_z \bar{f}^{(m+1)}(z) - \Pi_y \bar{f}^{(m)}(y), \varphi_z^{m+1}\rangle\langle \varphi_z^{m+1}, \varphi_y^m\rangle\langle \varphi_y^m, \eta_x^{2^{-n}}\rangle \Big| \\
&\quad \lesssim \sup_\zeta \sum_{y\in\Lambda_m^{C,x}} \sum_{h\in\mathcal{E}_{m+1}^{C,0}} |\bar{f}^{(m)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(m+1)}(y)|_\zeta 2^{-(m-n)|\mathfrak{s}|-m\zeta},
\end{aligned}
$$

where we have used the notation from the proof of Proposition 2.2 with n_0 replaced by n. In fact, following the proof of Proposition 2.2 for small scales, with $\kappa \in (0,1)$ such that $\gamma + \frac{\kappa-1}{\kappa} q|\mathfrak{s}| > 0$ and with the weight function

$$
\theta(z) = 2^{-z(\gamma + \frac{\kappa-1}{\kappa} q|\mathfrak{s}|)},
$$

where $z \geq 0$, we can now bound

$$
\begin{aligned}
&\Big\|\Big\| \sup_{\kappa\in\mathfrak{B}^r} \sum_{m\geq n} \frac{|\langle g_m, \kappa_x^{2^{-n}}\rangle|}{2^{-n\gamma}} \Big\|_{\ell^q(n)} \Big\|_{L^p} \\
&\qquad \lesssim \Big\| \Big(\sum_{n\geq 0} \Big| \sum_{y\in\Lambda_n} \sum_{h\in\mathcal{E}_n^M} \frac{|\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \chi_y^n(x) \Big|^q \Big)^{1/q} \Big\|_{L^p},
\end{aligned}
$$

which in turn is bounded by $\|\!|\bar{f}|\!\|_{\bar{\mathcal{F}}^\gamma_{p,q}}$ in view of Remark 3.4. Putting together these results we infer by the triangle inequality that

$$
\begin{aligned}
&\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\mathcal{R}f-\Pi_x f(x),\eta_x^{2^{-n}}\rangle|}{2^{-n\gamma}}\Big\|_{\ell^q}\Big\|_{L^p}\\
&\qquad\le\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\mathcal{R}_n f-\mathcal{P}_n\Pi_x f(x),\eta_x^{2^{-n}}\rangle|}{2^{-n\gamma}}\Big\|_{\ell^q}\Big\|_{L^p}\\
&\qquad\quad+\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\sum_{m\ge n}\frac{|\langle\delta f_m-\mathcal{P}_m^{\perp}\Pi_x f(x),\eta_x^{2^{-n}}\rangle|}{2^{-n\gamma}}\Big\|_{\ell^q}\Big\|_{L^p}\\
&\qquad\quad+\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r}\sum_{m\ge n}\frac{|\langle g_m,\eta_x^{2^{-n}}\rangle|}{2^{-n\gamma}}\Big\|_{\ell^q}\Big\|_{L^p}\\
&\qquad\lesssim |\!|\!| f|\!|\!|_{\mathcal{F}^{\gamma}_{p,q}}\|\Pi\|(1+\|\Gamma\|),
\end{aligned}
$$

uniformly over all models. If we take two different models, the reconstruction bound follows by the same argument as in [7].

Step 3: We prove the uniqueness of the reconstruction operator. For this purpose, we recall that for any Schwartz distribution ψ and for any smooth test function ρ supported in the unit ball, the convolution $\psi*\rho^{\delta}(x)=\langle\psi,\rho_x^{\delta}\rangle$ converges in distribution to ψ as ρ approximates a delta, i.e. if $\delta\to 0$. Now, assume there are two distributions ξ^1 and ξ^2 which both satisfy the reconstruction bound (5.1). Then, for any $\delta>0$ choose $n\in\mathbb{N}$ such that $\delta\in[2^{-n},2^{-n+1})$. Now, we can bound

$$
\Big\|\frac{|\langle\xi^1-\xi^2,\rho_x^{\delta}\rangle|}{\delta^{\gamma}}\Big\|_{L^p}\lesssim\Big\|\int_{2^{-n}}^{2^{-n+1}}\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\xi^1-\xi^2,\eta_x^{\lambda}\rangle|}{\lambda^{\gamma}}\,\frac{\mathrm{d}\lambda}{\lambda}\Big\|_{L^p}.
$$

Furthermore, let us write

$$
f_n=\int_{2^{-n}}^{2^{-n+1}}\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle\xi^1-\xi^2,\eta_x^{\lambda}\rangle|}{\lambda^{\gamma}}\,\frac{\mathrm{d}\lambda}{\lambda}.
$$

By the reconstruction bound, $\|\|f_n\|_{\ell^q}\|_{L^p}<\infty$. It is then an easy exercise to show that this implies $\|f_n\|_{L^p}\to 0$ as $n\to\infty$. Since taking $n\to\infty$ is equivalent to $\delta\to 0$, we find that $\langle\xi^1-\xi^2,\rho_x^{\delta}\rangle\to 0$ in $L^p(\mathrm{d}x)$. Since L^p convergence implies convergence in distribution, this tells us that $\xi^1-\xi^2=0$. Note that this argument does not rely on γ, and it shows that there is at most one distribution that can satisfy the reconstruction bound. ∎

Remark 5.2. For a given $C>0$, we denote more generally by $\mathfrak{B}^r_C(\mathbb{R}^d)$ the subspace of smooth functions on $\mathbb{R}^d$ with $\mathcal{C}^r$-norm bounded by 1 and supported in the cube $Q(0,C)$. A detailed inspection of the proof of the reconstruction theorem then shows that

$$(5.3)\qquad \Big\|\Big\|\sup_{\eta\in\mathfrak{B}^r_C(\mathbb{R}^d)}\frac{|\langle\mathcal{R}f-\Pi_x f(x),\eta_x^\lambda\rangle|}{\lambda^\gamma}\Big\|_{L^q_\lambda}\Big\|_{L^p}\lesssim_C |||f|||_{\mathcal{F}^\gamma_{p,q}}.$$

Of course, the respective bound is also true in the case of two models.

REMARK 5.3. There is another version of the reconstruction bound which we would like to mention at this point. It follows from the identity

$$\begin{aligned}&\langle\mathcal{R}f-\Pi_{x+h}f(x+h),\eta_{x+h}^{2^{-n}}\rangle\\ &\qquad=\langle\mathcal{R}f-\Pi_x f(x),\eta_{x+h}^{2^{-n}}\rangle+\langle\Pi_{x+h}(f(x+h)-\Gamma_{x+h,x}f(x)),\eta_{x+h}^{2^{-n}}\rangle\end{aligned}$$

and the bound of Remark 5.2 that

$$(5.4)\qquad \Big\|\Big\|\fint_{Q(0,C2^{-n})}\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle\mathcal{R}f-\Pi_{x+h}f(x+h),\eta_{x+h}^{2^{-n}}\rangle|}{2^{-n\gamma}}\,\mathrm{d}h\Big\|_{\ell^q}\Big\|_{L^p}\lesssim_C |||f|||_{\mathcal{F}^\gamma_{p,q}},$$

again uniformly over all $f\in\mathcal{F}^\gamma_{p,q}$.

The remainder of this section is devoted to proving a consequence of the reconstruction theorem when the regularity structure is the polynomial regularity structure $(\bar{\mathcal{A}},\bar{\mathcal{T}},\bar{\mathcal{G}})$, together with the model (Π,Γ) acting canonically on this structure. In this case, it turns out that modelled distributions correspond by means of the reconstruction operator bijectively to Triebel–Lizorkin distributions over $\mathbb{R}^d$:

PROPOSITION 5.4. *Let $\gamma\in\mathbb{R}_+\setminus\mathbb{N}$, $1\le p<\infty$ and $1\le q\le\infty$. Then the reconstruction operator yields an isomorphism between the Banach spaces $\mathcal{F}^\gamma_{p,q}(\bar{\mathcal{T}})$ and $F^\gamma_{p,q}(\mathbb{R}^d)$.*

We divide the proof into the following two lemmata. In a first step, we will prove that the reconstruction operator is an injection between the two spaces above. Then, we will show that $\mathcal{R}$ is invertible.

