THE LOCAL STRUCTURE OF THE ENERGY LANDSCAPE IN MULTIPHASE MEAN CURVATURE FLOW: WEAK-STRONG UNIQUENESS AND STABILITY OF EVOLUTIONS

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Abstract. We prove that in the absence of topological changes, the notion of BV solutions to planar multiphase mean curvature flow does not allow for a mechanism for (unphysical) non-uniqueness. Our approach is based on the local structure of the energy landscape near a classical evolution by mean curvature. Mean curvature flow being the gradient flow of the surface energy functional, we develop a gradient-flow analogue of the notion of calibrations. Just like the existence of a calibration guarantees that one has reached a global minimum in the energy landscape, the existence of a "gradient flow calibration" ensures that the route of steepest descent in the energy landscape is unique and stable.

1. Introduction

In evolution problems for interfaces, the occurrence of topology changes and the associated geometric singularities generally limits the applicability of classical solution concepts to a finite time horizon, depending on the initial data. The evolution beyond topology changes can only be described in the framework of suitably weakened solution concepts. However, weak concepts may in general suffer from an (unphysical) loss of uniqueness of solutions: For example, in the framework of Brakke solutions [6] to mean curvature flow (MCF), the interface may suddenly disappear at any time (see Figure 2 for an illustration). In particular, Brakke solutions fail to be unique, even prior to the onset of geometric singularities in the classical solution. With the exception of evolution equations subject to a comparison principle such as two-phase mean curvature flow [14, 22], only few positive results on uniqueness of weak solutions for interface evolution problems are known.

In the present work, we establish a weak-strong uniqueness principle for distributional solutions (in the framework of finite perimeter sets, a solution concept also known as “BV solutions”) to multiphase mean curvature flow: As long as a strong solution to planar multiphase mean curvature flow – in the sense of an evolution of smooth curves meeting at triple junctions at an angle of 120° – exists, any distributional solution starting from the same initial conditions must coincide with it. Note that for regular initial data, strong solutions are known to exist until a topology change in the network of evolving curves occurs, see for instance [36].
In particular, our result establishes uniqueness of distributional solutions to planar multiphase mean curvature flow in the absence of topology changes.

The key insight in our present work is the observation that in analogy to the notion of calibrations for minimizers of the surface energy functional, one may develop a notion of calibrations for its gradient flow. Just like classical calibrations carry information on the global structure of the energy landscape – namely, a global lower bound for the energy – , “gradient flow calibrations” carry information on the local structure of the energy landscape near a partition evolving by mean curvature: The existence of a gradient flow calibration implies that the path of steepest descent in the energy landscape of the surface energy functional is unique and stable with respect to perturbations of the initial condition.\(^1\)

We implement this strategy in general ambient dimension \(d \geq 2\) by proving that the existence of a gradient flow calibration implies an inclusion principle for BV solutions to multiphase mean curvature flow ensuring that they are contained in the calibrated flow. This essentially reduces proving the desired weak-strong uniqueness to the construction of such a gradient flow calibration for strong solutions, which we provide in the planar case \(d = 2\). However, we would like to emphasize that conceptually the approach carries over to multiple dimensions. In particular, with the techniques used in the present paper it is for example possible to calibrate the smooth evolution of a double bubble; the adaptation of our arguments will be elaborated on in the upcoming note [24]. However, as soon as quadruple junctions (as they typically occur in three spatial dimensions) are present in the initial data, an additional construction is needed; nevertheless, we expect the principles of our present construction to guide the construction also in this situation.

1.1. Multiphase mean curvature flow. Mathematically, mean curvature flow is one of the most studied geometric evolution equations. Being the gradient flow of the area functional with respect to the \(L^2(S_t)\) distance, it constitutes the perhaps most natural area-reducing flow for submanifolds. Its multiphase variant may be

\(^1\)While in the present work “path of steepest descent” is to be understood as “BV solution to multiphase mean curvature flow”, we will give a rigorous statement of this notion at the level of the energy functional in a future work.
Figure 2. Top: An initially circular interface evolving by mean curvature flow. In finite time the interface shrinks to a point and disappears, giving rise to a geometric singularity and a topology change. Bottom: In Brakke solutions to mean curvature flow, the interface may suddenly disappear at any time, leading to a drastic failure of uniqueness of solutions.

seen as the simplest case of mean curvature flow for a non-smooth surface, allowing for “branching” of the surface (see e.g. Figure 1).

Multiphase mean curvature flow also is an important phenomenological model for the motion of grain boundaries in polycrystals (“grains” being the domains in a polycrystal with a single crystallographic orientation): Their evolution may be approximated as the gradient flow of the surface energy between the different grains, see for instance the seminal work of Mullins [39]. While in principle the motion of grain boundaries is governed by anisotropic mean curvature flow or even more complex evolution equations [25, 13], isotropic multiphase mean curvature flow may be viewed as an important model case for these equations. For recent developments in anisotropic and crystalline curvature flows, we refer to Caselles and Chambolle [9] and Chambolle, Morini, and Ponsiglione [12].

The existence theory for solutions to multiphase mean curvature flow is quite well-developed: Classical solutions to planar multiphase mean curvature flow are known to exist (and to be unique) for short times, see Bronsard and Reitich [7]. For initial configurations close to an equilibrium state, classical solutions exist even globally in time, see Kinderlehrer and Liu [30]. In the higher-dimensional case, Depner, Garcke, and Kohsaka [19] have shown the local-in-time existence of classical solutions for the evolution of a double bubble. In principle, Brakke’s concept of varifold solutions [6] is applicable to multiphase mean curvature flow. However, it suffers from the well-known shortcoming of exhibiting a drastic and unphysical failure of uniqueness of solutions [6] as mentioned above; see Figure 2 for an illustration.

The existence of classical solutions to planar multiphase mean curvature flow up to finitely many singular times – a solution concept that we will refer to as “classical solutions with restarting” – has been established by Mantegazza, Novaga, Pluda, and Schulze [36] under the assumption that certain types of singularities do not accumulate, extending earlier results by Ilmanen, Neves, and Schulze [28] and Mantegazza, Novaga, and Tortorelli [37]. However, it is not evident how to generalize this notion of solutions to the higher-dimensional case, as it relies on the classification of potential singularities.
Figure 3. An example of a nonunique evolution of multiphase mean curvature flow, starting from an initial interface consisting only of smooth curves meeting at an angle of 120°.

In [31, 32], a conditional convergence result for an efficient numerical scheme for multiphase mean curvature flow – the thresholding scheme of Merriman, Bence, and Osher [38] – towards BV solutions of multiphase mean curvature flow has been shown by Otto and the third author, thereby also establishing a conditional existence result for BV solutions. In [33], a conditional convergence result for the Allen-Cahn approximation for multiphase mean curvature flow towards BV solutions has been derived by the third and the fourth author. Both results employ an assumption of convergence of the interface area, analogous to the one in Luckhaus-Sturzenhecker [35] for the implicit time discretization developed by Luckhaus-Sturzenhecker and Almgren-Taylor-Wang [2].

1.2. The uniqueness properties of multiphase mean curvature flow. The uniqueness properties of weak solution concepts for multiphase mean curvature flow have remained essentially unexplored. For two-phase mean curvature flow, a combination of the level-set formulation by Osher and Sethian [41] and Ohta, Jasnow, and Kawasaki [40], and the concept of viscosity solutions by Crandall and Lions [17] facilitates an existence and uniqueness theory for a weak notion of solutions, as shown by Chen, Giga, and Goto [14] and Evans and Spruck [22]. While these viscosity solutions to two-phase mean curvature flow are unique, a given level set may “fatten” [5], thereby failing to describe an interface and indicating the emergence of a non-unique evolution of the surface. Nevertheless, fattening is known to not occur prior to the first topology change, provided that one starts with a smooth initial surface. Unfortunately, the absence of a comparison principle for multiphase mean curvature flow a priori prevents the applicability of these techniques in the multiphase case.

Recently, a notion of solutions for multiphase mean curvature flow has been proposed by Kim and Tonegawa [29] which is likely to exclude the unphysical sudden vanishing of the interface in Brakke solutions. While we have not yet been able to establish a weak-strong uniqueness principle for their solution concept, it seems likely that our techniques may also contribute to the analysis of the uniqueness properties of this notion of solutions.

The example in Figure 3 shows that after topology changes, the uniqueness of BV solutions to planar multiphase mean curvature flow may fail. Note that in contrast to the sudden vanishing of the interface in Brakke solutions, this is a case of physical non-uniqueness: The failure of uniqueness is caused by a physically unstable situation – the symmetric configuration of four perfect squares –, starting
Table 1. An overview of solutions concepts for multiphase mean curvature flow.

<table>
<thead>
<tr>
<th>Solution concept</th>
<th>Topology changes</th>
<th>Uniqueness prior to topology changes</th>
<th>Existence theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kim-Tonegawa solutions</td>
<td>possible</td>
<td>? (likely yes(^2))</td>
<td>yes [29]</td>
</tr>
<tr>
<td>classical solutions with restarting (2D only)</td>
<td>possible</td>
<td>yes [36]</td>
<td>yes(^3) [36]</td>
</tr>
<tr>
<td>BV solutions</td>
<td>possible</td>
<td><strong>yes (Theorem 1)</strong></td>
<td>yes(^4) [31, 33]</td>
</tr>
</tbody>
</table>

from which infinitesimal perturbations may select either of the two evolutions. This example also shows that a principle of maximal dissipation of energy may fail to single out a unique evolution.

Our main result – a uniqueness theorem for BV solutions to planar multiphase mean curvature flow prior to the first topology change – is therefore not only the first positive result concerning uniqueness for a weak solution concept to multiphase mean curvature flow, but also optimal for general initial data. Nevertheless, let us mention that it has been suggested by Ilmanen (see e.g. [36]) that the uniqueness properties may be better if one restricts one’s attention to generic initial data: For initial data given by a small random perturbation of a fixed multiphase interface, the evolution by mean curvature in the plane is expected to be unique and stable with respect to perturbations for almost every perturbation. The argument in favor of this proposed phenomenon is based on a numerical study classifying the “stable” and therefore “generically occurring” singularities in planar mean curvature flow [26]. Evidence in favor of “generic” well-posedness is abundant in two-phase mean curvature flow: For instance, an infinitesimal amount of stochastic noise has been shown to yield selection principles for the evolution, see Dirr, Luckhaus, and Novaga [21] and Souganidis and Yip [44]. Furthermore, in the framework of viscosity solutions it is immediate that fattening of level sets must be absent in almost all levels. Finally, a classification of generic singularities has been achieved by Colding and Minicozzi [15, 16].

1.3. **Classical calibrations and gradient flow calibrations.** The key idea for our weak-strong uniqueness result is a gradient-flow analogue of the notion of calibrations. The classical concept of calibrations is an important tool to deduce lower bounds on the interface energy functional for fixed boundary conditions. Recall that a classical calibration for a candidate minimizer \((\bar{\chi}_1, \ldots, \bar{\chi}_P)\) of the interface energy functional (for given boundary conditions and with equal surface tensions) is a collection of vector fields \(\xi_i, 1 \leq i \leq N,\) subject to the following three properties:

- It holds that \(|\xi_i - \xi_j| \leq 1\) for all \(i\) and \(j\).
- The vector fields are solenoidal, i.e., \(\nabla \cdot \xi_i = 0\) for all \(i\).

\(^2\)Provided that one starts with a multiplicity one interface.

\(^3\)Global existence under the assumption that a certain type of singularities does not accumulate.

\(^4\)Global existence under an assumption as in Almgren-Taylor-Wang / Luckhaus-Sturzenhecker.
Calibrations

Existence implies global minimality of surface energy among all partitions

Shortness condition

Stationary situation

Vector fields solenoidal

Existence implies uniqueness of BV solutions to gradient flow

Coercivity condition

Advection equation

Motion by mean curvature

\begin{align*}
|\xi_{i,j}| & \leq 1 \\
|\xi_{i,j}| & \leq \max\{1 - c \text{dist}^2(x, \bar{I}_{ij}), 0\} \\
(\partial_t \xi_i \equiv 0, B \equiv 0) & \quad \partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^T \xi_{i,j} = O(\text{dist}(x, \bar{I}_{ij})) \\
\nabla \cdot \xi_i & = 0 \\
\xi_{i,j} \cdot B & = -\nabla \cdot \xi_{i,j} + O(\text{dist}(x, \bar{I}_{ij}))
\end{align*}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
Calibrations & Gradient flow calibrations \\
\hline
Existence implies global minimality & Existence implies uniqueness of BV solutions to gradient flow \\
Shortness condition & Coercivity condition \\
Stationary situation & Advection equation \\
Vector fields solenoidal & Motion by mean curvature \\
\hline
\end{tabular}
\caption{A comparison of the concept of calibrations for minimal partitions with the new concept of gradient flow calibrations.}
\end{table}

- On the interface $\partial\{\bar{\chi}_i = 1\} \cap \partial\{\bar{\chi}_j = 1\}$ between the phases $i$ and $j$ ($i \neq j$), the vector field $\xi_{i,j} := \xi_i - \xi_j$ coincides with the outer unit normal vector field of $\partial\{\bar{\chi}_i = 1\}$.

The existence of a calibration allows to infer that the partition $(\bar{\chi}_1, \ldots, \bar{\chi}_P)$ indeed minimizes the interface energy functional among all possible Caccioppoli partitions, see [4, Definition 4.16], of the underlying set $D \subset \mathbb{R}^d$, $d \geq 2$ with the same boundary conditions: For any competitor partition $(\chi_1, \ldots, \chi_P)$, one may compute using the first two defining conditions of a calibration (with the abbreviation $I_{i,j} := \partial^* \{\chi_i = 1\} \cap \partial^* \{\chi_j = 1\}$)

\begin{align*}
E[\chi] & = \frac{1}{2} \sum_{i,j=1, i \neq j}^P \int_{I_{i,j}} 1 \, d\mathcal{H}^{d-1} \geq \frac{1}{2} \sum_{i,j=1, i \neq j}^P \int_{I_{i,j}} (\xi_i - \xi_j) \cdot \frac{\nabla \chi_i}{|\nabla \chi_i|} \, d|\nabla \chi_i| \\
& = \sum_{i,j=1, i \neq j}^P \int_{I_{i,j}} \xi_i \cdot \frac{\nabla \chi_i}{|\nabla \chi_i|} \, d|\nabla \chi_i| = \sum_{i,j=1, i \neq j}^P \int_{I_{i,j}} \xi_i \cdot \frac{\nabla \chi_i}{|\nabla \chi_i|} \, d|\nabla \chi_i| \\
& = \sum_{i=1}^P \int_{\partial D} \chi_i n_{\partial D} \cdot \xi_i \, d\mathcal{H}^{d-1}.
\end{align*}

The third defining condition for a calibration shows that in the previous computation, equality is in fact achieved for $(\bar{\chi}_1, \ldots, \bar{\chi}_P)$. This proves $E[\chi] \geq E[\bar{\chi}]$ for all $\chi$ with the same boundary conditions $\chi = \bar{\chi}$ on $\partial D$.

We recall that a notion of calibrations is also available for free discontinuity problems [1], having important implications for the numerical computation of minimizers [11]. For further applications of the concept of calibrations to anisotropic or nonlocal perimeters, we refer to [3, 10] and [8, 42].

In the present work, we introduce a gradient-flow analogue of the notion of calibrations. As shown above, the existence of a (classical) calibration ensures that a certain configuration is a global minimizer of the energy functional. In a similar spirit, the existence of a gradient-flow calibration ensures that the path of steepest descent in the energy landscape of the surface energy functional is unique, and moreover that this path is stable with respect to perturbations in the initial
condition. In the case of equal surface tensions, a *gradient flow calibration* for a
given classical solution \( \tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_P) \) to multiphase mean curvature flow on \( \mathbb{R}^d \)
consists of the following objects:

- A vector field \( \xi_{i,j} \) for each pair of phases \( 1 \leq i, j \leq P, \ i \neq j \). Denoting by \( \bar{I}_{i,j} \) the interface between the phases \( i \) and \( j \) in the strong solution \( \tilde{\chi} \) and by \( \bar{n}_{i,j} \) its unit normal vector field pointing from phase \( i \) to phase \( j \), we require \( \xi_{i,j} \) to be an extension of \( \bar{n}_{i,j} \) subject to the coercivity condition

\[
|\xi_{i,j}| \leq \max\{1 - c \text{dist}^2(x, \bar{I}_{i,j}), 0\}
\]

for some \( c > 0 \).

- The extended normal vector fields \( \xi_{i,j} \) must have the structure \( \xi_{i,j} = \xi_j - \xi_i \) for some vector fields \( \xi_i, 1 \leq i \leq P \). (This structure is reminiscent of the corresponding condition for classical calibrations and in fact serves a similar purpose, see the explanation preceding (3) below.)

- A single velocity field \( B \), which approximately transports all extended normal vector fields \( \xi_{i,j} \) in the sense

\[
\partial_t \xi_{i,j} + (B \cdot \nabla) \xi_{i,j} + (\nabla B)^\top \xi_{i,j} = O(\text{dist}(x, \bar{I}_{i,j})).
\]

Furthermore, the length of the extended normal vector fields is transported to higher accuracy in the sense

\[
\partial_t |\xi_{i,j}|^2 + (B \cdot \nabla)|\xi_{i,j}|^2 = O(\text{dist}^2(x, \bar{I}_{i,j})).
\]

- Near the interfaces \( \bar{I}_{i,j} \) of the strong solution, the normal velocity \( \xi_{i,j} \cdot B \) is given by the mean curvature of \( \bar{I}_{i,j} \) in the sense

\[
\xi_{i,j} \cdot B = -\nabla \cdot \xi_{i,j} + O(\text{dist}(x, \bar{I}_{i,j})).
\]

Note that on the interface \( \bar{I}_{i,j} \), the expression \( -\nabla \cdot \xi_{i,j} \) is exactly equal to its mean curvature.

If a gradient flow calibration exists, we may introduce a measure for the difference between any BV solution \( \chi = (\chi_1, \ldots, \chi_P) \) to multiphase mean curvature flow and the strong solution \( \tilde{\chi} \) by defining (with \( \bar{I}_{i,j} := \partial^* \{\chi_i = 1\} \cap \partial^* \{\chi_j = 1\} \) denoting the interface between phases \( i \) and \( j \) and with \( \bar{n}_{i,j} \) being its normal pointing from phase \( i \) to phase \( j \))

\[
E[\chi|\xi] := \frac{1}{2} \sum_{i,j=1, i \neq j}^P \int_{\bar{I}_{i,j}} 1 - \xi_{i,j} \cdot \bar{n}_{i,j} d\mathcal{H}^{d-1}.
\]

Note that the condition (1a) then precisely ensures that \( E[\chi|\xi] \) is a suitable notion of error between the BV solution \( \chi \) and the strong solution \( \tilde{\chi} \): In addition to providing a tilt-excess-like control of the error, it also provides an estimate on the distance of the interfaces.

On the other hand, the calibration structure \( \xi_{i,j} = \xi_i - \xi_j \) ensures that the error functional (2) may be rewritten as an expression involving only two contributions: First, the total interface energy of the BV solution \( E[\chi] \) and, second, a linear...
functional of the characteristic functions $\chi_i$ of the phases: Indeed, we may compute

$$E[\chi|\xi] = \frac{1}{2} \sum_{i,j=1,i \neq j} P \int_{I_{i,j}} 1 - \xi_{i,j} \cdot n_{i,j} dH^{d-1}$$

$$= E[\chi] - \frac{1}{2} \sum_{i,j=1,i \neq j} \int_{I_{i,j}} (\xi_i - \xi_j) \cdot n_{i,j} dH^{d-1}$$

$$= E[\chi] - \sum_{i,j=1,i \neq j} \int_{I_{i,j}} \xi_i \cdot n_{i,j} dH^{d-1}$$

$$= E[\chi] + \sum_{i=1} P \int_{\mathbb{R}^d} \xi_i \cdot d\nabla \chi_i$$

$$= E[\chi] - \sum_{i=1} P \int_{\mathbb{R}^d} \chi_i \nabla \cdot \xi_i \, dx.$$  

This enables us to estimate the time evolution of the error functional $E[\chi|\bar{\chi}]$ using only two ingredients, namely, first, the sharp energy dissipation estimate (12d) for the interface energy $E[\chi]$ for BV solutions, and, second, the evolution equation (12b) for the phase indicator functions $\chi_i$ from the BV formulation of mean curvature flow. The equations (1b)–(1d) are crucial for deriving a Gronwall-type estimate for $E[\chi|\xi]$ in subsequent rearrangements. We remark that this approach may be regarded as an instance of the relative entropy method introduced independently by Dafermos [18] and Di Perna [20].