LEMMA 5.5. *Let $f\in\mathcal{F}^\gamma_{p,q}(\bar{\mathcal{T}})$ and for $k\in\mathbb{N}^d$ with $|k|_\mathfrak{s}<\gamma$, define $f_k(x)=\mathcal{Q}_k f(x)$. Then $k!f_k$ is the kth derivative of $\mathcal{R}f$ (in the sense of distributions) and $\mathcal{R}f=f_0\in F^\gamma_{p,q}(\mathbb{R}^d)$.*

Proof. Following the argument for the uniqueness of the reconstruction operator, we can see that there exists at most one distribution $\xi^{(k)}$ that satisfies

$$\Big\|\Big\|\sup_{\eta\in\mathfrak{B}^{r+|k|}}\frac{|\langle\xi^{(k)}-\partial^k\Pi_x f(x),\eta_x^\lambda\rangle|}{\lambda^{\gamma-|k|}}\Big\|_{L^q_\lambda}\Big\|_{L^p}<\infty.$$

From the identity $(k!f_k-\partial^k\Pi_x f(x))(y)=k!\mathcal{Q}_k(f(y)-\Gamma_{y,x}f(x))$ and the definition of modelled distributions it is easy to see that the above estimate is satisfied with $\xi^{(k)} = k! f_k$. Moreover, also $\xi^{(k)} = \partial^k \mathcal{R} f$ satisfies that bound. Therefore $\partial^k \mathcal{R} f = k! f_k$.

It remains to prove that f_0 lies in $F^\gamma_{p,q}(\mathbb{R}^d)$. Note that the reconstruction theorem only gives regularity strictly smaller than γ, at least for $q < \infty$. Since $f_0 \in L^p$ it is easy to show that

$$\Big\| \sup_{\eta \in \mathfrak{B}^r} |\langle f_0, \eta^\lambda_x \rangle| \Big\|_{L^p} \lesssim \|f_0\|_{L^p}.$$

Furthermore, the second bound in the definition of the Triebel–Lizorkin space $F^\gamma_{p,q}$ is an immediate consequence of the reconstruction bound, since $\Pi_x f(x)$ is a polynomial and the test functions from the definition annihilate polynomials up to scaled degree r. ■

LEMMA 5.6. *There exists a continuous injection $\iota\colon F^\gamma_{p,q} \to \mathcal{F}^\gamma_{p,q}(\bar{\mathcal{T}})$ such that $\mathcal{R}\iota\xi = \xi$ for any $\xi \in F^\gamma_{p,q}$.*

Proof. The proof follows step by step the one in [7]. We start by proving that there is a continuous injection into $\bar{\mathcal{F}}^\gamma_{p,q}(\bar{\mathcal{T}})$. We fix some integer q and $k \in \mathbb{N}^d$ with $|k|_\mathfrak{s} \le q$ and a function η. To lighten notation, we will write $|\cdot|$ for $|\cdot|_\mathfrak{s}$. We define

$$P^q_{k,y}(\eta, u) = \sum_{|\ell+k|\le q} (-1)^\ell \partial^\ell_u \bigg[\eta(u-y) \frac{(y-u)^\ell}{\ell! k!} \bigg].$$

For $\xi \in F^\gamma_{p,q}$ and $y \in \Lambda_n$, we also define $\bar{f}^{(n)}(y) \in \bar{\mathcal{T}}^-_\gamma$ by

$$\mathcal{Q}_k \bar{f}^{(n)}(y) = \langle \partial^k \xi(\cdot), P^{\lfloor\gamma\rfloor}_{k,y}(\rho^n, \cdot) \rangle,$$

where ρ is any smooth symmetric function with compact support in the unit ball, and which integrates to 1. In addition, ρ^n is shorthand for $\rho^{2^{-n}}$, i.e. we use L^1 scaling for ρ. This is in analogy to the definition given by Proposition 3.8 (cf. also [7]). Now, our task is to show that $\bar{f}$ lies in $\bar{\mathcal{F}}^\gamma_{p,q}(\bar{\mathcal{T}})$.

The idea is of course to transform the bounds in the definition of $F^\gamma_{p,q}$ into the bounds from the definition of $\bar{\mathcal{F}}^\gamma_{p,q}(\bar{\mathcal{T}})$. For this purpose, we define

$$\Phi^{k,n}_{y,h}(\cdot) = P^{\lfloor\gamma\rfloor}_{k,y+h}(\rho^n, \cdot) - \sum_{|\ell+k|\le\lfloor\gamma\rfloor} (-h)^\ell \frac{(\ell+k)!}{\ell! k!} \partial^\ell_u P^{\lfloor\gamma\rfloor}_{k+\ell,y}(\rho^n, \cdot),$$
$$\Psi^{k,n}_y(\cdot) = P^{\lfloor\gamma\rfloor}_{k,y}(\rho^n, \cdot) - P^{\lfloor\gamma\rfloor}_{k,y}(\rho^{n+1}, \cdot),$$

where $h \in \mathcal{E}_n$. These functions have been chosen so that

$$\langle \partial^k \xi, \Phi^{k,n}_{y,h} \rangle = \mathcal{Q}_k(\bar{f}^{(n)}(y) - \Gamma_{y+h,y} \bar{f}^{(n)}(y)),$$
$$\langle \partial^k \xi, \Psi^{k,n}_y \rangle = \mathcal{Q}_k(\bar{f}^{(n)}(y) - \bar{f}^{(n+1)}(y)).$$

One easily sees that $P^{\lfloor\gamma\rfloor}_{k,y}(\rho^n, \cdot)$, $\Phi^{k,n}_{y,h}$ and $\Psi^{k,n}_y$ are smooth functions with compact support in the ball of radius 2^{-n} about y. Thus, for $n = 0$ the first bound from Definition 3.2 follows immediately:

$$\sup_{\zeta}\Big(\sum_{y\in\Lambda_0}|\bar f^{(0)}(y)|_{\zeta}^{p}\Big)^{1/p}\lesssim |||\xi|||_{F^{\gamma}_{p,q}}.$$

Now, we turn to the translation and consistency bounds. Assuming for the moment that we can prove that these functions annihilate polynomials of degree smaller than $\gamma-|k|$, it follows that $\bar f\in\bar{\mathcal{F}}^{\gamma}_{p,q}$. Indeed, since $\partial^k\xi\in F^{\gamma-|k|}_{p,q}$,

$$\begin{aligned}\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}\frac{|\bar f^{(n)}(y+h)-\Gamma_{y+h,y}\bar f^{(n)}(y)|_{\zeta}}{2^{-n(\gamma-\zeta)}}\chi^n_y(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}&\\ \lesssim\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}\sum_{|k|=\zeta}\frac{|\langle\partial^k\xi,\Phi^{k,n}_{y,h}\rangle|}{2^{-n(\gamma-|k|)}}\chi^n_y(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}&\\ \lesssim\Big\|\Big(\sum_{n\geq 0}\Big|\sup_{\eta\in\mathfrak{B}^r_{\lfloor\gamma\rfloor-|k|}}\frac{|\langle\partial^k\xi,\eta^n_x\rangle|}{2^{-n(\gamma-|k|)}}\Big|^q\Big)^{1/q}\Big\|_{L^p}\lesssim |||\xi|||_{F^{\gamma}_{p,q}},&\end{aligned}$$

and similarly for the consistency bound. Thus, it remains to prove that Φ and Ψ annihilate polynomials of degree smaller than $\gamma-|k|$. Indeed,

$$\int P^{\lfloor\gamma\rfloor}_{k,y}(\rho^n,u)\,\mathrm{d}u=\frac{1}{k!},\tag{5.5}$$

and hence Φ and Ψ vanish when integrated against any constant, because up to some derivative terms they are made up of differences of P. By the arguments of [7] one can prove that

$$\langle P^{\lfloor\gamma\rfloor}_{k,y}(\rho^n,\cdot),(\cdot-y)^m\rangle=0$$

for $m\neq 0$ such that $|m+k|\leq\lfloor\gamma\rfloor$. Together with the fact that all P_k's have the same volume mean, this is sufficient in order to see that Ψ vanishes when integrated against the relevant polynomials. For completeness we treat the case of Φ, since it is left as an exercise in [7].

Fix m such that $|m+k|\leq\lfloor\gamma\rfloor$. We want to prove that

$$\langle\Phi^{k,n}_{y,h},(\cdot-(y+h))^m\rangle=0.$$

Indeed,

$$\begin{aligned}&\langle\Phi^{k,n}_{y,h},(\cdot-(y+h))^m\rangle\\ &\qquad=-\sum_{|\ell+k|\leq\lfloor\gamma\rfloor}(-h)^{\ell}\frac{(\ell+k)!}{\ell!k!}\int\partial^{\ell}_uP^{\lfloor\gamma\rfloor}_{k+\ell,y}(\rho^n,u)(u-(y+h))^m\,\mathrm{d}u\\ &\qquad=-\sum_{\ell\leq m}h^{\ell}\frac{(\ell+k)!}{\ell!k!}\int\frac{m!}{(m-\ell)!}P^{\lfloor\gamma\rfloor}_{k+\ell,y}(\rho^n,u)(-h)^{m-\ell}\,\mathrm{d}u\\ &\qquad=-\frac{h^m}{k!}\sum_{\ell\leq m}(-1)^{m-\ell}\frac{m!}{l!(m-l)!}=0,\end{aligned}$$

where we have repeatedly used the fact that $P^{\lfloor\gamma\rfloor}_{k+\ell,z}$ annihilates polynomials centred in z, and in the second line we applied integration by parts. In the last line we first used (5.5) and then the binomial formula. Hence $\bar{f} \in \bar{\mathcal{F}}^{\gamma}_{p,q}$.