Note that locally at a two-phase interface or at a triple junction of the strong solution, for any fixed time $t$ the blowups of our vector fields $\xi_i(t, t)$ turn out to precisely be calibrations of the planar interface or the triple junction, respectively. However, on a global (not blown-up) scale, the vector fields $\xi_i$ may be thought of as deformed variants of classical calibrations which follow the (smooth but typically curved) interface of the strong solution; we refer to Figure 7 and Figure 9c for the illustration of a vector field $\xi_{i,j} = \xi_i - \xi_j$ at a two-phase interface and at a triple junction, respectively.

Let us finally comment on the energy landscape interpretation of our approach, as illustrated in Figure 4. Let $S(t)$ be a classical solution to the gradient flow of the interface energy functional, i.e., a classical solution to multiphase mean curvature flow. For each point in time, the “calibration for the gradient flow” gives a smooth lower bound

$$\mathcal{F}_i := \frac{1}{2} \sum_{i,j=1,i \neq j} P \int_{I_{i,j}} \xi_{i,j} \cdot n_{i,j} dH^{d-1} \quad \text{(3)} \quad = \sum_{i=1} P \int_{\mathbb{R}^d} \chi_i \nabla \cdot \xi_i \, dx$$

(illustrated as the blue wireframe plot in Figure 4) for the rough landscape of the interface energy functional $E[\chi]$ (illustrated as the colored surface plot in Figure 4). This lower bound is sharp for the interface $S(t)$ in the sense $\mathcal{F}_i[S(t)] = E[S(t)]$, and it describes the local direction and speed of steepest descent of the energy functional $E[\chi]$ at $S(t)$ correctly in the sense $D\mathcal{F}_i[S(t)] \in DE[S(t)]$ (where heuristically $DE$ denotes the subdifferential of $E$). Moreover, for each $\chi$ the difference $E[\chi] - \mathcal{F}_i[\chi] = E[\chi|\xi]$ provides an estimate for the error between the smooth solution $S(t)$ and the configuration $\chi$, as measured in a tilt-excess-like quantity.
Figure 4. An illustration of the energy landscape interpretation of our construction: The gradient flow calibration provides a smooth lower bound for the rough energy landscape of the interface energy functional $E[\chi]$, capturing the energy and its subgradient at the current configuration correctly.

2. Main results

2.1. Weak-strong uniqueness principle. In the following, we present our weak-strong uniqueness principle for BV-solutions of multiphase mean curvature flow in the plane. In addition, we provide a quantitative stability estimate, i.e., as long as a strong solution exists, any solution to the BV formulation of multiphase mean curvature flow with slightly perturbed initial data remains close to it. Our results are valid under minimal assumptions on the surface tensions, see Definition 6.

Theorem 1 (Weak-strong uniqueness and quantitative stability). Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_P)$ be a strong solution of multiphase mean curvature flow on $\mathbb{R}^d$ in the sense of Definition 11 on some time interval $[0, T_{\text{strong}})$. Then any BV solution $\chi = (\chi_1, \ldots, \chi_P)$ to multiphase mean curvature flow on $\mathbb{R}^d$ in the sense of Definition 8 must coincide with the strong solution $\tilde{\chi}$ for all $0 \leq t < T_{\text{strong}}$, provided that it starts from the same initial data.

Furthermore, the evolution by mean curvature is stable with respect to perturbations in the initial data: For a general BV solution $\chi$ to multiphase mean curvature flow in the sense of Definition 8, the stability estimate

$$E[\chi|\xi](T) \leq E[\chi|\xi](0)e^{CT}$$

holds true for almost every $T \in [0, T_{\text{strong}})$. Here, $C > 0$ only depends on $\tilde{\chi}$. The interface error functional $E[\chi|\xi]$ is defined in (5), with $\xi_{i,j}$ denoting the gradient flow calibration for the classical solution $\tilde{\chi}$ as constructed in Proposition 4.
Proof. Theorem 1 is an immediate consequence of Proposition 3, Proposition 4, and Proposition 5. \qed

2.2. Calibrations and inclusion principle. The key ingredient for our uniqueness result prior to topology changes is the following gradient flow analogue of the notion of calibrations for minimal partitions. Our main result, Theorem 1, is then an immediate consequence of two implications: First, the existence of a gradient flow calibration guarantees uniqueness of the BV solution (see Proposition 3 and Proposition 5); second, the existence of a gradient flow calibration is guaranteed as long as a classical solution to multiphase mean curvature flow exists (see Proposition 4).

Definition 2 (Calibrations for the gradient flow). Let $d \geq 2$, $P \geq 2$ be integers and let $\sigma \in \mathbb{R}^{P \times P}$ be an admissible matrix of surface tensions in the sense of Definition 6. Let $T > 0$ and let $(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))_{t \in [0,T]}$ be such that for all $t \in [0,T]$ we have that $(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))$ is a partition of finite surface energy of $\mathbb{R}^d$ in the sense of Definition 7. For any $i,j \in \{1, \ldots, P\}$ with $i \neq j$ let $\bar{I}_{i,j} := \partial^+ \bar{\Omega}_i \cap \partial^- \bar{\Omega}_j$ be the interface between the phases $i$ and $j$. We then call a tuple consisting of vector fields $\xi_i \in C^1_{eq}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$, $1 \leq i \leq P$, and a vector field $B \in C^1_{eq}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$ a calibration for the gradient flow for the partition $(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))_{t \in [0,T]}$ if the following conditions are satisfied:

- For each pair of phases $i,j \in \{1, \ldots, P\}$, the vector field
  
  \begin{equation}
  \xi_{i,j} := \frac{1}{\sigma_{ij}} (\xi_i - \xi_j)
  \end{equation}

  coincides on $\bar{I}_{i,j}$ with the normal $\bar{n}_{ij}$ (with the convention that $\bar{n}_{ij}$ points from phase $i$ into phase $j$) and satisfies an estimate of the form

  \begin{equation}
  |\xi_{i,j}(x,t)| \leq \max\{1 - c \text{dist}^2(x, \bar{I}_{i,j}(t)), 0\}
  \end{equation}

  for some $c > 0$.

- The evolution of the vector fields $\xi_{i,j}$ is approximately transported by the velocity field $B$ in the sense

  \begin{equation}
  |\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^T \xi_{i,j}|(x,t) \leq C \text{dist}(x, \bar{I}_{i,j}(t))
  \end{equation}

  and

  \begin{equation}
  |\partial_t |\xi_{i,j}|^2 + (B \cdot \nabla)|\xi_{i,j}|^2|(x,t) \leq C \text{dist}^2(x, \bar{I}_{i,j}(t))
  \end{equation}

  for some $C > 0$.

- The normal component of the velocity field $B$ near the interface $\bar{I}_{i,j}$ is approximately given by the mean curvature of $\bar{I}_{i,j}$ in the sense that

  \begin{equation}
  |\xi_{i,j} \cdot B + \nabla \cdot \xi_{i,j}| \leq C \text{dist}(x, \bar{I}_{i,j}(t))
  \end{equation}

  for some $C > 0$.

Note that, at least heuristically, such a calibrated flow is a solution to mean curvature flow due to (4e) evaluated at $x \in \bar{I}_{i,j}$ and the interpretation of $B$ as the velocity field.

The next proposition states that for general $d \geq 2$ the existence of a gradient flow calibration for a given partition of $\mathbb{R}^d$ into $P$ domains $(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))$ constrains the possible locations of the interfaces in weak (BV) solutions to mean curvature flow to the corresponding interfaces of the partition $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$. This assertion
may be seen as a multiphase analogue of the varifold comparison principle by Ilmanen [27, Theorem 10.7], which provides a corresponding inclusion given a level set solution to two-phase mean curvature flow and any Brakke solution. Note that such an inclusion does not yet yield uniqueness of BV solutions, as it does not exclude the possibility that suddenly all except for one phase disappear.

Proposition 3 (Quantitative inclusion principle). Let \( d \geq 2 \) and \( P \geq 2 \) be integers and let \( \sigma \in \mathbb{R}^{P \times P} \) be an admissible matrix of surface tensions, see Definition 6. Let \( T > 0 \) and let \( (\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))_{t \in [0,T]} \) be an evolving partition of finite surface energy of \( \mathbb{R}^d \), see Definition 7. Let a calibration for the gradient flow in the sense of Definition 2 exist for the partition \((\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))_{t \in [0,T]}\). Then the interfaces \( I_{i,j}(t) := \partial^* \{ \chi_i(t) = 1 \} \cap \partial^* \{ \chi_j(t) = 1 \} \) of any BV solution to mean curvature flow \((\chi_1, \ldots, \chi_P)\) in the sense of Definition 8 with the same initial data as the calibrated partition must be contained in the corresponding interfaces \( I_{i,j}(t) := \partial^* \Omega_i(t) \cap \partial^* \Omega_j(t) \) for a.e. \( 0 < t < T \), i.e. it holds that \( I_{i,j}(t) \subset I_{i,j}(t) \) for all \( i, j \) with \( i \neq j \) up to \( \mathcal{H}^{d-1} \)-null sets.

Furthermore, the existence of a gradient flow calibration also implies a stability estimate: Introducing the interface error functional

\[
E[\chi|\xi](t) := \sum_{i,j=1, i \neq j}^P \sigma_{i,j} \int_{I_{i,j}(t)} 1 - \xi_{i,j}(\cdot, t) \cdot n_{i,j}(\cdot, t) \, d\mathcal{H}^{d-1},
\]

we have the stability estimate

\[
E[\chi|\xi](t) \leq E[\chi|\xi](0) e^{Ct}
\]

for almost every \( t \in [0,T] \) and for some \( C > 0 \) depending on the calibrated partition.

2.3. Gradient flow calibrations for regular networks. In fact, in the planar case the existence of a classical solution to mean curvature flow – in the sense of a smooth evolution of curves meeting at triple junctions with the correct contact angle, see Definition 11 – entails the existence of a calibration for the gradient flow:

Proposition 4. Let \( d = 2 \) and \( P \in \mathbb{N}, P \geq 2 \). Let \((\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))_{t \in [0,T]}\) be a smooth solution to multiphase mean curvature flow in the sense of Definition 11. Then there exists a gradient flow calibration in the sense of Definition 2.

We furthermore sharpen Proposition 3 in the planar case to also exclude the sudden vanishing of phases.

Proposition 5. Let \( d = 2 \) and \( P \in \mathbb{N}, P \geq 2 \). Let \((\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))_{t \in [0,T]}\) be a strong solution to multiphase mean curvature flow in the sense of Definition 11 and let \( \bar{\chi}_i := \chi_{\bar{\Omega}_i} \) denote the characteristic function of the \( \mathcal{L}^d \)-measurable set \( \Omega_i \) for \( i = 1, \ldots, P \). Let \((\chi_1, \ldots, \chi_P)\) be a BV solution to multiphase mean curvature flow in the sense of Definition 8. Introduce the bulk error functional

\[
E_{\text{volume}}[\chi|\bar{\Omega}](t) := \sum_{i=1}^P \int_{\mathbb{R}^d} \left| \chi_i - \bar{\chi}_i \right| \min \left\{ \text{dist} \left( x, \partial \bar{\Omega}_i \right), 1 \right\} \, dx.
\]

Then the stability estimate

\[
E_{\text{volume}}[\chi|\bar{\Omega}] \leq C e^{CT} \left( E_{\text{volume}}[\chi|\bar{\Omega}](0) + E[\chi|\xi](0) \right)
\]

holds, where \( E[\chi|\xi] \) is defined in (5) and \( C > 0 \) only depends on \( \Omega \).
2.4. Basic definitions. In the following, we recall the precise definitions of the solution concepts for multiphase mean curvature flow which our main results are concerned with. We begin with the notion of admissible surface tensions.

**Definition 6 (Admissible matrix of surface tensions)**. Let $P \geq 2$ be an integer and $\sigma = (\sigma_{i,j})_{i,j=1,\ldots,P} \in \mathbb{R}^{P \times P}$ be a matrix. The matrix $\sigma$ is called an admissible matrix of surface tensions if the following conditions are satisfied:

i) (Symmetry) It holds that $\sigma_{i,j} = \sigma_{j,i}$ and $\sigma_{i,i} = 0$ for every $i,j \in \{1,\ldots,P\}$.

ii) (Positivity) We have $\sigma_{\min} := \min\{\sigma_{i,j}: i,j \in \{1,\ldots,P\}, i \neq j\} > 0$.

iii) (Coercivity) For all choices of pairwise distinct $i,j,k \in \{1,\ldots,P\}$ the following strict triangle inequality holds true

\[ \sigma_{i,j} < \sigma_{i,k} + \sigma_{k,j}. \]  

Moreover, we assume the symmetric matrix $Q$ with entries

\[ Q_{i,j} := \sigma_{P,i}^2 + \sigma_{P,j}^2 - \sigma_{i,j}^2, \quad i,j = 1,\ldots,P-1 \]

to be strictly positive definite:

\[ z^T Q z > 0 \quad \text{for all } z \in \mathbb{R}^{P-1} \setminus \{0\}. \]

We labeled (9a) and (9b) as coercivity properties for the following reasons. First, the strict triangle inequality (9a) will ensure that our relative entropy functional provides control on wetting, i.e., the nucleation of a third phase at the smooth part of an interface between two phases. Second, the assumption (9b) turns out to be a condition for our relative entropy functional to prevent the nucleation of a fourth phase (or clusters of phases) at a triple junction. This condition also appeared in [34] and is in fact equivalent to the $l^2$-embeddability of the matrix of surface tensions $\sigma$ into $\mathbb{R}^{P-1}$, cf. [43], in the following sense: There exist points $q_1,\ldots,q_P \in \mathbb{R}^{P-1}$ such that $\sigma_{i,j} = |q_i - q_j|$ for all $i,j \in \{1,\ldots,P\}$, see Figure 5b.

**Definition 7 (Partitions with finite interface energy, cf. [4])**. Let $P \geq 2$ be an integer and let $\sigma = (\sigma_{i,j})_{i,j=1,\ldots,P} \in \mathbb{R}^{P \times P}$ be an admissible matrix of surface tensions in the sense of Definition 6. Let $(\Omega_1,\ldots,\Omega_P)$ be a partition of $\mathbb{R}^d$ in the sense that $\Omega_i \subset \mathbb{R}^d$ and $\mathbb{R}^d \setminus \bigcup_{i=1}^P \Omega_i$ is a set of $C^d$-measure zero. Let $\chi_i := \chi_{\Omega_i}$ denote the characteristic function of the $C^d$-measurable set $\Omega_i$ for $i = 1,\ldots,P$. We call $\chi = (\chi_1,\ldots,\chi_P)$ (or equivalently $(\Omega_1,\ldots,\Omega_P)$) a partition of $\mathbb{R}^d$ with finite interface energy if the
energy

\[
E[\chi] := \sum_{i,j=1, i \neq j}^{P} \sigma_{i,j} \int_{\mathbb{R}^d} \frac{1}{2} (d|\nabla \chi_i| + d|\nabla \chi_j| - d|\nabla (\chi_i + \chi_j)|)
\]

is finite.

Note that for a partition of \(\mathbb{R}^d\) with finite interface energy, each \(\Omega_i\) is a set of finite perimeter. By introducing the interfaces \(I_{i,j} := \partial^* \Omega_i \cap \partial^* \Omega_j\) as the intersection of the respective reduced boundaries, the energy of a partition \(\chi\) can be rewritten in the equivalent form

\[
E[\chi] = \sum_{i,j=1, i \neq j}^{P} \sigma_{i,j} \int_{I_{i,j}} 1 \, d\mathcal{H}^{d-1}.
\]

We next recall the notion of BV solutions to multiphase mean curvature flow, cf. [31, 32].

**Definition 8** (BV solution for multiphase mean curvature flow). Let \(d \geq 2\) and \(P \geq 2\) be integers. Let \(\sigma \in \mathbb{R}^{P \times P}\) be an admissible matrix of surface tensions in the sense of Definition 6, and let \(T_{\text{BV}} > 0\) be a finite time horizon. Let \(\chi_0 = (\chi_0, \ldots, \chi_P)\) be an initial partition of \(\mathbb{R}^d\) with finite interface energy in the sense of Definition 7.

A measurable map

\[
\chi = (\chi_1, \ldots, \chi_P) : \mathbb{R}^d \times [0, T_{\text{BV}}) \to \{0, 1\}^P
\]

(or the corresponding tupel of sets \(\Omega_i(t) := \{\chi_i(t) = 1\} \) for \(i = 1, \ldots, P\)) is called a BV solution for multiphase mean curvature flow with initial data \(\chi_0\) if the following conditions are satisfied:

i) (Partition with finite interface energy) For almost every \(T \in [0, T_{\text{BV}})\), \(\chi(T)\) is a partition of \(\mathbb{R}^d\) with finite interface energy in the sense of Definition 7 and

\[
\text{ess sup}_{T \in [0, T_{\text{BV}})} E[\chi(\cdot, T)] = \text{ess sup}_{T \in [0, T_{\text{BV}})} \sum_{i,j=1, i \neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(T)} 1 \, d\mathcal{H}^{d-1} < \infty,
\]

where \(I_{i,j} = \partial^* \Omega_i \cap \partial^* \Omega_j\) for \(i \neq j\) is the interface between \(\Omega_i\) and \(\Omega_j\).

ii) (Evolution equation) For all \(i \in \{1, \ldots, P\}\), there exist normal velocities \(V_i \in L^2(\mathbb{R}^d \times [0, T_{\text{BV}}), |\nabla \chi_i| \otimes L^1)\) in the sense that each \(\chi_i\) satisfies the evolution equation

\[
\int_{\mathbb{R}^d} \chi_i(\cdot, T) \varphi(\cdot, T) \, dx - \int_{\mathbb{R}^d} \chi_0 \varphi(\cdot, 0) \, dx
\]

\[
= \int_{0}^{T} \int_{\mathbb{R}^d} V_i \varphi \, d|\nabla \chi_i| \, dt + \int_{0}^{T} \int_{\mathbb{R}^d} \chi_i \partial_t \varphi \, dx \, dt
\]

for almost every \(T \in [0, T_{\text{BV}})\) and all \(\varphi \in C^\infty_c(\mathbb{R}^d \times [0, T_{\text{BV}}))\). Moreover, the (reflection) symmetry condition \(V_i \cdot \frac{\nabla \chi_i}{|\nabla \chi_i|} = V_j \cdot \frac{\nabla \chi_j}{|\nabla \chi_j|}\) shall hold \(\mathcal{H}^{d-1}\)-almost everywhere on the interfaces \(I_{i,j}\) for \(i \neq j\).
iii) (BV formulation of mean curvature) The normal velocities satisfy the equation
\[
\sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^{T_{BV}} \int_{I_{i,j}(t)} V_i \frac{\nabla \chi_i}{|\nabla \chi_i|} \cdot B \, dH^{d-1} \, dt
\]
\[
= - \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^{T_{BV}} \int_{I_{i,j}(t)} \left( \text{Id} - \frac{\nabla \chi_i}{|\nabla \chi_i|} \otimes \frac{\nabla \chi_i}{|\nabla \chi_i|} \right) : \nabla B \, dH^{d-1} \, dt
\]
for all \( B \in C^\infty_c(\mathbb{R}^d \times [0,T_{BV}); \mathbb{R}^d) \).

iv) (Energy dissipation inequality) The sharp energy dissipation inequality
\[
E[\chi(\cdot,T)] + \sum_{i,j=1,i \neq j}^{P} \int_0^T \int_{I_{i,j}(t)} |V_i|^2 \, dH^{d-1} \, dt \leq E[\chi_0]
\]
holds true for almost every \( T \in [0,T_{BV}) \).

Next, we give the definition of strong solutions to multiphase mean curvature flow. To this end, we first define a notion of regular partitions and regular networks of interfaces (cf. [36]).

**Definition 9 (Regular partitions and networks of interfaces).** Let \( d = 2 \), let \( P \geq 2 \) be an integer, and let \((\Omega_1, \ldots, \Omega_P)\) be a partition with finite interface energy of open subsets of \( \mathbb{R}^2 \) such that \( \partial^* \Omega_i = \partial \Omega_i \). Moreover, let \( \bar{\chi}_i := \chi_{\Omega_i} \) denote the characteristic function of the \( \mathcal{L}^d \)-measurable set \( \Omega_i \), and let \( \bar{I}_{i,j} := \partial \Omega_i \cap \partial \Omega_j \) denote the respective interfaces for \( i \neq j \). We call \( \bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P) \) (or equivalently \((\Omega_1, \ldots, \Omega_P)\) a regular partition of \( \mathbb{R}^2 \) and \((\bar{I}_{i,j})_{i \neq j} \) a regular network of interfaces in \( \mathbb{R}^2 \) if the following properties are satisfied:

i) (Regularity) Each interface \( \bar{I}_{i,j} \) is a one-dimensional manifold with boundary of class \( C^4 \). The interior of each interface is embedded.

ii) (Multi-points are triple junctions) Two different interfaces may intersect only at their boundary. Moreover, at each intersection point exactly three interfaces meet. In other words, all multi-points of the network of interfaces \((\bar{I}_{i,j})_{i \neq j} \) are triple junctions.

iii) (Balance-of-forces condition) Let \( p \in \mathbb{R}^2 \) be a triple junction present in the network. Assume for notational concreteness that at the triple junction \( p \), the three phases \( \Omega_i, \Omega_j, \) and \( \Omega_k \) meet. Then, the balance-of-forces condition
\[
\sigma_{i,j} \bar{n}_{i,j}(p) + \sigma_{j,k} \bar{n}_{j,k}(p) + \sigma_{k,i} \bar{n}_{k,i}(p) = 0
\]
has to be satisfied. Here, \( \bar{n}_{i,j}(x) \) denotes the unit-normal vector field of the interface \( \bar{I}_{i,j}(t) \) at \( x \in \bar{I}_{i,j} \) pointing from phase \( \Omega_i(t) \) towards phase \( \Omega_j(t) \).

Let \( \sigma \in \mathbb{R}^{P \times P} \) be an admissible matrix of surface tensions in the sense of Definition 6. We call \( \bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P) \) (or equivalently \((\Omega_1, \ldots, \Omega_P)\) a regular partition of \( \mathbb{R}^2 \) with finite interface energy, if \( \bar{\chi} \) satisfies in addition to the previous requirements
\[
E[\bar{\chi}] := \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_{\bar{I}_{i,j}} 1 \, dS < \infty.
\]
the previous definition, topological changes are excluded on the level of a strong solution.

**Definition 10** (Smoothly evolving partitions and networks of interfaces). Let $P \geq 2$ be an integer and let $\bar{x}(0) = (\bar{x}_1(0), \ldots, \bar{x}_P(0))$ be a regular partition of $\mathbb{R}^2$ with a regular network of interfaces $\{I_{i,j}(0)\}_{i \neq j}$ in the sense of Definition 9. Let $T_{\text{strong}} > 0$ be a finite time horizon and consider a family $(\Omega(t), \ldots, \Omega_P(t))_{t \in [0,T_{\text{strong}}]}$ of regular partitions of $\mathbb{R}^2$ in the sense of Definition 9. Let $\bar{x}_i(t) := \chi_{\Omega_i(t)}$ and $\bar{I}_{i,j}(t) := \partial\Omega_i(t) \cap \partial\Omega_j(t)$ for all $t \in [0,T_{\text{strong}}]$ and each pair $i \neq j$ with $i, j = 1, \ldots, P$.

We say that $\bar{x}$ (or equivalently $(\Omega_1, \ldots, \Omega_P)$) is a smoothly evolving regular partition of $\mathbb{R}^2$ and $(\bar{I}_{i,j})_{i \neq j}$ is a smoothly evolving regular network of interfaces in $\mathbb{R}^2$ if there exists a map

$$\Psi: \mathbb{R}^2 \times [0,T_{\text{strong}}] \to \mathbb{R}^2, (x,t) \to \Psi(x,t) = \Psi^t(x)$$

such that $\bar{x}_i(t) = \bar{x}_i(0) \circ \Psi^t$, $\bar{I}_{i,j}(t) = \Psi^t(\bar{I}_{i,j}(0))$ for all $t \in [0,T_{\text{strong}})$, $\Psi^0 = \text{Id}$, and which is in addition subject to the following regularity conditions:

i) For all $t \in [0,T_{\text{strong}})$, the map $\Psi^t: \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism.

ii) For all $t \in [0,T_{\text{strong}})$ and all $i \neq j$, the restriction $\Psi^t_{i,j}: \bar{I}_{i,j}(0) \to \bar{I}_{i,j}(t)$ is a $C^4$-diffeomorphism such that $\|\Psi^t_{i,j}\|_{L^\infty_t W^{4,\infty}} < \infty$.

iii) For all $i \neq j$, the restriction $\Psi_{i,j}: \bar{I}_{i,j}(0) \times [0,T_{\text{strong}}) \to \mathbb{R}^2$ satisfies $\partial_t \Psi_{i,j} \in C^0([0,T_{\text{strong}}); C^3(\bar{I}_{i,j}(0); \mathbb{R}^2))$ as well as $\|\partial_t \Psi_{i,j}\|_{L^\infty_t W^{2,\infty}} < \infty$.

iv) Moreover, we assume that there exists $r_c \in (0,\frac{1}{2}]$ with the following property:

For all $t \in [0,T_{\text{strong}})$, all $i \neq j$ and all points $x \in \bar{I}_{i,j}(t) \cup \partial\bar{I}_{i,j}(t)$ there exists a function $g_{i,j}: (-1,1) \to \mathbb{R}$ with $g_{i,j}'(0) = 0$ and $y_{\min} \in [-1,0]$, such that after a translation and a rotation, $\bar{I}_{i,j}(t) \cap B_{2r_c}(x)$ is given by the graph $\{(y,g_{i,j}(y)) : y \in (y_{\min},1)\} \cap B_{2r_c}(0)$, which we additionally ask to be a connected set. Furthermore, for any of these functions $g_{i,j}$ we ask the pointwise bounds $|g_{i,j}^{(m)}| \leq r_c^{-(m-1)}$ to hold for all $1 \leq m \leq 3$.

We have everything in place to proceed with the definition of strong solutions for multiphase mean curvature flow.
**Definition 11** (Strong solution for multiphase mean curvature flow). Let \( d = 2 \), let \( P \geq 2 \) be an integer, \( \sigma \in \mathbb{R}^{P \times P} \) be an admissible matrix of surface tensions in the sense of Definition 6, \( T_{\text{strong}} > 0 \) be a finite time horizon, and let \( \bar{\chi}_0 = (\bar{\chi}_1^0, \ldots, \bar{\chi}_P^0) \) be an initial regular partition of \( \mathbb{R}^2 \) with finite interface energy in the sense of Definition 9.

A measurable map
\[
\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P) : \mathbb{R}^d \times [0, T_{\text{strong}}) \to \{0, 1\}^P
\]
(or equivalently the tuple of time-dependent sets \( \Omega_i(t) := \{ \bar{\chi}_i(\cdot, t) = 1 \} \) for \( i = 1, \ldots, P \) and \( t \in [0, T_{\text{strong}}) \)) is called a strong solution for multiphase mean curvature flow with initial data \( \bar{\chi}_0 \) if the following conditions are satisfied:

i) (Smoothly evolving regular partition with finite interface energy) The family \( \bar{\chi} \) is a smoothly evolving regular partition of \( \mathbb{R}^2 \) and the family \( (\bar{I}_{i,j})_{i \neq j} \) is a smoothly evolving regular network of interfaces in \( \mathbb{R}^2 \) in the sense of Definition 10. In particular, for every fixed \( t \in [0, T_{\text{strong}}) \), \( \bar{\chi}(t) \) is a regular partition of \( \mathbb{R}^2 \) and \( (\bar{I}_{i,j}(t))_{i \neq j} \) is a regular network of interfaces in \( \mathbb{R}^2 \) in the sense of Definition 9 such that

\[
\sup_{t \in [0, T_{\text{strong}})} E[\bar{\chi}(t)] = \sup_{t \in [0, T_{\text{strong}})} \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_{I_{i,j}(t)} 1 \, dS < \infty.
\]

ii) (Evolution by mean curvature) For \( i, j = 1, \ldots, P \) with \( i \neq j \), \( x \in \bar{I}_{i,j} \) and \( t \in [0, T_{\text{strong}}) \) let \( \bar{V}_{i,j}(x, t) \) denote the normal speed of a point \( x \in \bar{I}_{i,j}(t) \), i.e., \( \bar{V}_{i,j}(x) := \bar{n}_{i,j}(x, t) \cdot \partial_t \Psi_{i,j}(y, t) \) at \( y = (\Psi_{i,j}^t)^{-1}(x) \in \bar{I}_{i,j}(0) \), where \( \Psi_{i,j} \) and \( \Psi_{i,j}^t \) are the maps from the definition of a smoothly evolving regular network of interfaces, see Definition 10. Denoting by \( \bar{H}_{i,j}(t) \) the mean curvature vector of \( \bar{I}_{i,j}(t) \), we then assume that

\[
\bar{V}_{i,j}(x, t) \bar{n}_{i,j}(x, t) = \bar{H}_{i,j}(x, t), \quad \text{for all } t \in [0, T_{\text{strong}}), x \in \bar{I}_{i,j}(t).
\]

iii) (Initial conditions) We have \( \bar{\chi}_i(x, 0) = \bar{\chi}_{0,i}(x) \) for all \( x \in \mathbb{R}^d \) and each \( i = 1, \ldots, P \).

2.5. **Relative entropy inequality.** The key ingredient for the proof of Proposition 3 is the derivation of a Gronwall-type inequality for the tilt-excess-like error functional (5): We aim to derive an estimate of the form

\[
E[\chi|\xi](T) \leq E[\chi|\xi](0) + C(\xi) \int_0^T E[\chi|\xi](t) \, dt
\]

for almost all admissible times \( T \geq 0 \) from which one may infer the desired stability estimate (6) by an application of Gronwall’s lemma; the weak-strong uniqueness principle then follows by means of the coercivity properties of the relative entropy error functional (5) and a subsequent estimate for \( E_{\text{volume}}[\chi|\bar{\chi}] \), see Proposition 5. The following result contains the first key step in the derivation of the Gronwall-type inequality (16): it is valid for general vector fields \( \xi \) and \( B \) with sufficient smoothness (not just for gradient flow calibrations).

**Proposition 12** (Relative entropy inequality). Let \( d \geq 2 \), \( P \geq 2 \) be integers, and let \( \sigma \in \mathbb{R}^{P \times P} \) be an admissible matrix of surface tensions in the sense of Definition 6.
Let $\chi = (\chi_1, \ldots, \chi_P)$ be a BV solution of multiphase mean curvature flow in the sense of Definition 8 on some time interval $[0, T_{\text{BV}})$. We denote by

\begin{equation}
\n_{i,j} := \frac{\nabla \chi_j}{|\nabla \chi_i|} = -\frac{\nabla \chi_i}{|\nabla \chi_i|}, \quad H^{d-1} \text{-a.e. on } I_{i,j},
\end{equation}

the (measure-theoretic) unit normal vector of the interface $I_{i,j}$ pointing from the $i$-th to the $j$-th phase of the BV solution. Moreover, let

\begin{equation}
V_{i,j} := V_i - V_j, \quad H^{d-1} \text{-a.e. on } I_{i,j}.
\end{equation}

Let $(\xi_{i,j})_{i,j\in\{1, \ldots, P\}}$ and $(\xi_i)_{i=1, \ldots, P}$ be families of compactly supported vector fields such that

$\xi_{i,j}, \xi_i \in L^\infty([0, T_{\text{strong}}); W^2, \infty(\mathbb{R}^d; \mathbb{R}^d)) \cap W^1, \infty([0, T_{\text{strong}}); C^0(\mathbb{R}^d; \mathbb{R}^d))$

as well as $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ for all $i \neq j$. Let

$B \in L^\infty([0, T_{\text{strong}}); W^1, \infty(\mathbb{R}^d; \mathbb{R}^d))$

be an arbitrary compactly supported vector field.

Consistently with (5), define the interface error control

\begin{equation}
E[\chi|\xi](t) := \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_{I_{i,j}(t)} 1 - \xi_{i,j}(\cdot, t) \cdot n_{i,j}(\cdot, t) \, dH^{d-1}.
\end{equation}

Then the interface error control is subject to the estimate

\begin{equation}
\begin{align*}
E[\chi|\xi](T) &+ \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} |V_{i,j} + \nabla \cdot \xi_{i,j}|^2 + |V_{i,j} n_{i,j} - (B \cdot \xi_{i,j}) \xi_{i,j}|^2 \, dH^{d-1} \, dt \\
&\leq E[\chi|\xi](0) + R_{\text{dt}} + R_{\text{dissip}}
\end{align*}
\end{equation}

for almost every $T \in (0, T_{\text{strong}})$. Here, we made use of the abbreviations

$R_{\text{dt}} := -\sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} (\partial_t |\xi_{i,j}|^2 + (B \cdot \nabla)|\xi_{i,j}|^2) \, dH^{d-1} \, dt$

$- \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^T \xi_{i,j} \cdot (n_{i,j} - \xi_{i,j}) \, dH^{d-1} \, dt,$

$R_{\text{dissip}} := \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} (|\nabla \cdot \xi_{i,j}| + B \cdot \xi_{i,j})^2 \, dH^{d-1} \, dt$

$- \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} |B \cdot \xi_{i,j}|^2 (1 - |\xi_{i,j}|^2) \, dH^{d-1} \, dt$

$- \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} (1 - n_{i,j} \cdot \xi_{i,j}) (\nabla \cdot \xi_{i,j}) (B \cdot \xi_{i,j}) \, dH^{d-1} \, dt$

$+ \sum_{i,j=1,i\neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \left( (\text{Id} - \xi_{i,j} \otimes \xi_{i,j})B \right) \cdot (V_{i,j} + \nabla \cdot \xi_{i,j}) n_{i,j} \, dH^{d-1} \, dt$. 

\[ + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} (1 - n_{i,j} : \xi_{i,j})(\nabla \cdot B) \, dH^{d-1} \, dt \]

\[ - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} (n_{i,j} - \xi_{i,j}) \otimes (n_{i,j} - \xi_{i,j}) : \nabla B \, dH^{d-1} \, dt. \]

2.6. Structure of the paper. The remaining part of the paper is organized as follows. Section 3 illustrates our strategy at the two most important examples, a smooth interface, and a triple junction.

The subsequent sections are then divided into two parts. In Section 4, we prove the stability of calibrated flows and exploit all properties of our gradient flow calibrations and the weak solution: In Subsection 4.1 we derive the relative entropy inequality in its full generality of Proposition 12; in Subsection 4.2, we prove the quantitative inclusion principle, Proposition 3; and Subsection 4.3 provides the ingredients for the control of the volume error in Proposition 5.

The remaining three sections of the manuscript are devoted to the construction of our gradient flow calibrations given a strong solution. First, we provide explicit constructions at a smooth interface (Section 5) and at a triple junction (Section 6). These cases do not only serve as model examples but they also form the building blocks for our general construction in Section 7, where we glue together these local constructions, to obtain a gradient flow calibration for regular networks, which establishes Proposition 4.

3. Outline of the strategy

3.1. Idea of proof for a smooth interface. Let us give a brief idea of the proof, ignoring technical difficulties in the simplest case of two phases sharing one single interface with \( \sigma = 1 \). In that case, it is sufficient to describe the weak solution and the calibrated flow by a single phase \( \Omega(t) \subset \mathbb{R}^d \), resp. \( \bar{\Omega}(t) \subset \mathbb{R}^d \) for \( t \in [0, T_{\text{strong}}) \), the first being a set of finite perimeter and the second being a simply connected, smooth set. The relative entropy is then simply given by

\[ E[\chi|\xi](t) = \int_{\partial^*\Omega(t)} (1 - n \cdot \xi) \, dH^{d-1}, \]

which has the interpretation of an oriented excess of the weak solution with respect to the strong one. Here \( \chi = \chi(x,t) \) denotes the characteristic function of \( \Omega = \Omega(t) \) and \( n = -\frac{d\chi}{\nabla \chi} \) denotes the (measure theoretic) exterior unit normal of \( \partial^*\Omega(t) \). Furthermore, the vector field \( \xi(\cdot, t) \) is an extension of the exterior unit normal \( \bar{n}(\cdot, t) \) of the calibrated, smooth interface \( \bar{I}(t) := \partial \bar{\Omega} \) necessitated by the fact that we evaluate it on the weak solution.

In order to relate the extension \( \xi \) to the evolution, we require it to be transported along an extension \( B \) of the velocity field of \( \bar{I} \) in the sense that

\[ \partial_t \xi = -(B \cdot \nabla) \xi - (\nabla B)^T \xi + O(\text{dist}(x, \bar{I}(t))), \]

which will help make the second term of \( R_{dH} \) small (see Proposition 12 for the definition). The extension for \( B \) will be done such that it is constant in the “normal” \( \xi \)-direction, meaning we have \((\xi \cdot \nabla)B = 0\), and such that the motion law \( \bar{n} \cdot B = \bar{V} = \bar{H} = -\nabla \tan \cdot \bar{n} \) is still approximately true away from the interface in the sense
that
\[
\xi \cdot B = -\nabla \cdot \xi + O(\text{dist}(x, \bar{I}(t))),
\]
helping with the first term of $R_{\text{dissip}}$.

As we also want the functional $E[\chi|\xi]$ to ensure that $\chi$ cannot be too far away from $\bar{\chi}$, we allow for $\xi$ to be short, i.e., we have $|\xi| \leq 1$, and we ask this effect to be transported by $B$ up to quadratic error
\[
\partial_t |\xi|^2 + (B \cdot \nabla)|\xi|^2 = O(\text{dist}^2(x, \bar{I}(t))),
\]
keeping the first term of $R_{\text{dissip}}$ small.

In the present case of a single interface, the construction of these vector fields is straightforward using the signed distance function $s = s(x, t)$ to the smooth interface $\bar{I}$: We set
\[
\xi := \zeta(s)\nabla s \quad \text{and} \quad B := - (\Delta s) \xi,
\]
where $\zeta$ is a suitable cut-off function such that $\zeta(s) = 1 - s^2$ close to $s = 0$. Note that since $|\nabla s| = 1$, this implies
\[
s^2 = 1 - \zeta(s) \leq 1 - \zeta(s) n \cdot \nabla s = 1 - n \cdot \xi
\]
for $s$ small, so that the relative entropy controls the (truncated) $L^2$ distance of the weak solution and the calibrated flow.

In the following heuristic derivation of the relative entropy inequality (from Proposition 12) in the case of a single interface, we will use the abbreviation $\int_{\partial^* \Omega} := \int_{\partial^* \Omega(t)} \cdot d\mathcal{H}^{d-1}$ for the integral along a time slice $\partial^* \Omega$ of the weak solution. Recall that $V$ denotes the normal velocity of the weak solution characterized by the distributional equation $\partial_t \chi = V |\nabla \chi|$, see (12b), so that the sign convention is $V > 0$ for expanding $\Omega$.