Finally we need to prove that $\mathcal{R}\iota\xi = \xi$, where $\iota\xi$ is of course given by the modelled distribution f corresponding to $\bar{f}$ through the isomorphism of Proposition 3.8. We observe that $\mathcal{R}i\xi = \mathcal{Q}_0 f$, and that Proposition 3.8 shows that $\Gamma_{x,x_n}\bar{f}^{(n)}(x_n) \to \mathcal{Q}_0 f(x)$ in L^p as $n \to \infty$. By the embedding $F^{\gamma}_{p,q} \subset L^p$, it is clear that $\xi = \lim_n \Gamma_{x,x_n}\bar{f}^{(n)}(x_n)$, because it is well known that $\xi * \rho^n \to \xi$ in L^p. Then, one can follow the arguments of [7] to conclude the proof. ■

6. Convolution against singular kernels. In this section, we prove Schauder type estimates at the level of modelled distributions of class $\mathcal{F}^{\gamma}_{p,q}$, i.e. given a kernel K which improves regularity by some $\beta > 0$ we construct a linear map $\mathcal{F}^{\gamma}_{p,q} \ni f \mapsto \mathcal{K}_\gamma f \in \mathcal{F}^{\gamma+\beta}_{p,q}$. In addition, we want to ensure that the construction behaves suitably under the action of the reconstruction operator, i.e.

$$\mathcal{R}\mathcal{K}_\gamma f = K * \mathcal{R}f. \tag{6.1}$$

Almost all non-trivial parts of this program were already carried out by Hairer [5, Sec. 5]. What is essentially left is to prove the Schauder estimate in the framework of the scale $\mathcal{F}^{\gamma}_{p,q}$ and to check the validity of (6.1) in this setting. First, let us recall the ingredients of the construction of the linear map $f \mapsto \mathcal{K}_\gamma f$.

We still assume that the regularity structure at hand contains the polynomial structure $(\bar{\mathcal{A}}, \bar{\mathcal{T}}, \bar{\mathcal{G}})$. Furthermore, let $K\colon Q(0,1) \to \mathbb{R}$ be a kernel which is smooth at every point in $Q(0,1) \setminus \{0\}$ and β-regularizing in the sense of [5, Assumptions 5.1, 5.4] for some $\beta > 0$. More precisely, we assume that for every $n \geq 0$ there exists a smooth kernel $K_n\colon \mathbb{R}^d \to \mathbb{R}$ such that the following properties are satisfied:

(i) The decomposition $K = \sum_{n\geq 0} K_n$ holds true on $Q(0,1)$.

(ii) The kernel K_0 has support in $Q(0,1)$. Furthermore, for every $n \geq 0$ and every $x \in \mathbb{R}^d$ we have the scaling relation

$$K_n(x) = 2^{-n(\beta-|\mathfrak{s}|)} K_0(2^{n\mathfrak{s}}x). \tag{6.2}$$

(iii) The kernel K_0 kills polynomials with scaled degree at most r.

Here and in what follows in this section, we let $r > \max|\mathcal{A}_{\gamma+\beta}|$.

Next, we want to recall the assumption that the polynomial structure only provides integer homogeneities for the regularity structure under consideration. It was pointed out by Hairer [5, Sec. 5] that in general it is necessary that $\mathcal{K}_\gamma f$ takes non-zero values in parts of the regularity structure which do not involve the polynomial structure. In order to encode the action of the kernel K in these parts with non-integer homogeneities, we assume that our regularity structure comes with an abstract integration map $\mathcal{I}\colon \mathcal{T} \to \mathcal{T}$ of order β, i.e. (cf. [5, Def. 5.7])

(i) $\mathcal{I}|_{\mathcal{T}_\zeta} : \mathcal{T}_\zeta \to \mathcal{T}_{\zeta+\beta}$,
(ii) $\mathcal{I}$ annihilates all elements in $\bar{\mathcal{T}}$,
(iii) $\mathcal{I}\Gamma\tau - \Gamma\mathcal{I}\tau \in \bar{\mathcal{T}}$ for all $\Gamma \in \mathcal{G}$ and $\tau \in \mathcal{T}$.

We refer again to [5, Rem. 5.8] for a discussion of this notion. Now, let $1 \le p < \infty$, $1 \le q \le \infty$ and $\gamma > 0$. Given $f \in \mathcal{F}^\gamma_{p,q}$, we define as in [5, (5.15)] the operator

$$\begin{aligned} \mathcal{K}_\gamma f(x) &= \mathcal{I}\big(f(x)\big) + \sum_{\zeta\in\mathcal{A}_\gamma} \sum_{|k|_{\mathfrak{s}}<\zeta+\beta} \sum_{n\ge 0} \frac{X^k}{k!} \langle \Pi_x \mathcal{Q}_\zeta f(x), D_1^k K_n(x,\cdot)\rangle \\ &\quad + \sum_{|k|_{\mathfrak{s}}<\gamma+\beta} \sum_{n\ge 0} \frac{X^k}{k!} \langle \mathcal{R}f - \Pi_x f(x), D_1^k K_n(x,\cdot)\rangle. \end{aligned} \tag{6.3}$$

Here, we have identified $K_n(x,z) = K_n(x-z)$. For convenience, let us also introduce for each $n \ge 0$ the operator

$$\begin{aligned} \mathcal{K}_{n,\gamma} f(x) = \sum_{\zeta\in\mathcal{A}_\gamma} \sum_{|k|_{\mathfrak{s}}<\zeta+\beta} &\frac{X^k}{k!} \langle \Pi_x \mathcal{Q}_\zeta f(x), D_1^k K_n(x,\cdot)\rangle \\ &+ \sum_{|k|_{\mathfrak{s}}<\gamma+\beta} \frac{X^k}{k!} \langle \mathcal{R}f - \Pi_x f(x), D_1^k K_n(x,\cdot)\rangle. \end{aligned}$$

Note that due to the bounds provided by a model, the reconstruction theorem and the scaling properties of the kernel K_0, the sums over $n \ge 0$ above converge absolutely in $L^p(\mathbb{R}^d)$. Of course, we are not only interested in this local bound but actually in the improved version concerning the whole norm for the scale $\mathcal{F}^\gamma_{p,q}$.

Apart from that, we also want to ensure that (6.1) holds true. Recall that this is of paramount importance when one wants to recast abstract solutions of fixed-point maps at the level of modelled distributions as mild solutions to regularized versions of an SPDE under consideration. Verifying the Schauder type estimates and (6.1) is in turn linked to finding a class of models which act appropriately on the abstract integration map $\mathcal{I}$.

To this end, the notion of an admissible model was introduced by Hairer [5, Def. 5.9]. For a model to be admissible it is required that the following relation (appropriately interpreted) holds true:

$$\Pi_x \mathcal{I}\tau(y) = \langle \Pi_x\tau, K(y,\cdot)\rangle - \sum_{|k|_{\mathfrak{s}}<\zeta+\beta} \frac{X^k}{k!} \langle \Pi_x\tau, D_1^k K(x,\cdot)\rangle,$$

where $\zeta \in \mathcal{A}_\gamma$ and $\tau \in \mathcal{T}_\zeta$. It is indeed a non-trivial result that this definition really satisfies the required analytic bounds of a model. For this, and a discussion of the notion of admissible models, we again refer to [5].