The optimal energy dissipation rate (12d) and the definition (12b) of $V$ imply
\[
\frac{d}{dt} E[\chi|\xi] = \frac{d}{dt} |\partial^* \Omega| - \frac{d}{dt} \int_{\Omega} (\nabla \cdot \xi) \, dx \leq - \int_{\partial^* \Omega} V^2 - \int_{\partial^* \Omega} V (\nabla \cdot \xi) - \int_{\partial^* \Omega} \partial_t \xi \cdot n.
\]
Testing the distributional mean curvature flow equation (12c) with the extended velocity field $B$ gives
\[
0 = \int_{\partial^* \Omega} V (n \cdot B) + \int_{\partial^* \Omega} (\text{Id} - n \otimes n) : \nabla B
\]
Adding these terms to the right-hand side of the previous inequality yields

\[
\frac{d}{dt} E[\chi|\xi] \leq -\int_{\partial^\ast \Omega} (V^2 + V (\nabla \cdot \xi) - V (n \cdot B)) + \int_{\partial^\ast \Omega} (\nabla \cdot B) - \int_{\partial^\ast \Omega} n \otimes n : \nabla B - \int_{\partial^\ast \Omega} \partial_t \xi \cdot n.
\]

Note that \( B = (\xi \cdot B) \xi + (\text{Id} - \xi \otimes \xi) B \), which we interpret as a decomposition of \( B \) into “normal” and “tangential” parts. Furthermore, as a result of \((\xi \cdot \nabla)B = 0\) we have \(- (n - \xi) \otimes (n - \xi) : \nabla B = - (n - \xi) \cdot (n - \xi) : \nabla B = - n \otimes n : \nabla B + n \cdot (\nabla B) \xi \). We then insert the first identity, complete the squares, and add and subtract \((B \cdot \nabla) \xi + (\nabla B)^T \xi \) to make \( \partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^T \xi \) appear. Together with the second identity we obtain

\[
\frac{d}{dt} E[\chi|\xi] \leq -\frac{1}{2} \int_{\partial^\ast \Omega} (V^2 + V n - (\xi \cdot B) \xi) + \frac{1}{2} \int_{\partial^\ast \Omega} (\nabla \cdot \xi)^2 + (\xi \cdot B)^2 + \int_{\partial^\ast \Omega} V n \cdot (\text{Id} - \xi \otimes \xi) B + \int_{\partial^\ast \Omega} (\nabla \cdot B) - \int_{\partial^\ast \Omega} (n - \xi) : \nabla B + \int_{\partial^\ast \Omega} n \cdot (B \cdot \nabla) \xi
\]

\[
\int_{\partial^\ast \Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^T \xi) \cdot n.
\]

where the second line collects precisely the terms left after completing the squares.

By symmetry considerations, we have

\[
0 = \int_{\Omega} \nabla \cdot [\nabla \cdot (B \otimes (\xi - \nabla B))] \, dx = \int_{\partial^\ast \Omega} [\nabla \cdot (B \otimes (\xi - \nabla B))] \cdot n
\]

\[
= \int_{\partial^\ast \Omega} [(\nabla \cdot \xi) n \cdot B - (\nabla \cdot B) n \cdot \xi - n \cdot (B \cdot \nabla) \xi],
\]

where for the second line we used \((\xi \cdot \nabla)B = 0\). Now we use \(|\xi| \leq 1\) to drop the prefactor \(|\xi|^2\) of \((\xi \cdot B)^2\) in the second right-hand side integral in inequality (25), complete the square, and add the above identity to obtain

\[
\frac{d}{dt} E[\chi|\xi] \leq -\frac{1}{2} \int_{\partial^\ast \Omega} (V^2 + V n - (\xi \cdot B) \xi) + \frac{1}{2} \int_{\partial^\ast \Omega} (\nabla \cdot \xi + (B \cdot \nabla) \xi)^2 + \int_{\partial^\ast \Omega} (\nabla \cdot \xi) \cdot (n - \xi) \cdot B
\]

\[
+ \int_{\partial^\ast \Omega} V n \cdot (\text{Id} - \xi \otimes \xi) B + \int_{\partial^\ast \Omega} (1 - n \cdot \xi) (\nabla \cdot B)
\]

\[
- \int_{\partial^\ast \Omega} (n - \xi) \otimes (n - \xi) : \nabla B - \int_{\partial^\ast \Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^T \xi) \cdot n.
\]
Once more, we decompose $B$ into “tangential” and “normal” components, and finally arrive at the entropy dissipation inequality
\[
\frac{d}{dt} E[\chi|\xi] \leq - \frac{1}{2} \int_{\partial^* \Omega} \left( (V + \nabla \cdot \xi)^2 + |Vn - (\xi \cdot B) \xi|^2 \right) \\
+ \frac{1}{2} \int_{\partial^* \Omega} (\nabla \cdot \xi + \xi \cdot B)^2 + \int_{\partial^* \Omega} (\nabla \cdot \xi) (n \cdot \xi - 1) (\xi \cdot B) \\
+ \int_{\partial^* \Omega} (\nabla \cdot \xi + V) n \cdot (\text{Id} - \xi \otimes \xi) B \\
+ \int_{\partial^* \Omega} (1 - n \cdot \xi) (\nabla \cdot B) - \int_{\partial^* \Omega} (n - \xi) \otimes (n - \xi) : \nabla B \\
- \int_{\partial^* \Omega} (\partial_t \xi + (B \cdot \nabla) \xi + (\nabla B)^T \xi) \cdot (n - \xi) \\
+ \int_{\partial^* \Omega} (\partial_t \xi + (B \cdot \nabla) \xi) \cdot \xi.
\]

Now let us briefly argue term-by-term that the right-hand side can be controlled by the relative entropy $E[\chi|\xi]$, which together with a Gronwall argument and Proposition 5 would yield Theorem 1 for $d = 2$. Thanks to (22), the first term of the second line is quadratic in $\text{dist}(x, \tilde{I})$ and therefore controlled by the relative entropy due to (24). The second integral of the second line is controlled by the relative entropy since $\nabla \cdot \xi$ and $\xi \cdot B$ are uniformly bounded. To handle the third line, we use Cauchy-Schwarz and Young, and absorb $\int (\nabla \cdot \xi + V)^2$ in the first integral. The remaining integral of $|\text{Id} - \xi \otimes \xi| n|^2 = |n - (\xi \cdot n)\xi|^2 \lesssim |n - \xi|^2 + (1 - n \cdot \xi)^2$ is controlled by the relative entropy. Clearly, both terms in the fourth line are controlled by the relative entropy. Finally, the integrals in the fifth and sixth lines are quadratic due to (21) and the factor $n - \xi$, and (23), respectively.

3.2. Idea of proof for a triple junction. The second model case is given by a triple junction, say, with equal surface tensions. To illustrate the additional difficulties, we also present the idea of our proof in this case. However, we restrict ourselves to the case $d = 2$.

We denote the phases of the weak solution by $\Omega_1$, $\Omega_2$, and $\Omega_3$ with characteristic functions $\chi_1$, $\chi_2$, and $\chi_3$. To simplify notation, we identify indices if they are equivalent mod 3, i.e., we define $\chi_4 := \chi_1$, $\chi_5 := \chi_2$, $\chi_6 := \chi_3$, and so on. Following the notation of Proposition 12, we denote the normal vector of the interface $I_{i,i+1} = \partial^* \Omega_i \cap \partial^* \Omega_{i+1}$ between phases $i$ and $i + 1$ for $i = 1, 2, 3$ in the weak solution by
\[
n_{i,i+1} := \frac{d\nabla \chi_{i+1}}{|d\nabla \chi_{i+1}|} = - \frac{d\nabla \chi_i}{|d\nabla \chi_i|} \quad H^1\text{-a.e. on } \partial^* \Omega_i \cap \partial^* \Omega_{i+1}.
\]
Its normal velocity is denoted by $V_i$, which is characterized by the distributional identity $\partial_t \chi_i = V_i |\nabla \chi_i|$. Additionally, we will consider its restriction $V_{i,i+1} := V_i |_{I_{i,i+1}}$ together with the symmetry condition $V_{i+1,i} := -V_{i,i+1}$. As before, the corresponding quantities in the calibrated solution will be indicated by an additional bar on top of the quantity, i.e., for example $\bar{\chi}_i$ for the indicator function of the corresponding phases, $\bar{n}_{i,i+1}$ for the corresponding normal, and so on.

The first key step is to construct extensions $\xi_{i,i+1}, i = 1, 2, 3$, of the unit normal vector field $\bar{n}_{i,i+1}$ of the calibrated interfaces $\bar{I}_{i,i+1}$. As in the case of a single
Figure 8. Sketch of a triple junction.

interface, the extensions $\xi_{i,i+1}$ and the velocity field $B$ are constructed to have the following properties:

- The time evolution of the vector fields $\xi_{i,i+1}$ is approximately described by transport along the flow of the velocity field $B$. More precisely, for the vector field $B$ we have for $i = 1, 2, 3$ that
  \[ \partial_t \xi_{i,i+1} = -(B \cdot \nabla)\xi_{i,i+1} - (\nabla B)^T \xi_{i,i+1} + O(\text{dist}(\cdot, I_{i,i+1})) \]

- On each interface $I_{i,i+1}$, $i = 1, 2, 3$, of the calibrated solution, the normal part of the velocity field $B$ must satisfy $\bar{n}_{i,i+1} \cdot B = \bar{H}_{i,i+1}$, where $\bar{H}_{i,i+1}$ is the scalar mean curvature of $I_{i,i+1}$. We strengthen this identity to approximately hold even away from the interface, in form of
  \[ \xi_{i,i+1} \cdot B = -\nabla \cdot \xi_{i,i+1} + O(\text{dist}(\cdot, I_{i,i+1})) \text{ for } i = 1, 2, 3. \]

- The vector fields $\xi_{i,i+1}$ have at most unit length $|\xi_{i,i+1}| \leq 1$.
- The length of the vector fields $\xi_{i,i+1}$ is advected with the flow of $B$ to higher order
  \[ \partial_t |\xi_{i,i+1}|^2 = -(B \cdot \nabla)|\xi_{i,i+1}|^2 + O(\text{dist}^2(\cdot, I_{i,i+1})) \text{ for } i = 1, 2, 3. \]

The new aspect of a triple junction as opposed to a single interface is that one also has to extend the normal of an interface to locations where a different interface may be closer. To this end, we turn to Herring’s angle condition (13), which in our case of equal surface tensions says that the three interfaces must meet at the triple junction to form equal angles of $120^\circ$ each, and require it to hold throughout the domain in the sense that

\[ \sum_{i=1}^{3} \xi_{i,i+1}(x,t) = 0 \text{ for all } x, t. \]

Furthermore, note carefully that we only define a single extension $B$ of the velocity field, and that $B$ is not necessarily a normal vector field on each interface $I_{i,i+1}$: Indeed, we expect the triple junction $p(t)$ to move according to $\frac{d}{dt} p = B(p(t), t)$, so that not allowing for tangential components would pin the triple junction in space. It turns out that in addition to Herring’s angle condition, which we take to be of first order, we require higher-order compatibility conditions of the interfaces at the triple junction. For instance, Definition 11 of a strong solution requires the validity of the evolution law (15b) also on the boundary points of $I_{i,i+1}$ for $i = 1, 2, 3$. Hence,
if we denote the tangent $\bar{\tau}_{i,i+1} := J^{-1} \bar{n}_{i,i+1}$ with $J := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\alpha_{i,i+1} := B \cdot \bar{\tau}_{i,i+1}$, we get $H_{i,i+1} \bar{n}_{i,i+1} + \alpha_{i,i+1} \bar{\tau}_{i,i+1} = H_{i,j+1} \bar{n}_{j,j+1} + \alpha_{j,j+1} \bar{\tau}_{j,j+1}$ for all $i \neq j$ at the triple point. Multiplying these identities with $\bar{n}_{i,i+1}$, then varying over $j$ as well as making use of Herring’s angle condition implies the second-order compatibility condition $H_{1,2} + H_{2,3} + H_{3,1} = 0$, see [37].

To construct the extensions $\xi_{i,i+1}$ of the normal vector fields $\bar{n}_{i,i+1}$, $i = 1, 2, 3$, we first partition space into six wedge-shaped sets around the triple junction: Three contain one strong interface each, while the remaining three wedges lie entirely within a single phase, see Figure 9a. On the mixed phase wedges, we will interpolate between the remaining vector fields to satisfy the identity (26) by 120° rotations of the ansatz, see Figure 9c. On the single phase wedges, we will interpolate between the competing definitions of the two adjacent mixed phase wedges.

All rigorous discussions of compatibility will be deferred to Section 6, and we will only describe the initial extension procedure here. Let us fix $i = 1, 2, 3$. In fact, it is more instructive to first extend the velocity field $B$ in the wedge-shaped neighborhood of the interface $\bar{I}_{i,i+1}$. To this end, we use the extension ansatz

$$B := \bar{H}_{i,i+1} \bar{n}_{i,i+1} + \alpha_{i,i+1} \bar{\tau}_{i,i+1} + \beta_{i,i+1} s_{i,i+1} \bar{\tau}_{i,i+1},$$

where $\bar{n}_{i,i+1}$ and $\bar{\tau}_{i,i+1}$ are extended to be constant in the $\bar{n}_{i,i+1}$-direction, $s_{i,i+1}$ is the signed distance function to $\bar{I}_{i,i+1}$ with the sign convention $\nabla s_{i,i+1} = \bar{n}_{i,i+1}$, and $\alpha_{i,i+1}$ and $\beta_{i,i+1}$ are still to be determined. All quantities except $B$ depend on the interface, we merely dropped the indices $i,i+1$ for ease of notation. As $\frac{d}{ds}p(t) = B(p(t),t)$, it is reasonable that $\alpha_{i,i+1}(p(t),t) := \bar{\tau}_{i,i+1}(p(t),t)$ since $B$ should be the tangential velocity of $p$ at the triple junction. It turns out to be convenient to extend $\alpha_{i,i+1}$ along the interface $\bar{I}_{i,i+1}$ by means of the ordinary differential equation $(\bar{\tau}_{i,i+1} \cdot \nabla) \alpha_{i,i+1} = \bar{H}_{i,i+1}^2$. It can also be seen that $\beta_{i,i+1}(x,t) := (\bar{\tau}_{i,i+1} \cdot \nabla) \bar{H}_{i,i+1}^2 + \alpha_{i,i+1} \bar{H}_{i,i+1}$ for $x \in I_{i,i+1}(t)$ is a good candidate to make $B$ independent of $i$. Indeed, the fact that the coefficient function $\beta_i(p(t),t)$ is independent of $i$ follows by differentiating Herring’s angle condition in time, see (111). It therefore represents another second-order compatibility condition. To define $\alpha_{i,i+1}$ and $\beta_{i,i+1}$ away from the interface, we once again require them to be constant in $\bar{n}_{i,i+1}$-direction.

It turns out that as the extension $\xi = \xi_{i,i+1}(x,t)$ of $\bar{n}_{i,i+1}$ one should take

$$\xi = \bar{n} + \alpha s \bar{\tau} - \frac{1}{2} \alpha^2 s^2 \bar{n} + \frac{1}{2} \beta s^2 \bar{\tau}, \tag{27}$$

where the functions $\alpha = \alpha_{i,i+1}(x,t)$ and $\beta = \beta_{i,i+1}(x,t)$ are as above and we dropped the indices $i,i+1$ for ease of notation. Note that in particular $\xi_{i,i+1} = \bar{n}_{i,i+1}$ on the interface $\bar{I}_{i,i+1}$ and we allow for linear corrections of the tangential component as we move away from the interface, but only for quadratic corrections of the normal component of $\xi$.

We then measure the error between the weak solution $\chi$ and the calibrated solution $\bar{\chi}$ by means of the relative entropy functional

$$E[\chi|\xi](t) := \sum_{i=1}^3 \int_{I_{i,i+1}(t)} (1 - n_{i,i+1} \cdot \xi_{i,i+1}) \, dH^1.$$ 

Let us use the abbreviation $\sum_i = \sum_{i=1}^3$ for the summation over the three relevant indices.
As in the two-phase case, we would like to make use of just two ingredients to evaluate the time evolution of the relative entropy: the energy dissipation inequality.
for the weak solution in the sharp form
\[
\frac{d}{dt} \sum_i \int_{I_{i,i+1}} \sigma^1 \leq - \sum_{i=1}^3 \int_{I_{i,i+1}} V_{i,i+1}^2 d\mathcal{H}^1,
\]
and the weak formulation of the evolution equation of the indicator functions \(\chi_i\)
\[
\frac{d}{dt} \int_{\mathbb{R}^d} \chi_i \varphi \, dx = \int_{\partial^* \Omega_i} V_i \varphi \, d\mathcal{H}^1 + \int_{\mathbb{R}^d} \chi_i \partial_t \varphi \, dx
\]
for compactly supported, smooth \(\varphi\). In order to make use of the latter equation, we have to rewrite the contributions
\[
\int_{I_{i,i+1}} n_{i,i+1} \cdot \xi_{i,i+1}(x,t)
\]
as a volume integral.

It turns out that the annihilation condition \(\sum_i \xi_{i,i+1}(x,t) = 0\) enables us to rewrite \(\xi_{i,i+1}\) as
\[
\xi_{i,i+1} = \xi_i - \xi_{i+1}
\]
by defining the vector field \(\xi_i\) as
\[
\xi_i := \frac{1}{3} (\xi_{i,i+1} - \xi_{i-1,i}).
\]
Combining (28) with the symmetry \(n_{i,i+1} = - \frac{\partial \mathcal{W}_{\chi_i}}{\partial \chi_i} = \frac{\partial \mathcal{W}_{\chi_{i+1}}}{\partial \chi_{i+1}}\) and the decomposition \(\partial^* \Omega_i = I_{i-1,i} \cup I_{i,i+1}\), we rewrite the second term in the relative entropy as
\[
- \sum_i \int_{I_{i,i+1}} n_{i,i+1} \cdot \xi_{i,i+1} \, d\mathcal{H}^1 = \sum_i \left( \int_{I_{i,i+1}} \xi_i \cdot d\nabla \chi_{i} + \int_{I_{i,i+1}} \xi_{i+1} \cdot d\nabla \chi_{i+1} \right)
\]
\[
= \sum_i \int_{\partial^* \Omega_i} \xi_i \cdot d\nabla \chi_{i}
\]
\[
= - \sum_i \int_{\mathbb{R}^d} \chi_i (\nabla \cdot \xi_i) \, dx.
\]

This indeed enables us to evaluate the time evolution of the relative entropy as
\[
\frac{d}{dt} E[\chi|\xi] \leq - \sum_i \int_{I_{i,i+1}} V_{i,i+1}^2 d\mathcal{H}^1
\]
\[
- \sum_i \int_{\partial^* \Omega_i} V_i (\nabla \cdot \xi_i) \, d\mathcal{H}^1 + \sum_i \int_{\partial^* \Omega_i} \partial_t \xi_i \cdot d\nabla \chi_{i} \, d\mathcal{H}^1.
\]

Arguing analogously to the previous computation in reverse order—that is, splitting the integrals into contributions \(\partial^* \Omega_i \cap \partial^* \Omega_{i+1} = I_{i,i+1}\), using (28) and the definitions of \(n_{i,i+1}\) and \(V_{i,i+1}\)—we obtain
\[
\frac{d}{dt} E[\chi|\xi] \leq - \sum_i \int_{I_{i,i+1}} V_{i,i+1}^2 d\mathcal{H}^1 - \sum_i \int_{I_{i,i+1}} V_{i,i+1} (\nabla \cdot \xi_{i,i+1}) \, d\mathcal{H}^1
\]
\[
- \sum_i \int_{I_{i,i+1}} \partial_t \xi_{i,i+1} \cdot n_{i,i+1} \, d\mathcal{H}^1.
\]

Now we proceed as in the two-phase case in the previous section: The BV formulation of mean curvature flow in this three-phase setting reads
\[
\sum_i \int_{I_{i,i+1}} V_{i,i+1} n_{i,i+1} \cdot B \, d\mathcal{H}^1 = - \sum_i \int_{I_{i,i+1}} (\text{Id} - n_{i,i+1} \otimes n_{i,i+1}) : \nabla B \, d\mathcal{H}^1.
\]
Following precisely the same algebraic manipulations as in the two-phase case we obtain
\[ \frac{d}{dt} E[\chi|\bar{\chi}] \leq - \frac{1}{2} \sum_i \int_{I_{i,i+1}} \left( (V_{i,i+1} + \nabla \cdot \xi_{i,i+1})^2 + |V_{i,i+1} n_{i,i+1} - (\xi_{i,i+1} \cdot B) \xi_{i,i+1}|^2 \right) d\mathcal{H}^1 
+ \frac{1}{2} \sum_i \int_{I_{i,i+1}} (\nabla \cdot \xi_{i,i+1} + \xi_{i,i+1} \cdot B)^2 d\mathcal{H}^1 
+ \sum_i \int_{I_{i,i+1}} (\nabla \cdot \xi_{i,i+1}) (n_{i,i+1} \cdot \xi_{i,i+1} - 1) (\xi_{i,i+1} \cdot B) d\mathcal{H}^1 
+ \sum_i \int_{I_{i,i+1}} (\nabla \cdot \xi_{i,i+1} + V_{i,i+1}) n_{i,i+1} \cdot (\text{Id} - \xi_{i,i+1} \otimes \xi_{i,i+1}) B d\mathcal{H}^1 
+ \sum_i \int_{I_{i,i+1}} (1 - n_{i,i+1} \cdot \xi_{i,i+1}) (\nabla \cdot B) d\mathcal{H}^1 
- \sum_i \int_{I_{i,i+1}} (n_{i,i+1} - \xi_{i,i+1}) \otimes (n_{i,i+1} - \xi_{i,i+1}) : \nabla B d\mathcal{H}^1 
- \sum_i \int_{I_{i,i+1}} (\partial_t \xi_{i,i+1} + (B \cdot \nabla) \xi_{i,i+1} + (\nabla B)^T \xi_{i,i+1}) \cdot (n_{i,i+1} - \xi_{i,i+1}) d\mathcal{H}^1 
+ \sum_i \int_{I_{i,i+1}} (\partial_t \xi_{i,i+1} + (B \cdot \nabla) \xi_{i,i+1}) \cdot \xi_{i,i+1} d\mathcal{H}^1. \]

With this inequality at our disposal we can conclude as in the two-phase case.