THEOREM 6.1. *Suppose that* $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ *is a regularity structure and* $K : Q(0,1) \to \mathbb{R}$ *is a kernel such that all the above assumptions are satisfied. Let* $1 \le p < \infty$ *and* $1 \le q \le \infty$. *In addition, assume that* $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}$ *and* $\zeta + \beta \notin \mathbb{N}$ *for all* $\zeta \in (\mathcal{A}_\gamma \cup \{\gamma\}) \setminus \mathbb{N}$. *Given an admissible model* (Π, Γ), *the linear operator* $\mathcal{K}_\gamma$ *maps* $\mathcal{F}^\gamma_{p,q}$ *continuously into* $\mathcal{F}^{\gamma+\beta}_{p,q}$, *and*

$$\|\!|\mathcal{K}_\gamma f\|\!|_{\mathcal{F}^{\gamma+\beta}_{p,q}} \lesssim \|\Pi\|(1 + \|\Gamma\|)\|\!|f\|\!|_{\mathcal{F}^{\gamma}_{p,q}}, \tag{6.4}$$

uniformly over all $f \in \mathcal{F}^\gamma_{p,q}$ *and all models* (Π, Γ). *Moreover,*

$$\mathcal{R}\mathcal{K}_\gamma f = K * \mathcal{R} f.$$

Furthermore, consider another model $(\bar{\Pi}, \bar{\Gamma})$, *and denote by* $\bar{\mathcal{K}}_\gamma$ *the associated abstract convolution operator. Then the quantity* $\|\!|\mathcal{K}_\gamma f; \bar{\mathcal{K}}_\gamma \bar{f}\|\!|_{\mathcal{F}^{\gamma+\beta}_{p,q}}$ *is bounded by*

$$\|\Pi\|(1+\|\Gamma\|)\|\!|f;\bar{f}\|\!|_{\mathcal{F}^\gamma_{p,q}} + \|\Pi - \bar{\Pi}\|(1+\|\Gamma\|)\|\!|\bar{f}\|\!|_{\mathcal{F}^\gamma_{p,q}} + \|\bar{\Pi}\|\,\|\Gamma - \bar{\Gamma}\|\,\|\!|\bar{f}\|\!|_{\mathcal{F}^\gamma_{p,q}},$$

uniformly over all $f \in \mathcal{F}^\gamma_{p,q}(\Pi, \Gamma)$, *all* $\bar{f} \in \mathcal{F}^\gamma_{p,q}(\bar{\Pi}, \bar{\Gamma})$ *and all models* (Π, Γ) *and* $(\bar{\Pi}, \bar{\Gamma})$.

Proof. We only focus on the bounds in the case of one model; the bound for two models then follows from the arguments in [7].

We begin with the local bound. First, we fix a non-integer homogeneity $\zeta < \gamma + \beta$. In this case $\mathcal{Q}_\zeta \mathcal{K}_\gamma f(x) = \mathcal{Q}_\zeta \mathcal{I}(f(x))$, from which the required bound immediately follows due to the properties of $\mathcal{I}$. Now, consider $k \in \mathbb{N}_0^d$ such that $|k|_\mathfrak{s} < \gamma + \beta$. We obviously have

$$\begin{aligned} &k!\mathcal{Q}_{X^k}\mathcal{K}_{n,\gamma} f(x) \\ &\quad = \langle \mathcal{R} f - \Pi_x f(x), D_1^k K_n(x,\cdot)\rangle + \sum_{\zeta > |k|_\mathfrak{s} - \beta} \langle \Pi_x \mathcal{Q}_\zeta f(x), D_1^k K_n(x,\cdot)\rangle. \end{aligned}$$

To bound the first term on the right hand side, we use the reconstruction bound to obtain

$$\begin{aligned} &\sum_{n\ge 0} \big\| |\langle \mathcal{R} f - \Pi_x f(x), D_1^k K_n(x,\cdot)\rangle| \big\|_{L^p} \\ &\qquad \lesssim \sum_{n \ge 0} 2^{-n(\gamma+\beta-|k|_\mathfrak{s})} \Big\| \sup_{\eta \in \mathfrak{B}^r(\mathbb{R}^d)} \frac{|\langle \mathcal{R} f - \Pi_x f(x), \eta_x^{2^{-n}}\rangle|}{2^{-n\gamma}} \Big\|_{L^p} \\ &\qquad \lesssim \|\!|f\|\!|_{\mathcal{F}^\gamma_{p,q}}. \end{aligned}$$

For the second term, we simply estimate

$$\sum_{\zeta>|k|_{\mathfrak{s}}-\beta}\sum_{n\geq 0}\big\||\langle\Pi_x\mathcal{Q}_\zeta f(x),D_1^kK_n(x,\cdot)\rangle|\big\|_{L^p}$$
$$\lesssim\sum_{\zeta>|k|_{\mathfrak{s}}-\beta}\sum_{n\geq 0}2^{-n(\zeta+\beta-|k|_{\mathfrak{s}})}\big\||f(x)|_\zeta\big\|_{L^p}$$
$$\lesssim\vert\!\vert\!\vert f\vert\!\vert\!\vert_{\mathcal{F}^\gamma_{p,q}}.$$

To prove the translation bound, we start again with a non-integer homogeneity $\zeta<\gamma+\beta$. Then (cf. [5])

$$\mathcal{Q}_\zeta\big(\mathcal{K}_\gamma f(x+h)-\Gamma_{x+h,x}\mathcal{K}_\gamma f(x)\big)=\mathcal{Q}_\zeta\mathcal{I}\big(f(x+h)-\Gamma_{x+h,x}f(x)\big)$$

for all $x\in\mathbb{R}^d$, all $n\geq 0$ and all $h\in Q(0,4\cdot 2^{-n})$. Hence, the desired bound follows from the translation bound for f and the properties of $\mathcal{I}$. Fix now some $k\in\mathbb{N}_0^d$ such that $|k|_{\mathfrak{s}}<\gamma+\beta$, i.e. we proceed with contributions in the polynomial structure. We also fix $x\in\mathbb{R}^d$, $n\geq 0$ and $h\in Q(0,4\cdot 2^{-n})$. In the following, we divide $K=\sum_{m\geq 0}K_m$ into the sum over $m<n$ and over $m\geq n$.

For $m\geq n$, we make use of the identity (cf. [5])

$$\begin{aligned}(6.5)\qquad & k!\mathcal{Q}_{X^k}\big(\mathcal{K}_{m,\gamma}f(x+h)-\Gamma_{x+h,x}\mathcal{K}_{m,\gamma}f(x)\big)\\ &=\langle\mathcal{R}f-\Pi_{x+h}f(x+h),D_1^kK_m(x+h,\cdot)\rangle\\ &\quad+\sum_{\substack{l\in\mathbb{N}_0^d\\ |k+l|_{\mathfrak{s}}<\gamma+\beta}}\frac{h^l}{l!}\langle\mathcal{R}f-\Pi_xf(x),D_1^{k+l}K_m(x,\cdot)\rangle\\ &\quad+\sum_{\zeta>|k|_{\mathfrak{s}}-\beta}\langle\Pi_{x+h}\mathcal{Q}_\zeta\big(f(x+h)-\Gamma_{x+h,x}f(x)\big),D_1^kK_m(x+h,\cdot)\rangle.\end{aligned}$$

For the first term on the right hand side, we have the bound

$$\sum_{m\geq n}\fint_{Q(0,4\cdot 2^{-n})}\frac{|\langle\mathcal{R}f-\Pi_{x+h}f(x+h),D_1^kK_m(x+h,\cdot)\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h$$
$$\lesssim\sum_{m\geq n}2^{-m(\beta-|k|_{\mathfrak{s}})}\fint_{Q(0,4\cdot 2^{-n})}\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle\mathcal{R}f-\Pi_{x+h}f(x+h),\eta_{x+h}^{2^{-m}}\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h$$
$$\lesssim\sum_{m\geq n}2^{(n-m)(\gamma+\beta-|k|_{\mathfrak{s}})}\fint_{Q(x,4\cdot 2^{-n})}\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle\mathcal{R}f-\Pi_zf(z),\eta_z^{2^{-m}}\rangle|}{2^{-m\gamma}}\,\mathrm{d}z.$$

We then estimate

$$\begin{aligned}
&\fint_{Q(x,4\cdot 2^{-n})} \sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)} \frac{|\langle \mathcal{R}f - \Pi_z f(z), \eta_z^{2^{-m}}\rangle|}{2^{-m\gamma}}\,\mathrm{d}z \\
&\lesssim \sum_{\substack{y\in\Lambda_m\\ \|y-x\|_{\mathfrak{s}}\le 4\cdot 2^{-n}}} 2^{(n-m)|\mathfrak{s}|} \fint_{Q(y,4\cdot 2^{-m})} \sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)} \frac{|\langle \mathcal{R}f - \Pi_z f(z), \eta_z^{2^{-m}}\rangle|}{2^{-m\gamma}}\,\mathrm{d}z \\
&\lesssim 2^{(n-m)\kappa'}\Big\{\mathcal{M}\Big(\Big|\sum_{y\in\Lambda_m}\fint_{Q(y,4\cdot 2^{-m})}\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle \mathcal{R}f - \Pi_z f(z), \eta_z^{2^{-m}}\rangle|}{2^{-m\gamma}}\chi_y^m\,\mathrm{d}z\Big|^{\kappa}\Big)(x)\Big\}^{1/\kappa},
\end{aligned}$$

uniformly over all $0<\kappa<1$, with the shorthand $\kappa' = |\mathfrak{s}|(\kappa-1)/\kappa$. Now, choosing κ sufficiently close to 1, we obtain, with the vector-valued maximal inequality,

$$\begin{aligned}
&\Big\|\Big\|\sum_{m\ge n}\fint_{Q(0,4\cdot 2^{-n})}\frac{|\langle \mathcal{R}f - \Pi_{x+h}f(x+h), D_1^k K_m(x+h,\cdot)\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h\Big\|_{\ell^q}\Big\|_{L^p} \\
&\qquad\lesssim \Big\|\Big\|\sum_{y\in\Lambda_n}\fint_{Q(y,4\cdot 2^{-n})}\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle \mathcal{R}f - \Pi_z f(z), \eta_z^{2^{-n}}\rangle|}{2^{-n\gamma}}\chi_y^n(x)\,\mathrm{d}z\Big\|_{\ell^q}\Big\|_{L^p} \\
&\qquad\lesssim \Big\|\Big\|\fint_{Q(0,5\cdot 2^{-n})}\sup_{\eta\in\mathfrak{B}^r}\frac{|\langle \mathcal{R}f - \Pi_{x+h}f(x+h), \eta_{x+h}^{2^{-n}}\rangle|}{2^{-n\gamma}}\,\mathrm{d}h\Big\|_{\ell^q}\Big\|_{L^p}.
\end{aligned}$$

Thus, thanks to Remark 5.3 we eventually arrive at a bound of required order. Concerning the second term on the right hand side of (6.5), we derive the bound

$$\begin{aligned}
&\sum_{m\ge n}\sum_{\substack{l\in\mathbb{N}_0^d\\ |k+l|_{\mathfrak{s}}<\gamma+\beta}}\fint_{Q(0,4\cdot 2^{-n})}\frac{|h^l|}{l!}\,\frac{|\langle \mathcal{R}f - \Pi_x f(x), D_1^{k+l}K_m(x,\cdot)\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h \\
&\quad\lesssim \sum_{m\ge n}\sum_{\substack{l\in\mathbb{N}_0^d\\ |k+l|_{\mathfrak{s}}<\gamma+\beta}}\fint_{Q(0,4\cdot 2^{-n})}\frac{\|h\|_{\mathfrak{s}}^{|l|_{\mathfrak{s}}}}{2^{m(\beta-|k+l|_{\mathfrak{s}})}}\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle \mathcal{R}f - \Pi_x f(x), \eta_x^{2^{-m}}\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h \\
&\quad\lesssim \sum_{\substack{l\in\mathbb{N}_0^d\\ |k+l|_{\mathfrak{s}}<\gamma+\beta}}\sum_{m\ge n}2^{(n-m)(\gamma+\beta-|k+l|_{\mathfrak{s}})}\sup_{\eta\in\mathfrak{B}^r(\mathbb{R}^d)}\frac{|\langle \mathcal{R}f - \Pi_x f(x), \eta_x^{2^{-m}}\rangle|}{2^{-m\gamma}}.
\end{aligned}$$

Thus, by virtue of Young's inequality and the reconstruction bound we again obtain a bound of required order. It remains to derive a suitable bound for the third term on the right hand side of (6.5). In this case, we have

$$\sum_{m\geq n}\sum_{\zeta>|k|_{\mathfrak{s}}-\beta}\fint_{Q(0,4\cdot 2^{-n})}\frac{|\langle \Pi_{x+h}\mathcal{Q}_\zeta\big(f(x+h)-\Gamma_{x+h,x}f(x)\big),D_1^kK_m(x+h,\cdot)\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h$$
$$\lesssim \sum_{\zeta>|k|_{\mathfrak{s}}-\beta}\sum_{m\geq n}2^{-m(\zeta+\beta-|k|_{\mathfrak{s}})}\fint_{Q(0,4\cdot 2^{-n})}\frac{|f(x+h)-\Gamma_{x+h,x}f(x)|_\zeta}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h$$
$$\lesssim \sum_{\zeta>|k|_{\mathfrak{s}}-\beta}\sum_{m\geq n}2^{(n-m)(\zeta+\beta-|k|_{\mathfrak{s}})}\fint_{Q(0,4\cdot 2^{-n})}\frac{|f(x+h)-\Gamma_{x+h,x}f(x)|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h,$$

i.e. the desired bound follows immediately. This concludes our discussion of the regime $m\geq n$.

For $m<n$, following Hairer [5] we introduce the test functions

$$K_{n,x,y}^{k,\gamma+\beta}=D_1^kK_n(y,\cdot)-\sum_{\substack{l\in\mathbb{N}_0^d\\ |k+l|_{\mathfrak{s}}<\gamma+\beta}}\frac{(y-x)^l}{l!}D_1^{k+l}K_n(x,\cdot).$$

We then make use of the identity (cf. [5])

$$k!\mathcal{Q}_{X^k}\big(\mathcal{K}_{m,\gamma}f(x+h)-\Gamma_{x+h,x}\mathcal{K}_{m,\gamma}f(x)\big)=\langle\mathcal{R}f-\Pi_xf(x),K_{m,x,x+h}^{k,\gamma+\beta}\rangle$$
$$+\sum_{\zeta\leq|k|_{\mathfrak{s}}-\beta}\langle \Pi_{x+h}\mathcal{Q}_\zeta\big(f(x+h)-\Gamma_{x+h,x}f(x)\big),D_1^kK_m(x+h,\cdot)\rangle.$$

As above, we bound each of the two terms on the right hand side separately. For the second term, we first observe that terms related to homogeneities with $\zeta+\beta=|k|_{\mathfrak{s}}$ actually do not contribute: our assumptions imply that $\zeta\in\mathbb{N}_0$, and as K_0 annihilates polynomials of scaled degree $\leq r$, these terms vanish due to the action of the model on the canonical structure. Hence, we can restrict the sum to homogeneities $\zeta\in\mathcal{A}_\gamma$ such that $\zeta<|k|_{\mathfrak{s}}-\beta$. Then we get

$$\sum_{\zeta<|k|_{\mathfrak{s}}-\beta}\sum_{m<n}\fint_{Q(0,4\cdot 2^{-n})}\frac{|\langle \Pi_{x+h}\mathcal{Q}_\zeta\big(f(x+h)-\Gamma_{x+h,x}f(x)\big),D_1^kK_m(x+h,\cdot)\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h$$
$$\lesssim \sum_{\zeta<|k|_{\mathfrak{s}}-\beta}\sum_{m<n}2^{-m(\zeta+\beta-|k|_{\mathfrak{s}})}\fint_{Q(0,4\cdot 2^{-n})}\frac{|f(x+h)-\Gamma_{x+h,x}f(x)|_\zeta}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\,\mathrm{d}h$$
$$\lesssim \sum_{\zeta<|k|_{\mathfrak{s}}-\beta}\sum_{m<n}2^{(n-m)(\zeta+\beta-|k|_{\mathfrak{s}})}\fint_{Q(0,4\cdot 2^{-n})}\frac{|f(x+h)-\Gamma_{x+h,x}f(x)|_\zeta}{2^{-n(\gamma-\zeta)}}\,\mathrm{d}h,$$

which immediately entails a bound of required type. It remains to bound the term involving $K_{n,x,y}^{k,\gamma+\beta}$. Here, we first recall the following scaled version of Taylor's theorem [5, Prop. A.1]. Let e_i denote the ith standard unit vector of $\mathbb{R}^d$ and let, for $\gamma' > 0$,

$$\partial\gamma' = \{l \in \mathbb{N}_0^d \setminus \{0\} \colon |l|_{\mathfrak{s}} \geq \gamma',\ |l - e_{\mathfrak{m}(l)}|_{\mathfrak{s}} < \gamma'\},$$

where $\mathfrak{m}(l) = \inf\{i \in \{1, \dots, d\} \colon l_i \neq 0\}$. Then

$$K^{k,\gamma+\beta}_{m,x,x+h} = \sum_{\substack{l \in \mathbb{N}_0^d \setminus \{0\} \\ k+l \in \partial(\gamma+\beta)}} \int_{\mathbb{R}^d} D_1^{k+l} K_m(x+z, \cdot)\, \mu^l(h, \mathrm{d}z), \tag{6.6}$$