## 4. Stability of calibrated flows

This section is devoted to the proof of the stability properties of calibrated flows. In the next three subsections, we derive the relative entropy inequality Proposition 12, the stability estimate Proposition 3, and the volume error control Proposition 5.

### 4.1. Relative entropy inequality: Proof of Proposition 12

Consider a BV solution $\chi = (\chi_1, \ldots, \chi_P)$ of multiphase mean curvature flow in the sense of Definition 8 on some time interval $[0, T_{\text{BV}})$, and let $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)$ be a strong solution of multiphase mean curvature flow in the sense of Definition 11 on some time interval $[0, T_{\text{strong}})$ with $T_{\text{strong}} \leq T_{\text{BV}}$. Recall from (19) the definition of our relative entropy functional $E[\chi|\bar{\chi}]$. The aim of this section is to provide the proof of the relative entropy inequality.

**Proof of Proposition 12.** In order to make use of the evolution equations (12b) for the indicator functions $\chi_i$ of the BV solution, we start by rewriting the interface error control of our relative entropy. Using $\sigma_{i,j} \xi_{i,j} = \xi_i - \xi_j$ from Definition 2 of a gradient flow calibration, the symmetry relation $n_{i,j} = -n_{j,i}$, the definition (17) of the measure theoretic normal as well as the representation of the energy (11), we
obtain by an application of the generalized divergence theorem

\[
\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(T)} 1 - \xi_{i,j}(\cdot, T) \cdot n_{i,j}(\cdot, T) \, d\mathcal{H}^{d-1}
\]

\[
= E[\chi(\cdot, T)] - \sum_{i,j=1,i\neq j}^{P} \int_{I_{i,j}(T)} (\xi_{i}(\cdot, T) - \xi_{j}(\cdot, T)) \cdot n_{i,j}(\cdot, T) \, d\mathcal{H}^{d-1}
\]

\[
= E[\chi(\cdot, T)] + \sum_{i=1}^{P} \sum_{j=1,j\neq i}^{P} \int_{I_{i,j}(T)} \xi_{i}(\cdot, T) \cdot \nabla \chi_{i}(\cdot, T) \, d\mathcal{H}^{d-1}
\]

\[
+ \sum_{i=1}^{P} \sum_{j=1,j\neq i}^{P} \int_{I_{i,j}(T)} \xi_{j}(\cdot, T) \cdot \nabla \chi_{j}(\cdot, T) \, d\mathcal{H}^{d-1}
\]

\[
= E[\chi(\cdot, T)] + 2 \sum_{i=1}^{P} \int_{\mathbb{R}^{d}} \xi_{i}(\cdot, T) \cdot \nabla \chi_{i}(\cdot, T) \, d|\nabla \chi_{i}(\cdot, T)|
\]

\[
(29) \quad E[\chi(\cdot, T)] = 2 \sum_{i=1}^{P} \int_{\mathbb{R}^{d}} \chi_{i}(\cdot, T)(\nabla \cdot \xi_{i}(\cdot, T)) \, dx.
\]

This enables us to compute by the sharp energy dissipation inequality (12d), the evolution equations (12b) for the indicator functions \( \chi_{i} \) of the BV solution, and definition (18) of the velocities \( V_{i,j} \) for almost every \( T \in (0, T_{\text{strong}}) \)

\[
E[\chi|\dot{\chi}](T)
\]

\[
\leq E[\chi(\cdot, 0)] - 2 \sum_{i=1}^{P} \int_{\mathbb{R}^{d}} \chi_{0,i}(\nabla \cdot \xi_{i}(\cdot, 0)) \, dx - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} |V_{i,j}|^{2} \, d\mathcal{H}^{d-1} \, dt
\]

\[
- 2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} \chi_{i} \partial_{t}(\nabla \cdot \xi_{i}) \, dx \, dt - 2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} V_{i}(\nabla \cdot \xi_{i}) \, d|\nabla \chi_{i}| \, dt.
\]

The first two terms combine to \( E_{\text{interface}}[\chi|\dot{\chi}](0) \) using (29) in reverse order. We aim to rewrite the latter two terms back to surface integrals over the interfaces as well.

To this end, we argue analogously to the computation in (29) but now in reverse order. Using first the generalized divergence theorem, then splitting the integrals over the reduced boundaries of the phases into contributions over the interfaces \( I_{i,j} = \partial^{\ast} \Omega_{i} \cap \partial^{\ast} \Omega_{j} \) by means of \( \sigma_{i,j} \xi_{i,j} = \xi_{i} - \xi_{j} \) from Definition 2 of a gradient flow calibration we obtain

\[
-2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} \chi_{i} \partial_{t}(\nabla \cdot \xi_{i}) \, dx \, dt = 2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} \nabla \chi_{i} \cdot \partial_{t} \xi_{i} \, d|\nabla \chi_{i}| \, dt
\]

\[
= \sum_{i=1}^{P} \sum_{j=1,j\neq i}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \frac{\nabla \chi_{i}}{|\nabla \chi_{i}|} \cdot \partial_{t} \xi_{i} \, d\mathcal{H}^{d-1} \, dt
\]

\[
+ \sum_{j=1}^{P} \sum_{i=1,i\neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \frac{\nabla \chi_{j}}{|\nabla \chi_{j}|} \cdot \partial_{t} \xi_{j} \, d\mathcal{H}^{d-1} \, dt
\]
\[ (17) \quad - \sum_{i,j=1,i \neq j}^{P} \int_0^T \int_{I_{i,j}(t)} n_{i,j} \cdot \partial_t (\xi_i - \xi_j) \, d\mathcal{H}^{d-1} \, dt \]

\[ = - \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} n_{i,j} \cdot \partial_t \xi_{i,j} \, d\mathcal{H}^{d-1} \, dt. \]

The term incorporating the normal velocities is treated similarly. In addition to the above ingredients, i.e., \( \sigma_{i,j} \xi_{i,j} = \xi_i - \xi_j \) from Definition 2 of a gradient flow calibration and splitting the integrals over the reduced boundaries of the phases into contributions over the interfaces \( I_{i,j} = \partial^* \Omega_i \cap \partial^* \Omega_j \), we also use that \( V_{i,j} = -V_{j,i} \) on \( I_{i,j} \) together with definition (18) to compute

\[
-2 \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} V_i(\nabla \cdot \xi_i) \, d|\nabla \chi_i| \, dt = - \sum_{i=1}^{P} \sum_{j=1,j \neq i}^{P} \int_0^T \int_{I_{i,j}(t)} V_{i,j}(\nabla \cdot \xi_i) \, d\mathcal{H}^{d-1} \, dt
\]

\[
+ \sum_{j=1}^{P} \sum_{i=1,i \neq j}^{P} \int_0^T \int_{I_{i,j}(t)} V_{i,j}(\nabla \cdot \xi_j) \, d\mathcal{H}^{d-1} \, dt
\]

\[
= - \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} V_{i,j}(\nabla \cdot \xi_{i,j}) \, d\mathcal{H}^{d-1} \, dt. \]

Combining the last two identities, we obtain for almost every \( T \in (0, T_{\text{strong}}) \)

\[
E[\chi|\xi|(T)]
\]

\[
\leq E[\chi|\xi|(0)] - \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} |V_{i,j}|^2 \, d\mathcal{H}^{d-1} \, dt
\]

\[
- \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} n_{i,j} \cdot \partial_t \xi_{i,j} \, d\mathcal{H}^{d-1} \, dt
\]

\[
- \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} V_{i,j}(\nabla \cdot \xi_{i,j}) \, d\mathcal{H}^{d-1} \, dt. \]

For the next step, we use the vector field \( B \) as a test function in the BV formulation of mean curvature flow (12c). Adding the resulting equation to the previous inequality, observing in the process that \( V_i \nabla \chi_i = -V_i n_{i,j} \) on \( I_{i,j} \) due to (17) and (18), as well as decomposing \( B = (\text{Id} - \xi_{i,j} \otimes \xi_{i,j}) B + (B \cdot \xi_{i,j}) \xi_{i,j} \), we obtain

\[
E[\chi|\xi](T)
\]

\[
\leq E[\chi|\xi](0) - \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} |V_{i,j}|^2 \, d\mathcal{H}^{d-1} \, dt
\]

\[
+ \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (B \cdot \xi_{i,j}) \xi_{i,j} \cdot V_{i,j} n_{i,j} \, d\mathcal{H}^{d-1} \, dt
\]

\[
- \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} V_{i,j}(\nabla \cdot \xi_{i,j}) \, d\mathcal{H}^{d-1} \, dt. \]
Furthermore, on the one hand, adding and subtracting \( (\nabla \cdot B) \xi_{i,j} \) yields for almost every term on the left hand side of the relative entropy inequality (20), we complete the squares yielding for almost every \( T \in (0, T_{\text{strong}}) \)
\[
+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} ((B \cdot \nabla)\xi_{i,j}) \cdot n_{i,j} \, d\mathcal{H}^{d-1} \, dt \\
- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} (\partial_{t}\xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{T}\xi_{i,j}) \cdot n_{i,j} \, d\mathcal{H}^{d-1} \, dt
\]

for almost every \( T \in (0, T_{\text{strong}}) \). On the other hand, we may exploit symmetry to obtain (relying again on the by now routine fact that one can switch back and forth between certain volume integrals and surface integrals over the individual interfaces by means of \( \sigma_{i,j}\xi_{i,j} = \xi_{i} - \xi_{j} \) from Definition 2 of a gradient flow calibration, the symmetry relation \( n_{i,j} = -n_{j,i} \) and the definition (17))

\[
\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} n_{i,j} \cdot (\nabla \cdot (B \otimes \xi_{i,j})) \, d\mathcal{H}^{d-1} \, dt \\
= \sum_{i,j=1,i\neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} n_{i,j} \cdot (\nabla \cdot (B \otimes (\xi_{i} - \xi_{j}))) \, d\mathcal{H}^{d-1} \, dt \\
= -2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} \frac{\nabla \chi_{i}}{|\nabla \chi_{i}|} \cdot (\nabla \cdot (B \otimes \xi_{i})) \, d\mathcal{H}^{d-1} \, dt \\
= 2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} \chi_{i} \nabla \cdot (\nabla \cdot (B \otimes \xi_{i})) \, dx \, dt \\
= 2 \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^{d}} \chi_{i} \nabla \cdot (\nabla \cdot (\xi_{i} \otimes B)) \, dx \, dt \\
= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} n_{i,j} \cdot (\nabla \cdot (\xi_{i,j} \otimes B)) \, d\mathcal{H}^{d-1} \, dt.
\]

Because of this identity, we can now compute

\[
0 = \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} n_{i,j} \cdot (\nabla \cdot (B \otimes (\xi_{i,j} - \xi_{i,j} \otimes B))) \, d\mathcal{H}^{d-1} \, dt \\
= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} (\nabla \cdot \xi_{i,j}) B \cdot n_{i,j} \, d\mathcal{H}^{d-1} \, dt \\
+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} n_{i,j} \cdot (\xi_{i,j} \cdot \nabla) B \, d\mathcal{H}^{d-1} \, dt \\
- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} n_{i,j} \cdot (B \cdot \nabla)\xi_{i,j} \, d\mathcal{H}^{d-1} \, dt \\
- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} (\nabla \cdot B)\xi_{i,j} \cdot n_{i,j} \, d\mathcal{H}^{d-1} \, dt.
\]

Making use of the identities (31) and (32) in the inequality (30) as well as adding (33) to the right hand side of (30), we arrive at the following bound for the time
evolution of the interface error control of our relative entropy functional

\[
E[\chi|\xi](T) + \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \left( \frac{1}{2} |\nabla \cdot \xi_{i,j}|^2 + \frac{1}{2} |(B \cdot \xi_{i,j})n_{i,j}|^2 \right) d\mathcal{H}^{d-1} dt
\]

\[
\leq E[\chi|\xi](0) + \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \left( \frac{1}{2} |\nabla \cdot \xi_{i,j}|^2 + \frac{1}{2} |(B \cdot \xi_{i,j})n_{i,j}|^2 \right) d\mathcal{H}^{d-1} dt
\]

which is valid for almost every \( T \in (0,T_{\text{strong}}) \). Completing squares and adding zero yields for the second, third and fourth term on the right hand side of (34)

\[
\sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \left( \frac{1}{2} |\nabla \cdot \xi_{i,j}|^2 + \frac{1}{2} |(B \cdot \xi_{i,j})n_{i,j}|^2 \right) d\mathcal{H}^{d-1} dt
\]

\[
+ \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla \cdot \xi_{i,j}) B \cdot n_{i,j} d\mathcal{H}^{d-1} dt
\]

\[
+ \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\text{Id} - \xi_{i,j} \otimes \xi_{i,j}) B \cdot V_{i,j} n_{i,j} d\mathcal{H}^{d-1} dt
\]

\[
- \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (n_{i,j} - \xi_{i,j}) \otimes n_{i,j} : \nabla B d\mathcal{H}^{d-1} dt
\]

\[
+ \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} n_{i,j} \otimes \xi_{i,j} : \nabla B d\mathcal{H}^{d-1} dt
\]

\[
- \sum_{i,j=1,i\neq j} P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\partial_t \xi_{i,j} + (B \cdot \nabla) \xi_{i,j} + (\nabla B)^T \xi_{i,j}) \cdot n_{i,j} d\mathcal{H}^{d-1} dt,
\]
Adding zero in the last term on the right hand side of (34) in order to generate the transport equation for the length of the vector fields $\xi_{i,j}$, we observe that the last three terms on the right hand side of (34) combine to

$$
- \sum_{i,j=1,i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (n_{i,j} - \xi_{i,j}) \otimes n_{i,j} : \nabla B \, d\mathcal{H}^{d-1} \, dt
$$

$$
+ \sum_{i,j=1,i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} n_{i,j} \otimes \xi_{i,j} : \nabla B \, d\mathcal{H}^{d-1} \, dt
$$

$$
- \sum_{i,j=1,i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^T \xi_{i,j}) \cdot n_{i,j} \, d\mathcal{H}^{d-1} \, dt
$$

(36)

Employing the notation of Proposition 12 as well as using (35) and (36) in (34), we deduce that the right hand side of (34) indeed reduces to

$$
E[\chi^2(T)]
$$

$$
+ \sum_{i,j=1,i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \left[ \frac{1}{2} |V_{i,j} + \nabla \cdot \xi_{i,j}|^2 + \frac{1}{2} |V_{i,j} n_{i,j} - (B \cdot \xi_{i,j}) \xi_{i,j}|^2 \right] \, d\mathcal{H}^{d-1} \, dt
$$

$$
\leq E[\chi^2(0)] + R_{dt} + R_{\text{dissip}},
$$

which is valid for almost every $T \in (0, T_{\text{strong}})$. This concludes the proof of (20). \qed

4.2. Quantitative inclusion principle: Proof of Proposition 3. We now prove the inclusion principle stating that interfaces of any BV solution must be contained in the corresponding interfaces of a calibrated flow, provided both start with the same initial data.

Proof of Proposition 3. Step 1: The stability estimate (6). The starting point is the estimate on the evolution of the interface error functional (5) from Proposition 12. In the following, we estimate the terms appearing on the right hand side one-by-one. Due to (4c), (4d), as well as the trivial relation

$$
|n_{i,j} - \xi_{i,j}|^2 \leq 2(1 - n_{i,j} \cdot \xi_{i,j})
$$

(37)
(which follows by $|\xi_{i,j}| \leq 1$), we immediately deduce

\begin{equation}
|R_{dt}| \leq C \int_0^T E[\chi|\xi](t) \, dt.
\end{equation}

Making use of the simple estimate $1 - |\xi_{i,j}|^2 \leq 2(1 - |\xi_{i,j}|) \leq 2(1 - n \cdot \xi_{i,j})$ and again the bound (37), we also obtain

\begin{align*}
|R_{\text{diss}}| &\leq \sum_{i,j=1, i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} |(\nabla \cdot \xi_{i,j}) + B \cdot \xi_{i,j}|^2 \, d\mathcal{H}^{d-1} \, dt \\
&\quad + \sum_{i,j=1, i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\text{Id} - \xi_{i,j} \otimes \xi_{i,j}) B \cdot (V_{i,j} + \nabla \cdot \xi_{i,j}) n_{i,j} \, d\mathcal{H}^{d-1} \, dt \\
&\quad + C \int_0^T E[\chi|\xi](t) \, dt \\
&=: I + II + C \int_0^T E[\chi|\xi](t) \, dt.
\end{align*}

By means of (4e), we may directly estimate

\begin{equation}
|I| \leq C \int_0^T E[\chi|\xi](t) \, dt.
\end{equation}

Furthermore, by an application of Hölder’s and Young’s inequality we deduce

\begin{align*}
|II| &\leq \delta \sum_{i,j=1, i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} (V_{i,j} + \nabla \cdot \xi_{i,j})^2 \, d\mathcal{H}^{d-1} \, dt \\
&\quad + C \delta^{-1} \int_0^T E[\chi|\xi](t) \, dt,
\end{align*}

uniformly over all $\delta \in (0, 1)$. Hence, we get the bound

\begin{equation}
|R_{\text{diss}}| \leq \delta \sum_{i,j=1, i \neq j}^P \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2} (V_{i,j} + \nabla \cdot \xi_{i,j})^2 \, d\mathcal{H}^{d-1} \, dt \\
+ C \delta^{-1} \int_0^T E[\chi|\xi](t) \, dt.
\end{equation}

Plugging in the bounds from (38) and (39) into the relative entropy inequality from Proposition 12, and then choosing $\delta \in (0, 1)$ sufficiently small in order to absorb the first right-hand side term, we therefore get constants $C > 0$ and $c > 0$
such that the estimate

\[ E[\chi|\xi](T) \]

\[ + c \sum_{i,j=1,i\neq j} P \int_0^T \int_{I_{i,j}(t)} \left( \frac{1}{2}(V_{i,j} + \xi_{i,j})^2 + \frac{1}{2}|V_{i,j} n_{i,j} - (B \cdot \xi_{i,j})\xi_{i,j}|^2 \right) d\mathcal{H}^{d-1} dt \]

\[ \leq C \int_0^T E[\chi|\xi](t) dt \]

holds true for almost every \( T \in (0, T_{\text{strong}}) \). By an application of Gronwall’s lemma, the asserted stability estimate (6) from Proposition 3 follows.