where $\mu^l(h, \mathrm{d}z)$ is a signed measure on $\mathbb{R}^d$ with total mass $h^l/l!$ and support in $\{z \in \mathbb{R}^d \colon z_i \in [0, h_i]\}$. Since $\gamma + \beta \notin \mathbb{N}$ by assumption, the set $\partial(\gamma+\beta)$ is actually identical to $\{l \in \mathbb{N}_0^d \setminus \{0\} \colon |l|_{\mathfrak{s}} > \gamma + \beta,\ |l - e_{\mathfrak{m}(l)}|_{\mathfrak{s}} < \gamma + \beta\}$. Now, one can find an absolute constant $C > 0$ such that

$$\begin{aligned}
&\sum_{m<n} \fint_{Q(0,4\cdot 2^{-n})} \frac{|\langle \mathcal{R}f - \Pi_x f(x), K^{k,\gamma+\beta}_{m,x,x+h}\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}}\, \mathrm{d}h \\
&\quad\lesssim \sum_{\substack{l \in \mathbb{N}_0^d \setminus \{0\} \\ k+l \in \partial(\gamma+\beta)}} \sum_{m<n} 2^{-m(\beta-|k+l|_{\mathfrak{s}})} 2^{-n|l|_{\mathfrak{s}}} \sup_{\eta \in \mathfrak{B}^r_C(\mathbb{R}^d)} \frac{|\langle \mathcal{R}f - \Pi_x f(x), \eta_x^{2^{-m}}\rangle|}{2^{-n(\gamma-|k|_{\mathfrak{s}})}} \\
&\quad\lesssim \sum_{\substack{l \in \mathbb{N}_0^d \setminus \{0\} \\ k+l \in \partial(\gamma+\beta)}} \sum_{m<n} 2^{(n-m)(\gamma+\beta-|k+l|_{\mathfrak{s}})} \sup_{\eta \in \mathfrak{B}^r_C(\mathbb{R}^d)} \frac{|\langle \mathcal{R}f - \Pi_x f(x), \eta_x^{2^{-m}}\rangle|}{2^{-m\gamma}}.
\end{aligned}$$

By Young's inequality and Remark 5.2, this is again sufficient to obtain a bound of desired order. In particular, the asserted bound on the $\mathcal{F}^{\gamma+\beta}_{p,q}$-norm of $\mathcal{K}_\gamma f$ follows.

It remains to verify (6.1). Of course, one could adapt the arguments of [7] to the scale $\mathcal{F}^\gamma_{p,q}$, but we prefer to make use of the embeddings $\mathcal{F}^{\gamma+\beta}_{p,q} \subset \mathcal{B}^{\gamma+\beta}_{p,q\vee p}$ and $\mathcal{F}^\gamma_{p,q} \subset \mathcal{B}^\gamma_{p,q\vee p}$. Indeed, just note that all the constructions involved coincide on the respective spaces, i.e. the identity is satisfied as it holds true for the $\mathcal{B}^\gamma_{p,q}$ spaces. ∎

7. Products of modelled distributions. An essential tool in the theory of regularity structures is the possibility to build products of modelled distributions. In this section we address this issue in the framework of Triebel–Lizorkin spaces. We recall the definition of an abstract product in a regularity structure [5].

DEFINITION 7.1 (Sector). Given a regularity structure $(\mathcal{A}, \mathcal{T}, \mathcal{G})$, a set $V \subset \mathcal{T}$ is called a *sector* if $\Gamma\tau \in V$ for any $\tau \in V$, $\Gamma \in \mathcal{G}$ and we can write $V = \bigoplus_{\alpha\in\mathcal{A}} V_\alpha$ with $V_\alpha \subset \mathcal{T}_\alpha$. The *regularity* of V is

$$\alpha_V = \min\{\alpha \in \mathcal{A} : V_\alpha \neq \emptyset\}.$$

Definition 7.2 (Product). Consider a regularity structure $(\mathcal{T},\mathcal{A},\mathcal{G})$ with two sectors $V,\overline{V}$. A *product* between V and $\overline{V}$ is a map

$$\star\colon V\times\overline{V}\to\mathcal{T}$$

such that, for any $\alpha\in\mathcal{A}$ and $\beta\in\mathcal{A}$, its restriction to $V_\alpha\times\overline{V}_\beta$ is a continuous bilinear map $\star\colon V_\alpha\times\overline{V}_\beta\to\mathcal{T}_{\alpha+\beta}$ and

$$\Gamma(\tau\star\bar\tau)=\Gamma\tau\star\Gamma\bar\tau\quad\text{ for all } \Gamma\in\mathcal{G},\ \tau\in V_\alpha \text{ and } \bar\tau\in\overline{V}_\beta.$$

Finally, for a given α we say that $f\in\mathcal{F}^{\gamma,\alpha}_{p,q}$ if it lies in $\mathcal{F}^\gamma_{p,q}$ and $f(x)\in\mathcal{T}_{\geq\alpha}$ for all x. For a given sector V we say that $f\in\mathcal{F}^\gamma_{p,q}(V)$ if $f\in\mathcal{F}^\gamma_{p,q}$ and $f(x)\in V$ for all x.

Proposition 7.3. *Consider a regularity structure with two sectors V_1,V_2 of regularity α_1,α_2 respectively which are endowed with a product between them. For any $f\in\mathcal{F}^{\gamma_1}_{p_1,q_1}(V_1)$ and $g\in\mathcal{F}^{\gamma_2}_{p_2,q_2}(V_2)$ with $1\leq p_i<\infty$ and $1\leq q_i\leq\infty$ the pointwise product*

$$fg(x):=f(x)\star g(x)$$

lies in $\mathcal{F}^{\gamma,\alpha}_{p,q}$ with

$$\alpha=\alpha_1+\alpha_2,\qquad \gamma=(\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1),$$
$$p=\frac{p_1p_2}{p_1+p_2},\qquad q=q_1\vee q_2,$$

provided that $p\geq 1$. In addition,

$$|||fg|||_{\mathcal{F}^\gamma_{p,q}}\lesssim |||f|||_{\mathcal{F}^{\gamma_1}_{p_1,q_1}}|||g|||_{\mathcal{F}^{\gamma_2}_{p_2,q_2}}.$$

Proof. First we consider a sequence of functions $\pi^{(n)}:\Lambda_n\to\mathcal{T}$ which is a discrete version of the product:

$$\pi^{(n)}(x)=\bar f^{(n)}(x)\star\bar g^{(n)}(x).$$

Note that this definition may produce contributions in $\mathcal{T}_{\geq\gamma}$. Since the statement of the proposition only requires us to encode the product up to homogeneities $<\gamma$, we set all contributions in $\mathcal{T}_{\geq\gamma}$ to zero, i.e. we will actually study

$$\pi^{(n)}_{<\gamma}(x):=\sum_{k+l<\gamma}\mathcal{Q}_k\bar f^{(n)}(x)\star\mathcal{Q}_l\bar g^{(n)}(x).$$

Thus, we aim at proving that $\pi^{(n)}_{<\gamma}\in\bar{\mathcal{F}}^\gamma_{p,q}$, since $\pi^{(n)}_{<\gamma}\in\mathcal{T}_{\geq\alpha}$ by construction. Then, Proposition 3.9 implies that $\pi^{(n)}_{<\gamma}$ corresponds to a function $\pi_{<\gamma}\in\mathcal{F}^\gamma_{p,q}$. It thus suffices to show that $\pi_{<\gamma}=fg$ to conclude the proof.

We will use the fact that uniformly over ζ in a bounded set, there exist only a finite number of homogeneities $\zeta_1,\zeta_2\in\mathcal{A}$ with $\zeta_i\geq\alpha_i$ such that $\zeta_1+\zeta_2=\zeta$. The local bound follows directly from Hölder's inequality, so we concentrate on the translation bound for $\pi^{(n)}_{<\gamma}$. Fix $\zeta < \gamma$. Then

$$
\begin{aligned}
&\pi^{(n)}_{<\gamma}(y+h) - \Gamma_{y+h,y}\pi^{(n)}_{<\gamma}(y)\\
&= \sum_{k+l<\gamma} \big\{ \mathcal{Q}_k \bar{f}^{(n)}(y+h) \star \mathcal{Q}_l \bar{g}^{(n)}(y+h) - \Gamma_{y+h,y}\mathcal{Q}_k \bar{f}^{(n)}(y) \star \Gamma_{y+h,y}\mathcal{Q}_l \bar{g}^{(n)}(y) \big\}\\
&= \sum_{k+l<\gamma} \mathcal{Q}_k \big(\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(n)}(y)\big) \star \mathcal{Q}_l \bar{g}^{(n)}(y+h)\\
&\quad + \sum_{k+l<\gamma} \mathcal{Q}_k \Gamma_{y+h,y}\bar{f}^{(n)}(y) \star \mathcal{Q}_l \big(\bar{g}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{g}^{(n)}(y)\big)\\
&\quad + \mathrm{Res}_\gamma(\bar{f}^{(n)}, \bar{g}^{(n)}),
\end{aligned}
$$

where we introduced the "error term"

$$
\begin{aligned}
\mathrm{Res}_\gamma(\bar{f}^{(n)}, \bar{g}^{(n)}) := &\sum_{k+l<\gamma} \mathcal{Q}_k \Gamma_{y+h,y}\bar{f}^{(n)}(y) \star \mathcal{Q}_l \Gamma_{y+h,y}\bar{g}^{(n)}(y)\\
&- \sum_{k+l<\gamma} \Gamma_{y+h,y}\mathcal{Q}_k \bar{f}^{(n)}(y) \star \Gamma_{y+h,y}\mathcal{Q}_l \bar{g}^{(n)}(y)\\
&= \mathcal{Q}_{<\gamma}\Big(\sum_{k+l\geq\gamma} \Gamma_{y+h,y}\mathcal{Q}_k \bar{f}^{(n)}(y) \star \Gamma_{y+h,y}\mathcal{Q}_l \bar{g}^{(n)}(y) \Big).
\end{aligned}
$$

To obtain the first identity, we made use of the property that the product commutes with the action of $\Gamma \in \mathcal{G}$. Hence, we may bound

$$
\begin{aligned}
&|\pi^{(n)}_{<\gamma}(y+h) - \Gamma_{y+h,y}\pi^{(n)}_{<\gamma}(y)|_\zeta\\
&\quad\leq \sum_{\zeta_1+\zeta_2=\zeta} |\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{f}^{(n)}(y)|_{\zeta_1} |\bar{g}^{(n)}(y+h)|_{\zeta_2}\\
&\qquad + \sum_{\zeta_1+\zeta_2=\zeta} |\Gamma_{y+h,y}\bar{f}^{(n)}(y)|_{\zeta_1} |\bar{g}^{(n)}(y+h) - \Gamma_{y+h,y}\bar{g}^{(n)}(y)|_{\zeta_2}\\
&\qquad + |\mathrm{Res}_\gamma(\bar{f}^{(n)}, \bar{g}^{(n)})|_\zeta.
\end{aligned}
\tag{7.1}
$$

For the last term, note first that due to Assumption 3.6 the sum defining $\mathrm{Res}_\gamma(\bar{f}^{(n)}, \bar{g}^{(n)})$ actually runs over all homogeneities $k \in \mathcal{A}_{\gamma_1}$ and $l \in \mathcal{A}_{\gamma_2}$ such that $k + l > \gamma$. Hence, we find $\varepsilon > 0$ such that

$$
\begin{aligned}
|\mathrm{Res}_\gamma(\bar{f}^{(n)}, \bar{g}^{(n)})|_\zeta &\lesssim \sum_{k+l>\gamma} \sum_{\zeta_1+\zeta_2=\zeta} \|h\|_{\mathfrak{s}}^{k+l-\zeta_1-\zeta_2} |\bar{f}^{(n)}(y)|_k |\bar{g}^{(n)}(y)|_l\\
&\lesssim 2^{-n\varepsilon} \sum_{k+l>\gamma} 2^{-n(\gamma-\zeta)} |\bar{f}^{(n)}(y)|_k |\bar{g}^{(n)}(y)|_l,
\end{aligned}
$$

uniformly over all $n \geq 0$, all $y \in \Lambda_n$, all $h \in \mathcal{E}_n$ and all $\zeta \in \mathcal{A}_\gamma$. From this we can deduce with the aid of Hölder's inequality the bound

$$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}\frac{|\mathrm{Res}_\gamma(\bar f^{(n)},\bar g^{(n)})|_\zeta}{2^{-n(\gamma-\zeta)}}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}\lesssim\Big(\sum_{n\geq 0}2^{-n\varepsilon q}\Big)^{1/q}$$

$$\times\sup_{k+l>\gamma}\Big\|\sup_{n\geq 0}\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}|\bar f^{(n)}(y)|_k\chi_y^n(x)\Big\|_{L^{p_1}}\Big\|\sup_{n\geq 0}\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}|\bar g^{(n)}(y)|_l\chi_y^n(x)\Big\|_{L^{p_2}},$$

which is of required order due to Remark 3.7. Regarding the first term on the right hand side of (7.1), we immediately obtain the bound

$$\Big\|\Big(\sum_{n\geq 0}\Big|\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}\frac{|\bar f^{(n)}(y+h)-\Gamma_{y+h,y}\bar f^{(n)}|_{\zeta_1}}{2^{-n(\gamma-\zeta)}}|\bar g^{(n)}(y+h)|_{\zeta_2}\chi_y^n(x)\Big|^q\Big)^{1/q}\Big\|_{L^p}$$

$$\leq\Big\|\sup_{n\geq 0}\sum_{y\in\Lambda_n}\sum_{h\in\mathcal{E}_n}|\bar g^{(n)}(y+h)|_{\zeta_2}\chi_y^n(x)\Big\|_{L^{p_2}}|||\bar f|||_{\bar F^{\gamma_1}_{p_1,q_1}},$$

uniformly over $\zeta_1+\zeta_2=\zeta$; it is therefore of required order due to Remark 3.7 and Proposition 3.8. Analogously, one obtains a bound of required order for the second term on the right hand side of (7.1). This establishes the translation bound.

Since the consistency bound follows in exactly the same way, we conclude that $\pi^{(n)}_{<\gamma}\in\bar{\mathcal{F}}^\gamma_{p,q}$. This implies in particular that there exists a function $\pi:=\pi_{<\gamma}\in\mathcal{F}^\gamma_{p,q}$ such that

$$\lim_{n\to\infty}\mathcal{Q}_\zeta\pi^{(n)}_{<\gamma}=\mathcal{Q}_\zeta\pi\quad\text{ in } L^p.$$

Now, since $\pi^{(n)}_{<\gamma}=\sum_{k+l<\gamma}\mathcal{Q}_l f^{(n)}\star\mathcal{Q}_k g^{(n)}$ and

$$\mathcal{Q}_\zeta f^{(n)}\to\mathcal{Q}_\zeta f\quad\text{in } L^{p_1},$$
$$\mathcal{Q}_\zeta g^{(n)}\to\mathcal{Q}_\zeta g\quad\text{in } L^{p_2},$$

it follows from Hölder's inequality that $\mathcal{Q}_\zeta\pi^{(n)}_{<\gamma}\to\mathcal{Q}_\zeta fg$ in L^p for all homogeneities $\zeta\in[\alpha,\gamma)\cap\mathcal{A}$. ■

Remark 7.4. The condition $p\geq 1$ in the previous theorem is not necessary. However, stating the theorem in the general case would require defining Triebel–Lizorkin spaces for $p,q\in(0,1)$. Since the article is already quite long, we refrained from doing so.

Remark 7.5. The previous result together with Theorem 5.1 applied to the polynomial regularity structure allows one to deduce that the product of two Triebel–Lizorkin distributions is a well defined continuous bilinear map for $\gamma>0$, thus recovering a well known result from harmonic analysis.

8. A note on the Besov scale. In this section, we discuss a phenomenon which is well known from the classical theory of Besov spaces: that one obtains the same space whether one considers differences of a function or volume means of differences. We will show that this is still the case in the framework of regularity structures.

DEFINITION 8.1. Let $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ be a regularity structure and (Π, Γ) a model. Let $1 \le p, q \le \infty$ and $\gamma > 0$. We define $\mathcal{D}^{\gamma}_{p,q}$ to be the (Banach) space of all functions $f\colon \mathbb{R}^d \to \mathcal{T}^{-}_{\gamma}$ such that

(i) $\sup_{\zeta \in \mathcal{A}_\gamma} \big\| |f(x)|_\zeta \big\|_{L^p} < \infty$,

(ii) $\sup_{\zeta \in \mathcal{A}_\gamma} \Big\| \Big\| ⨍_{Q(0,4\lambda)} \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_\zeta}{\lambda^{\gamma-\zeta}} \, \mathrm{d}h \Big\|_{L^p} \Big\|_{L^q_\lambda} < \infty.$

The associated norm of $f \in \mathcal{D}^{\gamma}_{p,q}$ is denoted by $|||f|||_{\mathcal{D}^{\gamma}_{p,q}}$.

For $p < \infty$ and $q = \infty$, the space $\mathcal{D}^{\gamma}_{p,q}$ was already introduced and studied in the work of Hairer and Labbé [6] on multiplicative stochastic heat equations.