**Step 3: Weak-strong comparison.** In the case of coinciding initial conditions, i.e. \( E[\chi|\xi](0) = 0 \), the stability estimate (6) entails \( E[\chi|\xi] = 0 \) for almost every \( t \in [0, T_{\text{strong}}) \). From this and (4b), it immediately follows that \( I_{i,j}(t) \subset \tilde{I}_{i,j}(t) \) holds up to an \( \mathcal{H}^{d-1} \)-null set for almost every \( t \in [0, T_{\text{strong}}) \). This proves the weak-strong uniqueness principle for BV solutions of multiphase mean curvature flow in the plane.

□

4.3. **Weighted volume error control: Proof of Proposition 5.** Unfortunately, the interface error control provided by the functional from (19) suffers from a lack of coercivity in the case of vanishing interface length for a BV solution. It is for this reason that our relative entropy functional not only consists of the interface error control but also of a lower-order term \( E_{\text{volume}}[\chi|\bar{\chi}] \), which provides an error control in the volume occupied by the grains of the strong solution (and weighted by the distance to the respective grain boundary). The aim of this section is to derive the main ingredient for the stability estimate (8), and therefore Proposition 5. We start our discussion with the following pendant of Proposition 12. In analogy to Proposition 12, we will only require minimal assumptions on the weight functions defining the functional \( E_{\text{volume}}[\chi|\bar{\chi}] \). The construction of such weight functions with sufficiently good coercivity properties is deferred to the subsequent result.

**Lemma 13.** Let \( d = 2, P \geq 2 \) be an integer and \( \sigma \in \mathbb{R}^{P \times P} \) be an admissible matrix of surface tensions in the sense of Definition 6. Let \( \chi = (\chi_1, \ldots, \chi_P) \) be a BV solution of multiphase mean curvature flow in the sense of Definition 8 on some time interval \([0, T_{\text{BV}})\), and let \( \bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P) \) be a strong solution of multiphase mean curvature flow in the sense of Definition 11 on some time interval \([0, T_{\text{strong}})\) with \( T_{\text{strong}} \leq T_{\text{BV}} \). Recall from (17) resp. (18) the definitions of the (measure-theoretic) unit normal vectors \( n_{i,j} \) resp. of the normal velocities \( V_{i,j} \) of a BV solution.

Let \( 0 < r < R \) be two radii such that \( \bigcup_{i,j=1,i\neq j} I_{i,j}(t) + B_{2r}(0) \) is compactly supported in \( B_R(0) \) for all times \( t \in [0, T_{\text{strong}}) \). Let then \( (\vartheta_i)_{i=1,\ldots,P} \) be a family of functions satisfying

\[ \vartheta_i \in L^\infty([0, T_{\text{strong}}); W^{1,\infty}(\mathbb{R}^2)) \cap W^{1,\infty}([0, T_{\text{strong}}); L^\infty(\mathbb{R}^2)), \]

\[ \text{supp} \vartheta_i \subset B_{2r}(0), \vartheta_i \equiv 0 \text{ on the phase boundary } \bigcup_{j=1}^P \tilde{I}_{i,j} \text{ of the } i\text{-th phase } \{ \tilde{\chi}_i = 1 \} \text{ of the strong solution, } \vartheta_i \leq 0 \text{ in } \{ \tilde{\chi}_i = 1 \} \text{ and } \vartheta_i \geq 0 \text{ in } \{ \tilde{\chi}_i = 0 \}. \]

Define then the
weighted volume error control

\begin{equation}
E_{\text{volume}}[\chi|\bar{\chi}](t) := \sum_{i=1}^{P} \int_{\mathbb{R}^2} |\chi_i(\cdot, t) - \bar{\chi}_i(\cdot, t)||\partial_i(\cdot, t)| \, dx.
\end{equation}

Moreover, let \((\xi_{i,j})_{i\neq j \in \{1, \ldots, p\}}\) be a family of compactly supported vector fields such that

\[ \xi_{i,j} \in L^\infty([0, T_{\text{strong}}); W^{2,\infty}(\mathbb{R}^2; \mathbb{R}^2)) \cap W^{1,\infty}([0, T_{\text{strong}}); C^0(\mathbb{R}^2; \mathbb{R}^2)) \]

and let

\[ B \in L^\infty([0, T_{\text{strong}}); W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^2)) \]

be an arbitrary compactly supported vector field.

Then the weighted volume error control is subject to the following identity

\begin{equation}
E_{\text{volume}}[\chi|\bar{\chi}](T) = E_{\text{volume}}[\chi|\bar{\chi}](0) + R_{\text{volume}}.
\end{equation}

for almost every \( T \in (0, T_{\text{strong}}) \). Here, we made use of the abbreviation

\[ R_{\text{volume}} := - \sum_{i=1, i \neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \partial_i (B \cdot \xi_{i,j} - V_{i,j}) \, d\mathcal{H}^1 \, dt \]

\[ - \sum_{i=1, i \neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \partial_i B \cdot (\nu_{i,j} - \xi_{i,j}) \, d\mathcal{H}^1 \, dt \]

\[ + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \partial_i (\nabla \cdot B) \, dx \, dt \]

\[ + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) (\partial_t \chi_i + (B \cdot \nabla) \chi_i) \, dx \, dt. \]

**Proof.** We first observe that \( E_{\text{volume}}[\chi|\bar{\chi}] < \infty \) because of \( \text{supp} \partial_i \subset B_{2R}(0) \). To compute the time evolution, note that the sign condition on \( \partial_i \) is precisely what is needed to have

\[ E_{\text{volume}}[\chi|\bar{\chi}](T) = \sum_{i=1}^{P} \int_{\mathbb{R}^2} (\chi_i(\cdot, T) - \bar{\chi}_i(\cdot, T)) \partial_i(\cdot, T) \, dx. \]

Hence, we may make use of the evolution equations \((12b)\) for the indicator functions \( \chi_i \) of the BV solution as well as the ones for the indicator functions \( \bar{\chi}_i \) of the strong solution, yielding for almost every \( T \in (0, T_{\text{strong}}) \)

\[ E_{\text{volume}}[\chi|\bar{\chi}](T) \]

\[ = E_{\text{volume}}[\chi|\bar{\chi}](0) + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \partial_t \chi_i \, dx \, dt + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} V_i \partial_i d|\nabla \chi_i| \, dt. \]

Here we have used that \( \partial_i |\nabla \bar{\chi}_i| \equiv 0 \), since \( \partial_i \) vanishes on \( \partial\{\bar{\chi}_i = 1\} \). We next use \((18)\) and rewrite

\[ \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} V_i \partial_i d|\nabla \chi_i| \, dt = \sum_{i=1, i \neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \partial_i V_{i,j} \, d\mathcal{H}^1 \, dt. \]
Furthermore, by adding and subtracting \((B \cdot \nabla) \vartheta_i\), an integration by parts, \(\vartheta_i \equiv 0\) on \(\partial \{ \bar{\chi}_i = 1 \}\), and the definition (17) of the measure theoretic normal, we obtain

\[
\sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \partial_t \vartheta_i \, dx \, dt
\]

\[
= - \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) (B \cdot \nabla) \vartheta_i \, dx \, dt + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) (\partial_t \vartheta_i + (B \cdot \nabla) \vartheta_i) \, dx \, dt
\]

\[
= - \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \nabla \cdot (\vartheta_i B) \, dx \, dt + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \vartheta_i (\nabla \cdot B) \, dx \, dt
\]

\[
+ \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) (\partial_t \vartheta_i + (B \cdot \nabla) \vartheta_i) \, dx \, dt
\]

\[
= \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} \frac{\nabla \chi_i}{|\nabla \chi_i|} \cdot \vartheta_i B \, d|\nabla \chi_i| \, dt + \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \vartheta_i (\nabla \cdot B) \, dx \, dt
\]

\[
+ \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) (\partial_t \vartheta_i + (B \cdot \nabla) \vartheta_i) \, dx \, dt
\]

\[
= - \sum_{i=1, i \neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \vartheta_i B \cdot \xi_{i,j} \, dH^1 \, dt
\]

\[
- \sum_{i=1, i \neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \vartheta_i B \cdot (\eta_{i,j} - \xi_{i,j}) \, dH^1 \, dt
\]

\[
+ \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) \vartheta_i (\nabla \cdot B) \, dx \, dt
\]

\[
+ \sum_{i=1}^{P} \int_{0}^{T} \int_{\mathbb{R}^2} (\chi_i - \bar{\chi}_i) (\partial_t \vartheta_i + (B \cdot \nabla) \vartheta_i) \, dx \, dt
\]

for almost every \(T \in (0, T_{\text{strong}})\). All in all, we therefore get for almost every \(T \in (0, T_{\text{strong}})\) the identity (42) as asserted. \(\square\)

In order to infer the stability estimate (8) based on the identity (42), we have to construct — next to a gradient flow calibration — a family of suitable weight functions \(\vartheta_i\) with sufficiently good coercivity properties. As the following result shows, the existence of a strong solution always guarantees this.

Lemma 14. Let \(\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)\) be a strong solution of multiphase mean curvature flow in the sense of Definition 11 on some time interval \([0, T_{\text{strong}})\). Let \(\bar{\gamma}_c > 0\) be the localization parameter from Lemma 28. Choose \(R = R(\bar{\chi}) > 0\) such that \(\bigcup_{i,j=1, i \neq j}^{P} I_{i,j}(t) + B_{2\bar{\gamma}_c}(0)\) is compactly supported in \(B_R(0)\) for all times \(t \in [0, T_{\text{strong}})\).

Then, for every phase \(i \in \{1, \ldots, P\}\) there exist weight functions

\(\vartheta_i \in L^\infty([0, T_{\text{strong}}); W^{1,\infty}(\mathbb{R}^2; [-1, 1])) \cap W^{1,\infty}([0, T_{\text{strong}}); L^\infty(\mathbb{R}^2; [-1, 1]))\)

with \(\text{supp} \vartheta_i \subset B_{2R}(0)\) subject to the following additional conditions:
i) It holds \( \partial_i < 0 \) in \( \{ \bar{\chi}_i = 1 \} \cap B_{2R}(0) \setminus \partial \{ \bar{\chi}_i = 1 \} \) as well as \( \partial_i > 0 \) in \( \{ \bar{\chi}_i = 0 \} \cap B_{2R}(0) \setminus \partial \{ \bar{\chi}_i = 1 \} \).

ii) It holds \( \partial_i \equiv 0 \) on \( \partial \{ \bar{\chi}_i = 1 \} \).

iii) We have the bound \( |\text{dist}(\cdot, \partial \{ \bar{\chi}_i = 1 \})| \leq C|\partial_i| \) on \( \partial \{ \bar{\chi}_i = 1 \} + B_{2\bar{r}}(0) \).

Moreover, let the global velocity field \( B \) be defined as in Construction 33. Then, we in addition have the following estimate on the advective derivative

\[
|\partial_t \phi_i + (B \cdot \nabla) \phi_i| \leq C|\partial_i|
\]

for all \( i \in \{1, \ldots, P\} \).

\textbf{Proof.} Let \( \vartheta : \mathbb{R} \to \mathbb{R} \) be a truncation of the identity with \( \vartheta(r) = r \) for \( |r| \leq \frac{1}{2} \), \( \vartheta(r) = -1 \) for \( r \leq -1 \), \( \vartheta(r) = 1 \) for \( r \geq 1 \), \( 0 \leq \vartheta' \leq 2 \) and \( |\vartheta''| \leq C \). Let \( \eta_R \in C^\infty_{cpt}(\mathbb{R}^2; [0, 1]) \) be any smooth cutoff such that \( \eta_R \equiv 1 \) on \( B_R \) and \( \text{supp} \eta_R \subset B_{2R} \).

The idea for what follows is to construct in a first step a family of weight functions \( \hat{\vartheta}_i \) which satisfy all the requirements of Lemma 14 except for \( \text{supp} \hat{\vartheta} \subset B_{2R}(0) \).

Once we succeeded with that we may define \( \hat{\vartheta}_i := \eta_R \vartheta_i \) in a second step.

\textbf{Step 1: Construction of} \( \hat{\vartheta}_i \). If \( \partial \{ \bar{\chi}_i = 1 \} \) consists of only one interface, we then simply define

\[
\hat{\vartheta}_i(x, t) := \vartheta \left( \frac{\text{sdist}(x, \partial \{ \bar{\chi}_i(t) = 1 \})}{\bar{r}_c} \right), \quad (x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}).
\]

If \( \partial \{ \bar{\chi}_i = 1 \} \) consists of more than one two-phase interface, we first define the weight function \( \vartheta_i \) away from the triple junctions by the same formula:

\[
\hat{\vartheta}_i(x, t) := \vartheta \left( \frac{\text{sdist}(x, \partial \{ \bar{\chi}_i(t) = 1 \})}{\bar{r}_c} \right), \quad t \in [0, T_{\text{strong}}), x \in \mathbb{R}^2 \setminus \bigcup_{k \in K^3} B_{2\bar{r}_c}(T_k(t)).
\]

Now, let \( k \in K^3 \) be a triple junction \( T_k \subset \partial \{ \bar{\chi}_i = 1 \} \) with the other two majority phases being \( j, p \in \{1, \ldots, P\} \). As outlined in Definition 20, \( B_{2\bar{r}_c}(T_k) \) decomposes into six wedges. The three wedges \( W_{i,j}, W_{j,p}, \text{resp. } W_{p,i} \) contain the interfaces \( \bar{I}_{i,j}, \bar{I}_{j,p}, \text{resp. } \bar{I}_{p,i} \). The other three are interpolation wedges denoted by \( W_i, W_j, \text{resp. } W_p \).

Moreover, for the interpolation wedge \( W_i \), say, we denote by \( \lambda_i^j \) the interpolation function as built in Lemma 27 and which is equal to one on \( \partial W_{i,j} \cap \partial W_i \setminus T_k \) and vanishes on \( \partial W_{p,i} \cap \partial W_i \setminus T_k \). We also define \( \lambda_i^p := 1 - \lambda_i^j \). Analogously, one introduces the interpolation functions on \( W_j \) and \( W_p \).

We now define the weight function \( \hat{\vartheta}_i \) on the ball \( B_{2\bar{r}_c}(T_k) \) as follows:

\[
\hat{\vartheta}_i(x, t) := \vartheta \left( \frac{\text{sdist}(x, \bar{I}_{i,j}(t))}{\bar{r}_c} \right), \quad t \in [0, T_{\text{strong}}), x \in W_{i,j}(t) \cap B_{2\bar{r}_c}(T_k(t)),
\]

and analogously on \( W_{i,p} \), whereas we interpolate for \( t \in [0, T_{\text{strong}}) \) and \( x \in W_i(t) \cap B_{2\bar{r}_c}(T_k(t)) \) by means of

\[
\hat{\vartheta}_i(x, t) := \lambda_i^j(x, t) \vartheta \left( \frac{\text{sdist}(x, \bar{I}_{i,j}(t))}{\bar{r}_c} \right) + \lambda_i^p(x, t) \vartheta \left( \frac{\text{sdist}(x, \bar{I}_{i,p}(t))}{\bar{r}_c} \right).
\]

Furthermore, we define

\[
\hat{\vartheta}_i(x, t) := \vartheta \left( \frac{\text{dist}(x, T_k(t))}{\bar{r}_c} \right), \quad t \in [0, T_{\text{strong}}), x \in W_{i,j}(t) \cap B_{2\bar{r}_c}(T_k(t)),
\]
whereas we again interpolate for \( t \in [0, T_{\text{strong}}) \) and \( x \in W_p(t) \cap B_{2\tilde{r}_c}(T_k(t)) \) via

\[
\hat{\vartheta}(x, t) := \lambda^p(x, t) \left[ \frac{|x - p_k(t)|}{\tilde{r}_c} \right] + \lambda^i(x, t) \vartheta \left( \text{sdist}(x, \tilde{I}_{i,j}(t)) \right),
\]

and analogously for \( t \in [0, T_{\text{strong}}) \) and \( x \in W_p(t) \cap B_{2\tilde{r}_c}(T_k(t)) \).

First of all, it is immediate from the definitions that properties i), ii) and iii) hold true as required. Furthermore, the asserted regularity for the weight functions \( \hat{\vartheta} \) is a consequence of the regularity of the signed distance to the interfaces as well as the controlled blowup (130) and (131) of the derivatives of the interpolation parameter.

For (43), we may restrict ourselves to the tubular neighborhood \( \partial \{ \tilde{\chi}_i = 1 \} + B_{2\tilde{r}_c}(0) \). By the choice of the localization parameter \( \tilde{r}_c \) > 0 from Lemma 28, we have that for all \( x \in (\partial \{ \tilde{\chi}_i = 1 \} + B_{2\tilde{r}_c}(0)) \setminus \bigcup_{k \in \mathcal{K}^{\mathcal{N}}} B_{2\tilde{r}_c}(T_k) \) there is exactly one \( m \in \mathcal{K}^{\mathcal{N}} \) such that

\[
\text{sdist}(\cdot, \partial \{ \tilde{\chi}_i = 1 \}) = \text{sdist}(\cdot, T_m) \text{ in a small neighborhood around that point. In other words, because of the localization properties (141)–(145) for the partition of unity constructed in Lemma 28, we have in a neighborhood of such points } B(x) = \eta_m(x)B_m(x) \text{ by the definition (168) of } B. \text{ It therefore follow from (45), (55) and (59) that}
\]

\[
\partial_t \hat{\vartheta}_i(x) = -(B^m(x) \cdot \nabla) \hat{\vartheta}_i(x) = -(B(x) \cdot \nabla) \hat{\vartheta}_i(x) - (1 - \eta_m)(B^m(x) \cdot \nabla) \hat{\vartheta}_i(x)
\]

on \( (\partial \{ \tilde{\chi}_i = 1 \} + B_{2\tilde{r}_c}(0)) \setminus \bigcup_{k \in \mathcal{K}^{\mathcal{N}}} B_{2\tilde{r}_c}(T_k) \). Since we have the estimate \( |1 - \eta_m(x)| \leq C |\hat{\vartheta}_i(x)| \) on \( (\partial \{ \tilde{\chi}_i = 1 \} + B_{2\tilde{r}_c}(0)) \setminus \bigcup_{k \in \mathcal{K}^{\mathcal{N}}} B_{2\tilde{r}_c}(T_k) \) by definition (156) and property iii) of \( \hat{\vartheta} \), we indeed have (43) for points \( x \in (\partial \{ \tilde{\chi}_i = 1 \} + B_{2\tilde{r}_c}(0)) \setminus \bigcup_{k \in \mathcal{K}^{\mathcal{N}}} B_{2\tilde{r}_c}(T_k) \) away from the triple junctions.

We move on with checking (43) on \( B_{2\tilde{r}_c}(T_k) \). On the wedge \( W_{i,j} \), we make use of the evolution equation \( \frac{d}{dt} p_k(t) = B(p_k(t), t) \) (where \( T_k(t) = \{ p_k(t) \} \)) to obtain

\[
\partial_t \hat{\vartheta}_i = -(B(p_k(t), t) \cdot \nabla) \hat{\vartheta}_i = -(B \cdot \nabla) \hat{\vartheta}_i - ((B(p_k(t), t) - B) \cdot \nabla) \hat{\vartheta}_i,
\]

from which (43) follows on \( W_{i,j} \cap B_{2\tilde{r}_c}(T_k) \) due to the Lipschitz continuity of the velocity field \( B \) and property iii) of \( \hat{\vartheta} \). On the wedge \( W_{i,j} \), we may compute using (59) and (61) (as well as assuming for notational convenience that \( T_m \subset \tilde{I}_{i,j} \))

\[
\partial_t \hat{\vartheta}_i = -(B^m \cdot \nabla) \hat{\vartheta}_i
\]

\[
= -(B \cdot \nabla) \hat{\vartheta}_i - (1 - \eta_m - \eta_k)(B^m \cdot \nabla) \hat{\vartheta}_i - \eta_k (B^m - B^k) \cdot \nabla \hat{\vartheta}_i.
\]

Since \( |1 - \eta_m - \eta_k| \leq C |\text{dist}(\cdot, \tilde{I}_{i,j})|^2 \) on \( W_{i,j} \cap B_{2\tilde{r}_c}(T_k) \), and also (164) holds true, the bound (43) on \( W_{i,j} \cap B_{2\tilde{r}_c}(T_k) \) thus again follows from property iii) of \( \hat{\vartheta} \). The validity of (43) on \( W_{i,j} \cap B_{2\tilde{r}_c}(T_k) \) follows analogously.