DEFINITION 8.2. Let $(\mathcal{A}, \mathcal{T}, \mathcal{G})$ be a regularity structure and (Π, Γ) a model. Let $1 \le p, q \le \infty$ and $\gamma > 0$. We denote by $\bar{\mathcal{B}}^{\gamma}_{p,q}$ the (Banach) space of all sequences of maps

$$\bar{f}^{(n)}\colon \Lambda_n \to \mathcal{T}^{-}_{\gamma}, \qquad n \ge 0,$$

such that, uniformly over all $\zeta \in \mathcal{A}_\gamma$,

(i) $\Big(\sum_{y \in \Lambda_0} |\bar{f}^{(0)}(y)|^p_\zeta \Big)^{1/p} < \infty$,

(ii) $\Big(\sum_{n \ge 0} \sum_{h \in \mathcal{E}_n} \Big\| \frac{|\bar{f}^{(n)}(y+h) - \Gamma_{y+h,y} \bar{f}^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \Big\|^q_{\ell^p_n} \Big)^{1/q} < \infty$,

(iii) $\Big(\sum_{n \ge 0} \Big\| \frac{|\bar{f}^{(n)}(y) - \bar{f}^{(n+1)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}} \Big\|^q_{\ell^p_n} \Big)^{1/q} < \infty.$

The associated norm of $\bar{f} \in \bar{\mathcal{B}}^{\gamma}_{p,q}$ will be denoted by $|||\bar{f}|||_{\bar{\mathcal{B}}^{\gamma}_{p,q}}$.

Again, there is essentially no difference between the spaces $\mathcal{D}$ and $\bar{\mathcal{B}}$:

PROPOSITION 8.3. *Let $\gamma > 0$ and $1 \le p, q \le \infty$. Given $f \in \mathcal{D}^{\gamma}_{p,q}$ and $n \ge 0$, define $\bar{f}^{(n)}\colon \Lambda_n \to \mathcal{T}^{-}_{\gamma}$ via*

$$\bar{f}^{(n)}(y) = ⨍_{Q(y,2^{-n})} \Gamma_{y,z} f(z) \, \mathrm{d}z. \tag{8.1}$$

Then $\bar f \in \bar{\mathcal{B}}^\gamma_{p,q}$ *and* $\|\!|\bar f\|\!|_{\bar{\mathcal{B}}^\gamma_{p,q}} \lesssim \|\!|f\|\!|_{\mathcal{D}^\gamma_{p,q}}$. *In addition,*

$$\Gamma_{x,x_n} \bar f^{(n)}(x_n) \to f(x) \quad \textit{in } L^p.$$

Conversely, for any $\bar f \in \bar{\mathcal{B}}^\gamma_{p,q}$ *there is* $f\colon \mathbb{R}^d \to \mathcal{T}^-_\gamma$ *such that*

$$f_n(x) := \Gamma_{x,x_n} \bar f^{(n)}(x_n) \to f(x) \quad \textit{in } L^p,$$

where actually $f \in \mathcal{D}^\gamma_{p,q}$ *with* $\|\!|f\|\!|_{\mathcal{D}^\gamma_{p,q}} \lesssim \|\!|\bar f\|\!|_{\bar{\mathcal{B}}^\gamma_{p,q}}$.

Proof. Let us prove the first assertion, i.e. we fix $f \in \mathcal{D}^\gamma_{p,q}$ and we show that $\bar f \in \bar{\mathcal{B}}^\gamma_{p,q}$ where $\bar f$ is defined in (8.1). The respective local bound follows directly. For the translation bound, we observe that

$$\begin{aligned}
&\frac{|\bar f^{(n)}(y+h) - \Gamma_{y+h,y} \bar f^{(n)}(y)|_\zeta}{2^{-n(\gamma-\zeta)}}\\
&\lesssim \fint_{Q(y,2^{-n})} \fint_{Q(x,2^{-n})} \frac{|\Gamma_{y+h,z+h+y-x}(f(z+h+y-x) - \Gamma_{z+h+y-x,x} f(x))|_\zeta}{2^{-n(\gamma-\zeta)}} \,\mathrm{d}z \,\mathrm{d}x\\
&\lesssim \sum_{\beta\ge\zeta} \fint_{Q(y,2^{-n})} \fint_{Q(x,2\cdot 2^{-n})} \frac{|f(z+h) - \Gamma_{z+h,x} f(x)|_\beta}{2^{-n(\gamma-\beta)}} \,\mathrm{d}z \,\mathrm{d}x\\
&\lesssim \sum_{\beta\ge\zeta} \fint_{Q(y,2^{-n})} \fint_{Q(0,2\cdot 2^{-n})} \frac{|f(x+z+h) - \Gamma_{x+z+h,x} f(x)|_\beta}{2^{-n(\gamma-\beta)}} \,\mathrm{d}z \,\mathrm{d}x\\
&\lesssim \sum_{\beta\ge\zeta} \fint_{Q(y,2^{-n})} \fint_{Q(0,3\cdot 2^{-n})} \frac{|f(x+h') - \Gamma_{x+h',x} f(x)|_\beta}{2^{-n(\gamma-\beta)}} \,\mathrm{d}h' \,\mathrm{d}x,
\end{aligned}$$

which already leads to a bound of desired order. The consistency bound can be derived analogously. The convergence assertion follows from the same argument as for the Triebel–Lizorkin scale $\mathcal{F}^\gamma_{p,q}$.

Let us turn to the second statement, i.e. consider $\bar f \in \bar{\mathcal{B}}^\gamma_{p,q}$. Along the same lines as for the Triebel–Lizorkin scale, one obtains

$$\sum_{n\ge n_0} \big\| |f_{n+1}(x) - f_n(x)|_\zeta \big\|_{L^p} \lesssim 2^{-n_0(\gamma-\zeta)} \|\!|\bar f\|\!|_{\bar{\mathcal{B}}^\gamma_{p,q}},$$

uniformly over all $n_0 \ge 0$. Denote by $f\colon \mathbb{R}^d \to \mathcal{T}^-_\gamma$ the associated limit in L^p. We aim to show that $f \in \mathcal{D}^\gamma_{p,q}$ with a corresponding bound for its norm. To this end, it is again convenient to bound

$$\begin{aligned}
&\bigg\|\bigg\| \fint_{Q(0,4\lambda)} \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_\zeta}{\lambda^{\gamma-\zeta}} \,\mathrm{d}h \bigg\|_{L^p} \bigg\|_{L^q_\lambda}\\
&\qquad\qquad \lesssim \bigg(\sum_{n\ge 0} \bigg\| \fint_{Q(0,4\cdot 2^{-n})} \frac{|f(x+h) - \Gamma_{x+h,x} f(x)|_\zeta}{2^{-n(\gamma-\zeta)}} \,\mathrm{d}h \bigg\|^q_{L^p} \bigg)^{1/q}.
\end{aligned}$$

Now, we make use of the decomposition (3.5). Contributions from the last two terms can be treated along the same lines as in the case of the Triebel–Lizorkin scale. Thus, let us only discuss the contribution due to the first term, i.e. $f(x+h) - f_n(x+h)$ with $x \in \mathbb{R}^d$, $n \geq 0$ and $h \in Q(0, 4 \cdot 2^{-n})$. We bound (with obvious modification if $p = \infty$ and/or $q = \infty$)

$$\bigg\| \fint_{Q(0,4\cdot 2^{-n})} \frac{|f(x+h) - f_n(x+h)|_\zeta}{2^{-n(\gamma-\zeta)}} \, \mathrm{d}h \bigg\|_{L^p}^q \\ \lesssim \bigg(\fint_{Q(0,4\cdot 2^{-n})} \bigg\| \frac{|f(x+h) - f_n(x+h)|_\zeta}{2^{-n(\gamma-\zeta)}} \bigg\|_{L^p}^p \mathrm{d}h \bigg)^{q/p} = \bigg\| \frac{|f(x) - f_n(x)|_\zeta}{2^{-n(\gamma-\zeta)}} \bigg\|_{L^p}^q .$$

Thus, we have reduced the argument to the case of the third term on the right hand side of (3.5), which concludes the proof. ∎

Recall from [7] that $|||f|||_{\mathcal{B}^\gamma_{p,q}} \sim |||\bar{f}|||_{\bar{\mathcal{B}}^\gamma_{p,q}}$, where $\bar{f}$ is defined by the local averages as in (8.1). We therefore obtain the result announced at the beginning of this section.

COROLLARY 8.4. *Let* $1 \leq p, q \leq \infty$. *Then* $\mathcal{D}^\gamma_{p,q} = \mathcal{B}^\gamma_{p,q}$ *in the sense of equivalent norms.*

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