It remains to discuss the interpolation parameter. To this end, we only give the argument for \( \hat{\vartheta} \). The corresponding bound on the other two interpolation wedges \( W_j \) resp. \( W_p \) follows similarly. If the advective derivative drops on the interpolation parameter, we may rely on (134), \( \hat{\vartheta}(0) = 0 \) as well as property iii) of \( \hat{\vartheta} \) to obtain on \( W_{i} \cap B_{2\tilde{r}_c}(T_k) \) the identity

\[
\hat{\vartheta} \left( \text{sdist}(\cdot, \tilde{I}_{i,j}) \right) \partial_t \lambda^i + \hat{\vartheta} \left( \text{sdist}(\cdot, \tilde{I}_{i,p}) \right) \partial_t \lambda^p
\]

\[
= -\hat{\vartheta} \left( \text{sdist}(\cdot, \tilde{I}_{i,j}) \right) (B \cdot \nabla) \lambda^i - \hat{\vartheta} \left( \text{sdist}(\cdot, \tilde{I}_{i,j}) \right) (B \cdot \nabla) \lambda^p + O(\hat{\vartheta}).
\]
Furthermore, denoting again $T_m \subset \mathcal{I}_{i,j}$ and introducing for notational convenience the shorthand $\vartheta^m := \vartheta\left(\frac{\mathrm{dist}(-\mathcal{I}_{i,j})}{\tau_c}\right)$, we may compute due to (59) on the interpolation wedge $W_i \cap B_{2r_i}(T_k)$

$$\partial_t \vartheta^m_i = -(B^m \cdot \nabla)\vartheta^m_i = -(B \cdot \nabla)\hat{\vartheta}_i - \left((B^m - B^k) \cdot \nabla\right)\hat{\vartheta}_i - \left((B^k - B) \cdot \nabla\right)\hat{\vartheta}_i.$$

The latter two terms are again of required order. Since an analogous computation can be carried out for $\vartheta\left(\frac{\mathrm{dist}(-\mathcal{I}_{i,j})}{\tau_c}\right)$, we infer that (43) indeed holds true on the interpolation wedge $W_i \cap B_{2r_i}(T_k)$.

**Step 2: Construction and properties of $\vartheta_i$.** As already mentioned at the beginning of the proof, we may now define $\vartheta_i := \eta_R \hat{\vartheta}_i$. The regularity, the support property as well as the requirements $i\cdot ii$) for $\vartheta_i$ are then immediate consequences of its definition and the previous step. The estimate (43) on the advective derivative also carries over since $\eta_R$ is time-independent and $(B \cdot \nabla)\eta_R \equiv 0$ holds true. The latter is a consequence of $B$ being supported in $\bigcup_{i,j=1,i \neq j}^{P} \mathcal{I}_{i,j}(t) + B_{2r_i}(0) \subset B_R(0)$ (which follows from the definition (168) and the localization properties of the partition of unity from Lemma 28) and that $\text{supp} \nabla \eta_R \subset B_{2R}(0) \setminus B_R(0)$. This concludes the proof of Lemma 14. \hfill \square

We have now everything in place to post-process the right-hand side in (42).

**Lemma 15.** Let $d = 2$, $P \geq 2$ be an integer and $\sigma \in \mathbb{R}^{P \times P}$ be an admissible matrix of surface tensions in the sense of Definition 6. Let $\chi = (\chi_1, \ldots, \chi_P)$ be a $BV$ solution of multiphase mean curvature flow in the sense of Definition 8 on some time interval $[0, T_{\text{BV}})$, and let $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)$ be a strong solution of multiphase mean curvature flow in the sense of Definition 11 on some time interval $[0, T_{\text{strong}})$ with $T_{\text{strong}} \leq T_{\text{BV}}$.

Let $(\xi_{i,j}, \hat{B})$ be the gradient flow calibration provided by Construction 30 and Construction 33. Consider moreover the family of weight functions $\vartheta_i$ as constructed by Lemma 14. Finally, recall the definition of the interface error control $E[\chi|\bar{\chi}]$ from (19) and the definition of the weighted volume error control $E_{\text{volume}}[\chi|\bar{\chi}]$ from (41). Then, uniformly over all $\delta \in (0, 1)$, it holds

$$E_{\text{volume}}[\chi|\bar{\chi}](T) \leq E_{\text{volume}}[\chi|\bar{\chi}](0) + \frac{C}{\delta} \int_0^T E_{\text{volume}}[\chi|\bar{\chi}](t) + E[\chi|\bar{\chi}](t) \, dt$$

$$+ \delta \sum_{i,j=1,i \neq j}^{P} \sigma_{i,j} \int_0^T \int_{\mathcal{I}_{i,j}(t)} \frac{1}{2} (\nabla \mathcal{H}^1 - B \cdot \nabla) \chi_{i,j}^2 \, dH^1 \, dt$$

for almost every $T \in [0, T_{\text{strong}})$.

**Proof.** Starting point is (42) meaning that we need to estimate the term $R_{\text{volume}}$. Note that Lemma 13 is indeed applicable with $(\xi_{i,j}, \hat{B})$ being the gradient flow calibration provided by Construction 30 and Construction 33, and $\vartheta_i$ being the family of weight functions as constructed by Lemma 14.
First, we may infer because of \((43)\), Hölder’s and Young’s inequality as well as the bound \((37)\) that

\[
|R_{\text{volume}}| \leq C\delta^{-1} \sum_{i,j=1,i\neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t) \cap \{\text{dist}(\cdot, \partial \{\bar{x}_i(\cdot, t) = 1\}) \leq 2\bar{r}_c\}} \varrho_i^2 \, d\mathcal{H}^1 \, dt
\]

\[
+ C\delta^{-1} \sum_{i,j=1,i\neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t) \cap \{\text{dist}(\cdot, \partial \{\bar{x}_i(\cdot, t) = 1\}) > 2\bar{r}_c\}} \varrho_i^2 \, d\mathcal{H}^1 \, dt
\]

\[
+ \delta \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{0}^{T} \int_{I_{i,j}(t)} \frac{1}{2} (V_{i,j} - B \cdot \xi_{i,j})^2 \, d\mathcal{H}^1 \, dt
\]

\[
+ C\delta^{-1} \int_{0}^{T} E[\chi|\xi](t) + E_{\text{volume}}[\chi|\bar{x}](t) \, dt
\]

holds true, uniformly over all \(\delta \in (0, 1)\). Since \(\varrho_i \equiv 0\) on \(\partial \{\bar{x}_i = 1\}\) by Lemma 14, we may estimate \(|\varrho_i|^2 \leq C |\text{dist}(\cdot, \partial \{\bar{x}_i = 1\})|^2\) on \(\{\text{dist}(\cdot, \partial \{\bar{x}_i = 1\}) \leq 2\bar{r}_c\}\). To discuss the contribution on \(\{\text{dist}(\cdot, \partial \{\bar{x}_i = 1\}) > 2\bar{r}_c\}\) we at least have \(|\xi_{i,j}| \leq c < 1\) due to \((160)\) and definition \((162)\). In summary, we may estimate \(|\varrho_i|^2 \leq C(1-|\xi_{i,j}|)\) on \(\{\text{dist}(\cdot, \partial \{\bar{x}_i(\cdot, t) = 1\}) > 2\bar{r}_c\}\).

It is then a consequence of \((163)\) that we may estimate

\[
|R_{\text{weightVol}}| \leq \delta \sum_{i,j=1,i\neq j}^{P} \int_{0}^{T} \int_{I_{i,j}(t)} \frac{1}{2} (V_{i,j} - B \cdot \xi_{i,j})^2 \, d\mathcal{H}^1 \, dt
\]

\[
+ C\delta^{-1} \int_{0}^{T} E[\chi|\xi](t) + E_{\text{volume}}[\chi|\bar{x}](t) \, dt
\]

uniformly over all \(\delta \in (0, 1)\). This concludes the proof. 

We are finally in position to provide a proof for the stability estimate \((8)\).

**Proof of Proposition 5.** As a combination of Lemma 15, the inequality \((40)\) and an absorption argument, we obtain for the sum of the two functionals \(E_{\text{volume}}[\chi|\bar{x}]\) resp. \(E[\chi|\xi]\) defined in \((41)\) resp. \((19)\) the estimate

\[
E_{\text{volume}}[\chi|\bar{x}](T) + E[\chi|\xi](T) \leq C \int_{0}^{T} E_{\text{volume}}[\chi|\bar{x}](t) + E[\chi|\xi](t) \, dt
\]

for almost every \(T \in [0, T_{\text{strong}}]\). Since the properties of the weight \(\varrho_i\), see Lemma 14, ensure that \(|\varrho_i(\cdot, t)|\) is comparable to \(\text{dist}(\cdot, \partial \{\bar{x}_i(\cdot, t) = 1\})\) \(\land 1\) this concludes the proof of Proposition 5 by an application of Gronwall’s lemma. 

5. **Gradient flow calibrations at a smooth manifold**

The aim of this section is to construct a gradient flow calibration in the simplest situation of one single manifold (with or without boundary) evolving by mean curvature, see Lemma 17 for the main result of this section. For the sake of simplicity, we stick to the case \(d = 2\), but the construction in this section immediately carries over to arbitrary dimensions. In terms of a gradient flow calibration for a whole network of interfaces in the sense of Definition 2, the vector fields constructed in
Lemma 17 provide the local building block at a smooth two-phase interface of the network. These vector fields therefore only live in a small tubular neighborhood around the evolving interface. In order to make use of the constructions outlined in this section in the general network case, one has to multiply with a suitably defined cutoff function localizing to the tubular neighborhood for a given two-phase interface of the network. In light of the coercivity condition in Definition 2 of a gradient flow calibration, this cutoff shall decrease quadratically in the distance to the interface. The construction of such a set of cutoff functions is deferred to Section 7.1.

**Definition 16.** Let $T_{\text{strong}} > 0$ be a finite time horizon. A one-parameter family $(\bar{I}(t))_{t \in [0,T_{\text{strong}}]}$ of embedded, connected and orientable one-dimensional $C^4$-manifolds (with or without boundary) in $\mathbb{R}^2$ is called a regular interface evolving by mean curvature if there exists a map

$$
\Psi : \bar{I}(0) \times [0,T_{\text{strong}}) \to \mathbb{R}^2, \quad (x,t) \mapsto \Psi(x,t) = \Psi^t(x)
$$

with the following properties:

1. $\Psi^0 = \text{Id}$, and for all $t \in [0,T_{\text{strong}})$ the map $\Psi^t : \bar{I}(0) \to \bar{I}(t)$ is a $C^4$-diffeomorphism such that $\|\Psi\|_{L^\infty W_4^{3,\infty}} < \infty$.

2. We have $\partial_t \Psi \in C^0([0,T_{\text{strong}}); C^3(\bar{I}(0); \mathbb{R}^2))$ and $\|\partial_t \Psi\|_{L^\infty W_4^{3,\infty}} < \infty$.

3. We assume that there exists $r_c \in (0, \frac{1}{2}]$ with the following property: For all $t \in [0,T_{\text{strong}})$ and all $x \in \bar{I}(t)$ there exists a function $g : (-1,1) \to \mathbb{R}$ with $g'(0) = 0$ and $y_{\text{min}} \in [-1,0]$ such that after a translation and a rotation, $\bar{I}(t) \cap B_{2r_c}(x)$ is given by the graph $\{(y,g(y)) : y \in (y_{\text{min}},1]\}$, which we require to be a connected set. Furthermore, for any of these functions $g$ we require the pointwise bounds $|g^{(m)}| \leq r_c^{-1}m!$ to hold for all $1 \leq m \leq 3$.

Let $\bar{n}(\cdot,t)$ denote a unit normal vector field of $\bar{I}(t)$, arising by choosing an orientation for $\bar{I}(0)$ and transporting it along $\Psi$. Let $\bar{V}(x,t)$ denote the normal speed of a point $x \in \bar{I}(t)$, i.e., the normal component of $\partial_t \Psi(y,t)$ at $y = (\Psi^t)^{-1}(x) \in \bar{I}(0)$. Denoting by $H(t)$ the mean curvature vector field of $\bar{I}(t)$, we then assume that

$$
\bar{V}(x,t)\bar{n}(x,t) = H(x,t) \quad \text{for all } t \in [0,T_{\text{strong}}), x \in \bar{I}(t).
$$

As usual, tangential velocities can be chosen arbitrarily for a smooth manifold evolving by its mean curvature. Note also that the preceding definition allows the boundary points of the manifold, if present, to move.

It follows from our assumptions in Definition 16 that for all $t \in [0,T_{\text{strong}})$ the maps

$$
\Phi^t : \bar{I}(t) \times (-r_c,r_c) \to \mathbb{R}^2, \quad (x,s) \mapsto x + s\bar{n}(x,t)
$$

are $C^3$-diffeomorphisms onto their image. Exploiting this property, we may define a signed distance function in the image of $\Phi^t$ via the second component of its inverse function

$$
s(x,t) := (\Phi^t)^{-1}(x,t) = \begin{cases} 
\text{dist}(x,\bar{I}(t)), & x \in \Phi^t(\bar{I}(t) \times [0,r_c]), \\
-\text{dist}(x,\bar{I}(t)), & x \in \Phi^t(\bar{I}(t) \times (-r_c,0)).
\end{cases}
$$

Let us denote in the following by $\mathcal{U}$ the space-time domain $\bigcup_{t \in [0,T_{\text{strong}})} \text{im}(\Phi^t) \times \{t\}$. For a point $(x,t) \in \mathcal{U}$, the projection of $x$ onto the nearest point on the interface
\( \bar{t}(t) \) is well-defined and given by

\[
P_I(x, t) := P_{\bar{t}(t)}x := x - s(x, t)\bar{n}(x, t).
\]

As a consequence of our regularity assumptions on the evolving family of interfaces \((\bar{t}(t))_{t \in [0, T_{\text{strong}})}\), we infer that the signed distance \( s \) (resp. its time derivative \( \partial_t s \)) are of class \( C^1_u \) in \( L^\infty L^\infty_x \) (resp. \( C^0_uC^3_x \) with \( \nabla^3 s \in L^\infty L^\infty_x \)) in the space-time domain \( U \), see Lemma 18. In particular, the projection map \( P_I \) is of class \( C^0 \) in \( \nabla^3 P_I \in L^\infty L^\infty_x \) in \( U \).

The identities needed for the estimation of the terms occurring on the right-hand side of the relative entropy inequality (20) in Proposition 12, at least on the level of the two-phase problem considered in this section, are collected in the following result.

**Lemma 17.** Let \((\bar{t}(t))_{t \in [0, T_{\text{strong}})}\) be a regular interface evolving by mean curvature in the sense of Definition 16. Let \( \alpha \in L^\infty W^{1,\infty}_x \) be an arbitrary map, and define the tangent vector field \( \bar{\tau} := J^\bar{n} \) where \( J \) denotes the counter-clockwise rotation by an angle of 90°. Then the vector fields \( \xi \in L^\infty W^{2,\infty}_x \cap W^{1,\infty}_x \) and \( B \in L^\infty W^{1,\infty}_x \) given by

\[
(54) \quad \xi(x, t) := \nabla s(x, t) = \bar{n}(P_{\bar{t}(t)}x, t),
\]

\[
(55) \quad B(x, t) := H(P_{\bar{t}(t)}x, t) + \alpha(P_{\bar{t}(t)}x, t)\bar{\tau}(P_{\bar{t}(t)}x, t)
\]

for \((x, t) \in \mathcal{U} := \bigcup_{t \in [0, T_{\text{strong}})} \text{im}(\Phi^t) \times \{ t \} \) satisfy

\[
(56) \quad \partial_t \xi(x, t) + (B(x, t) \cdot \nabla)\xi(x, t) + (\nabla B(x, t))^T \xi(x, t) = 0,
\]

\[
(57) \quad \partial_t |\xi(x, t)|^2 + (B(x, t) \cdot \nabla)|\xi(x, t)|^2 = 0,
\]

\[
(58) \quad B(x, t) \cdot \xi(x, t) + \nabla \cdot \xi(x, t) = O(\text{dist}(x, \bar{t}(t)))
\]

for all \((x, t) \in \mathcal{U} \).

**Proof.** It follows from (59) and (63) below, as well as from the orthogonality \( \bar{\tau} \cdot \bar{n} = 0 \) that the tangential term in the definition of \( B \) does not have an effect on the transport equation for the signed distance \( s \), i.e., we have

\[
\partial_t s(x, t) = -(H(P_{\bar{t}(t)}x, t) \cdot \nabla)s(x, t) = -(B(x, t) \cdot \nabla)s(x, t).
\]

We may take the gradient of this identity so that by definition of \( \xi \) we have

\[
\partial_x \xi(x, t) = \nabla \partial_t s(x, t) = -(B(x, t) \cdot \nabla)\xi(x, t) - (\nabla B(x, t))^T \xi(x, t),
\]

which proves (56). The validity of (57) is evident from the fact that \( |\xi|^2 \equiv 1 \). For the identity (58), note first that \( B(x, t) \cdot \xi(x, t) = H(P_{\bar{t}(t)}x, t) \cdot \xi(x, t) \) as a consequence of the orthogonality \( \bar{\tau} \cdot \bar{n} = 0 \). By means of the identity (62) below,

\[
\nabla \bar{n}(y, t)|_{y = P_{\bar{t}(t)}x} = -(H(P_{\bar{t}(t)}x, t) \cdot \bar{n}(x, t))\bar{\tau}(P_{\bar{t}(t)}x, t) \otimes \bar{\tau}(P_{\bar{t}(t)}x, t)
\]

and the definition (54) of the vector field \( \xi \) we compute

\[
\nabla \cdot \xi(x, t) = -H(P_{\bar{t}(t)}x, t) \cdot \xi(x, t)(1 - s(x, t))\nabla^2 s(x, t) : \bar{\tau}(P_{\bar{t}(t)}x, t) \otimes \bar{\tau}(P_{\bar{t}(t)}x, t).
\]

This concludes the proof. \( \square \)

The preceding Lemma relies on a number of well-known properties of the signed distance and the nearest point projection. For further reference, we present them here in a separate statement.
Lemma 18. Let \((\bar{I}(t))_{t \in [0,T_{\text{strong})}}\) be a smoothly evolving interface moving by mean curvature in the sense of Definition 16. The signed distance function then satisfies \(s \in C^0_t C^3_x\) with \(\nabla s \in L^\infty_t L^2_x\) and \(\partial_t s \in C^0_t C^3_x\) with \(\nabla^3 \partial_t s \in L^\infty_t L^2_x\) in \(U\). Its time evolution is given by transport along the flow of the mean curvature vector field:

\[
\partial_t s(x, t) = -(\nabla H(\bar{I}(t)x, t) \cdot \nabla) s(x, t), \quad (x, t) \in U.
\]

Furthermore, the trivial extensions of the normal and tangent fields

\[
\bar{n}(x, t) := \bar{n}(P_{\bar{I}(t)}x, t), \\
\bar{\tau}(x, t) := \bar{\tau}(P_{\bar{I}(t)}x, t) = J^T \bar{n}(P_{\bar{I}(t)}x, t),
\]

where \(J\) denotes the counter-clockwise rotation by 90°, are transported as well:

\[
\partial_t \bar{n} = -\left(\nabla (P_{\bar{I}(t)}x, t) \cdot \nabla\right) \bar{n} - \left(\nabla (H(P_{\bar{I}(t)}x, t))\right)^T \bar{n}, \quad (x, t) \in U,
\]

\[
\partial_t \bar{\tau} = -\left(\nabla (P_{\bar{I}(t)}x, t) \cdot \nabla\right) \bar{\tau} - J^T \left(\nabla (H(P_{\bar{I}(t)}x, t))\right)^T J \bar{\tau}, \quad (x, t) \in U.
\]

The gradient of the projection map is given by

\[
\nabla P_{\bar{I}}(x, t) = \text{Id} - \bar{n}(P_{\bar{I}(t)}x, t) \otimes \bar{n}(P_{\bar{I}(t)}x, t) - s(x, t) \nabla^2 s(x, t), \quad (x, t) \in U.
\]

Finally, the derivatives of the signed distance \(s\) are subject to the relations

\[
\nabla s(x, t) = \bar{n}(P_{\bar{I}(t)}x, t) = \nabla s(y, t)|_{y = P_{\bar{I}(t)}x},
\]

\[
\nabla s(x, t) \cdot \partial_i \nabla s(x, t) = 0, \\
\nabla s(x, t) \cdot \partial_j \nabla s(x, t) = 0, \quad j = 1, \ldots, d,
\]

\[
\partial_t s(x, t) = \partial_i s(y, t)|_{y = P_{\bar{I}(t)}x},
\]

for all points \((x, t) \in U\).

Proof. The representation of \(s\) as a component of the inverse of \(\Phi^t\) and the first two parts of Definition 16 initially give the regularity \(s \in C^0_t C^3_x \cap L^\infty_t W^{3,\infty}_x\) and \(\partial_t s \in C^0_t C^2_x \cap L^\infty_t W^{2,\infty}_x\). Apart from identities (60) and (61), a proof of the remaining, well-known identities was given for instance in [23, Lemma 10] with the only difference being the precise form of the normal velocity of the evolving family of interfaces. The equality (60) then follows from the identity (59) by differentiation, which in turn implies the equation (61) by rotation. The higher regularity for the signed distance \(s\) and its time derivative \(\partial_t s\) finally follows from the identity (63) and the first two parts of Definition 16.

6. Gradient flow calibrations at a triple junction

The aim of this section is to construct a gradient flow calibration in the case of three interfaces meeting at a single triple junction. This geometric setting will be referred to in the following as a regular triod moving by mean curvature. All relevant local, geometric properties of a regular triod moving by mean curvature are collected in Definition 20. We then state the main result of this section, Proposition 21, which provides all relevant properties of the constructed fields.

The construction of \(\xi_{i,j}\) proceeds in three steps. First, we extend the normal of the interface \(\bar{I}_{i,j}\) of the strong solution to auxiliary vector fields \(\xi_{i,j}\) defined on suitably chosen half-spaces \(H_{i,j}\), see Figure 10a, on which the nearest point-projection onto \(\bar{I}_{i,j}\) is well-defined and regular. One should think of \(\xi_{i,j}\) as the main building block for the vector field \(\xi_{i,j}\) on the half-space \(H_{i,j}\) containing the corresponding interface \(\bar{I}_{i,j}\).
In the second step, we aim to identify a candidate vector field for the definition of \( \xi_{i,j} \) outside the corresponding half-space \( \mathbb{H}_{i,j} \). The guiding principle for this step is to arrange the constructions in such a way that the Herring angle condition at the triple junction

\[
\sigma_{1,2} \bar{n}_{1,2} + \sigma_{2,3} \bar{n}_{2,3} + \sigma_{3,1} \bar{n}_{3,1} = 0,
\]

where the indices 1, 2, 3 correspond to the phases present at the triple junction, is satisfied by the family of vector fields \((\xi_{1,2}, \xi_{2,3}, \xi_{3,1})\) in the whole neighborhood of the triple junction:

\[
\sigma_{1,2} \xi_{1,2} + \sigma_{2,3} \xi_{2,3} + \sigma_{3,1} \xi_{3,1} = 0.
\]

The motivation for having (68) comes from the observation that this condition will allow us to define vector fields \((\xi_1, \xi_2, \xi_3)\) such that \(\sigma_{i,j} \xi_{i,j} = \xi_i - \xi_j\) holds true. The latter identity in turn is precisely what is needed to compute the time derivative of the relative entropy functional, see the proof of Proposition 12 in Section 4.1.

How do we achieve (68)? Imagine we want to find candidate vector fields for \(\xi_{2,3}\) and \(\xi_{3,1}\) in the half-space \(\mathbb{H}_{1,2}\) containing the interface \(\tilde{I}_{1,2}\) such that (68) is satisfied. Recall that in the first step we already constructed a candidate vector field \(\xi_{1,2}\) for \(\xi_{1,2}\) in the half space \(\mathbb{H}_{1,2}\). However, since the desired condition (68) really represents an angle condition, the natural candidates for \(\xi_{2,3}\) and \(\xi_{3,1}\) in \(\mathbb{H}_{1,2}\) are given by appropriate rotations of \(\tilde{\xi}_{1,2}\) in order to recover (68); see Figure 9b and Figure 9c above. The same procedure is repeated on the other two half-spaces \(\mathbb{H}_{2,3}\) and \(\mathbb{H}_{3,1}\) by appropriately rotating the vector fields \(\xi_{2,3}\) and \(\xi_{3,1}\), respectively.

To summarize, we outlined so far the construction of candidate vector fields \(\xi_{i,j}\) in each of the half-spaces \(\mathbb{H}_{i',j'}\). However, in the regions where the half-spaces overlap (see Figure 10a) we now have two competing building blocks for each of the desired vector fields \(\xi_{i,j}\). To overcome this issue, we partition the neighborhood of the triple junction into six wedges as indicated in Figure 10b, three of which are denoted by \(W_{i,j} = W_{j,i}\) and the remaining three by \(W_i\). We will require that \(\tilde{I}_{i,j} \subset W_{i,j} \subset \mathbb{H}_{i,j}\), see Figure 10b. The first inclusion corresponds to a geometric smallness condition away from the triple junction. For the remaining three wedges it is required that \(W_i \subset \{\tilde{\chi}_i = 1\}\) and \(W_i \subset \bigcap_{j \neq i} \mathbb{H}_{i,j}\), see again Figure 10b. We refer to these wedges from now on as interpolation wedges as they serve as the domains on which we interpolate between the two competing candidate vector fields.

The following lemma ensures that we can indeed find wedges with the desired properties. Its proof is deferred to the end of the section.

**Lemma 19.** For an admissible matrix of surface tensions \((\sigma_{i,j})_{i,j=1,2,3}\) in the sense of Definition 6, there exist coefficients \(a_1^+, a_1^-, b_1^+, b_1^- \in \mathbb{R}\) for \(i = 1, 2, 3\) such that the following holds: Given a point \(p\) and vectors \(\tau_{1,2}, \tau_{2,3}, \tau_{3,1} \in S^1\), which we will below consider to be tangents at the triple junction \(p\) (cf. Figure 11 below), satisfying

\[
\sigma_{1,2} \tau_{1,2} + \sigma_{2,3} \tau_{2,3} + \sigma_{3,1} \tau_{3,1} = 0
\]

we define the half-spaces

\[
\mathbb{H}_{i,i+1} := \{ x \in \mathbb{R}^2 : (x - p) \cdot \tau_{i,i+1} > 0 \}\]
and the vectors $v_i := a_i^+ \tau_{i,i+1} + a_i^- \tau_{i-1,i}$ and $w_i := b_i^+ \tau_{i,i+1} + b_i^- \tau_{i-1,i}$ for all $i \in \{1, 2, 3\}$. Then these vectors satisfy $v_i, w_i \in S^1$ and $v_i \cdot w_i \in (0, 1)$, and the interior of the conical hulls spanned by $(v_i, w_i)$ and $(v_i, w_{i+1})$ define pairwise disjoint, non-empty, open wedges

\begin{align}
W_i &:= \{ \gamma_1 v_i + \gamma_2 w_i : \gamma_1, \gamma_2 \in (0, \infty) \} \\
W_{i,i+1} &:= \{ \gamma_1 v_i + \gamma_2 w_{i+1} : \gamma_1, \gamma_2 \in (0, \infty) \}
\end{align}

satisfying

\begin{align}
\bigcup_{i=1,2,3} (W_i \cup W_{i,i+1}) &= \mathbb{R}^2, \\
W_i &\subset \mathbb{H}_{i,i+1} \cap \mathbb{H}_{i-1,i}, \\
W_{i,i+1} &\subset \mathbb{H}_{i,i+1}, \\
\tau_{i,i+1} &\in W_{i,i+1}.
\end{align}

In order to not rely on cyclical notation in later sections, we set $W_{i+1,i} := W_{i,i+1}$ for all $i \in \{1, 2, 3\}$.

We have everything in place to sketch the third and final step in the construction of the vector fields $\xi_{i,j}$. To minimize confusion with the indices, let us concentrate for the sake of the discussion on just one of these vector fields, say $\xi_{1,2}$. On each of the wedges $W_{i,j}$ we define $\xi_{1,2}$ by means of the first two steps. I.e., we define $\xi_{1,2} := \tilde{\xi}_{1,2}$ on $W_{1,2}$ and $\xi_{1,2} := R \tilde{\xi}_{2,3}$ (resp. $\xi_{1,2} := R' \tilde{\xi}_{3,1}$) on $W_{2,3}$ (resp. on $W_{3,1}$) for an appropriate rotation matrix $R$ (resp. $R'$) in order to satisfy (68). Note that these definitions are justified because of (73).
On the interpolation wedges we proceed by gluing the two competing constructions. For example, on $W_1$ we define $\tilde{\xi}_{1,2} := (1-\lambda_1)\xi_{1,2} + \lambda_1 R^t \xi_{3,1}$ whereas on $W_2$ (resp. $W_3$) we set $\xi_{1,2} := (1-\lambda_2)R^t \xi_{2,3} + \lambda_2 \xi_{1,2}$ (resp. $\xi_{1,2} := (1-\lambda_3)R^t \xi_{3,1} + \lambda_3 R^t \xi_{2,3}$). The functions $(\lambda_1, \lambda_2, \lambda_3)$ are suitably chosen interpolation functions, see Lemma 27. Because of (72) these definitions are again justified.

As a final remark concerning the construction of the vector fields $\xi_{i,j}$, it is not clear at this point whether it is possible to carry out the above program together with having sufficiently high regularity for the vector fields $\xi_{i,j}$ at the triple junction. The naive candidate for the auxiliary vector fields $\tilde{\xi}_{i,j}$ from the first step of the construction are the gradient flow calibrations at a smooth manifold from the previous section, i.e., $\tilde{\xi}_{i,j}(x) := \tilde{n}_{i,j}(P_{I_{i,j}}(x))$ on $\tilde{H}_{i,j}$ where $P_{I_{i,j}}$ denotes the nearest point projection onto $I_{i,j}$. However, this ansatz together with the other two steps only provides continuous vector fields $\tilde{\xi}_{i,j}$ which already fail to be Lipschitz at the triple junction. Hence, we have to employ in the first step of the construction a careful adaption of the naive candidate for the auxiliary vector fields $\tilde{\xi}_{i,j}$, see (86).

Following the work in [36], we now define a notion of a regular triod moving by mean curvature. In addition to the framework outlined in [36], we further impose certain “smallness” conditions on the geometry. In the case of a general network moving by its mean curvature, these conditions will be satisfied after localizing around the triple junctions on a sufficiently small scale, see Lemma 28.

**Definition 20.** Let $r > 0$ be a radius and $T_{\text{strong}} > 0$ be a finite time horizon. Let $p: [0, T_{\text{strong}}) \to \mathbb{R}^2$ be a $C^1$-map. We call a measurable map

$$\bar{\chi} = (\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3): \bigcup_{t \in [0, T_{\text{strong}})} B_r(p(t)) \times \{t\} \to \{0, 1\}^3$$

a regular triod with triple junction $p(t)$ evolving by mean curvature in the space-time domain $\bigcup_{t \in [0, T_{\text{strong}})} B_r(p(t)) \times \{t\}$, if it satisfies the following requirements:

i) The map $\bar{\chi} = (\bar{\chi}_1, \bar{\chi}_2, \bar{\chi}_3)$ is a strong solution for multiphase mean curvature flow in the space-time domain $\bigcup_{t \in [0, T_{\text{strong}})} B_r(p(t)) \times \{t\}$ in the sense of Definition 11.

ii) The phases $\{\bar{\chi}_i(\cdot, t) = 1\} \subset B_r(p(t))$ are open, non-empty and simply connected sets for all $i \in \{1, 2, 3\}$ and all $t \in [0, T_{\text{strong}})$. Define for all $i, j \in \{1, 2, 3\}$ the interfaces $I_{i,j}(t) := \partial \{\bar{\chi}_i = 1\} \cap \partial \{\bar{\chi}_j = 1\} \setminus \partial B_r(p(t))$. Then we assume that the three interfaces intersect only in the single point $p(t)$.

iii) We choose the tangent vector $\bar{\nu}_{i,j}(p(t), t) := J^{-1} \bar{n}_{i,j}(p(t), t)$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$, where

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and denote the corresponding wedges constructed in Lemma 19 by $W_i(t)$ and $W_{i,i+1}(t)$. In this notation, we require $I_{i,i+1}(t) \subset W_{i,i+1}(t) \cup \{p(t)\}$.

iv) We assume that for all $t \in [0, T_{\text{strong}})$ and all $i \in \{1, 2, 3\}$ the maps

$$\Phi_{i,i+1}^t: I_{i,i+1}(t) \times (-2r, 2r) \to \mathbb{R}^2, \quad (x, s) \mapsto x + s\bar{n}_{i,i+1}(x, t),$$

are $C^3$-diffeomorphisms onto their image with bounded and continuous third derivatives. Moreover we require that the image of $\Phi_{i,i+1}^t$ contains $\overline{W}_{i,i+1}(t) \cap \mathbb{R}^2$.
Proposition 21. Let \( \tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3) \) be a regular triod with triple junction \( p(t) \) evolving by mean curvature in the sense of Definition 20 on the space-time domain \( \bigcup_{I \in [0,T_{\text{strong}}]} B_r(p(t)) \times \{t\} \). Denote by \( \bar{I}(t) := \bigcup_{I \neq j} I_{i,j}(t) \) the union of the three interfaces. There then exists a radius \( \tilde{r} = \tilde{r}(\tilde{\chi}) \leq r \), only depending on the data of \( \tilde{\chi} \), with the following property: for all \( i, j \in \{1, 2, 3\} \) with \( i \neq j \), there exist extensions \( \xi_{i,j} \in L_t^\infty W_x^2 \cap W_t^{1,\infty} C_0^0 \) of the unit-normal vector fields, and a velocity field \( B \in L_t^\infty W_x^1 \), which are defined on the space-time domain \( \bigcup_{I \in [0,T_{\text{strong}}]} B_r(p(t)) \times \{t\} \) and subject to the following properties:

i) It holds \( \xi_{i,j}(x,t) = \tilde{n}_{i,j}(x,t) \) for all \( t \in [0,T_{\text{strong}}] \) and for all \( x \in \bar{I}_{i,j}(t) \). We also have \( |\xi_{i,j}(x,t)| = 1 \) for all \( t \in [0,T_{\text{strong}}] \) and for all \( x \in B_r(p(t)) \).

ii) We have the skew-symmetry relation \( \xi_{i,j} = -\xi_{j,i} \).

iii) The family of vector fields \( (\xi_{i,j})_{i \neq j} \) satisfies the Herring angle condition (67) in the entire neighborhood of the triple junction, i.e., it holds

\[
\sigma_{1,2} \xi_{1,2}(x,t) + \sigma_{2,3} \xi_{2,3}(x,t) + \sigma_{3,1} \xi_{3,1}(x,t) = 0
\]

for all \( t \in [0,T_{\text{strong}}] \) and all \( x \in B_r(p(t)) \).

iv) Throughout \( \bigcup_{I \in [0,T_{\text{strong}}]} B_r(p(t)) \times \{t\} \), we have the bounds

\[
|\partial_t \xi_{i,j} + (B \cdot \nabla) \xi_{i,j} + (\nabla B)^T \xi_{i,j}| \leq C \text{dist}(x, \bar{I}(t)),
\]

\[
|\nabla \cdot \xi_{i,j} + B \cdot \xi_{i,j}| \leq C \text{dist}(x, \bar{I}(t)),
\]

\[
\xi_{i,j} \cdot \partial_t \xi_{i,j} + \xi_{i,j} \cdot (B \cdot \nabla) \xi_{i,j} = 0
\]

for all \( i, j \in \{1, 2, 3\} \) with \( i \neq j \). The constant \( C > 0 \) only depends on the regular triod \( \tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3) \), and therefore in particular on the parameter \( r_c > 0 \) from Definition 10 of a smoothly evolving network of interfaces.

v) Finally, the following estimates on the derivatives of \( (\xi_{i,j}, B) \) hold true

\[
|\nabla^k \xi_{i,j}| \leq Cr_c^{k-2}, \quad k \in \{1, 2\},
\]

\[
|\nabla B| \leq Cr_c^{-3}
\]

throughout the space-time domain \( \bigcup_{I \in [0,T_{\text{strong}}]} B_r(p(t)) \times \{t\} \).

When applying these bounds in Section 7 the radius \( \tilde{r} > 0 \) will always be fixed and proportional to \( r_c \).

6.1. Local construction close to individual interfaces. Let \( \tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3) \) be a regular triod with triple junction \( p(t) \) moving by mean curvature in the sense of Definition 20 on the space-time domain \( \bigcup_{I \in [0,T_{\text{strong}}]} B_r(p(t)) \times \{t\} \). To simplify notation, we will from now on identify indices if they are equivalent mod 3, i.e., we define \( \chi_4 := \chi_1, \chi_5 := \chi_2, \chi_0 := \chi_3 \), and so on. In this subsection, we first introduce certain auxiliary vector fields \( \tilde{\xi}_{i,i+1} \) as extensions of the normal \( \tilde{n}_{i,i+1} \) to the closure of the half-spaces \( \mathbb{H}_{i,i+1} \). Their restriction to the wedge \( W_{i,i+1} \) will later serve as the definition of \( \xi_{i,i+1} \) on \( W_{i,i+1} \). Additionally, they will also serve as building blocks for the interpolation on the wedges \( W_i \), see Section 6.2.
We would like to define $\tilde{\xi}_{i,i+1}$, and later also the velocity field $B$, by an expansion ansatz in terms of the signed distance function $s_{i,i+1}$ to the interface $I_{i,i+1}$. Making use of the diffeomorphisms from property \textit{iv}) of Definition 20 of a regular triod, in analogy to Section 5 we define the signed distance function as

$$s_{i,i+1}(x, t) := (\Phi_{i,i+1}^t)^{-1}(x, t)$$

(83)

for $x \in \mathbb{H}_{i,i+1}(t) \cap B_r(p(t))$. Note that with this choice we have $\tilde{n}_{i,i+1} = \nabla s_{i,i+1}$ on the interface $I_{i,i+1}$. We extend $\tilde{n}_{i,i+1}$ to a vector field on $\mathbb{H}_{i,i+1}(t) \cap B_r(p(t))$ by means of this relation, and denote this extension still by $\tilde{n}_{i,i+1}$. The nearest point projection $P_{i,i+1}$ onto $I_{i,i+1}$ is given by

$$P_{i,i+1}(x, t) := x - s_{i,i+1}(x, t)\tilde{n}_{i,i+1}(x, t),$$

and its gradient by

$$\nabla P_{i,i+1}(x, t) = \text{Id} - \tilde{n}_{i,i+1}(x, t) \otimes \tilde{n}_{i,i+1}(x, t) - s_{i,i+1}(x, t)\nabla \tilde{n}_{i,i+1}(x, t).$$

The ansatz for the extension $\tilde{\xi}_{i,i+1}$ of the normal vector field $\tilde{n}_{i,i+1}|_{I_{i,i+1}}$ then is

$$\tilde{\xi}_{i,i+1}(x, t) := \tilde{n}_{i,i+1}(P_{i,i+1}(x, t), t)$$

(86)

$$+ \alpha_{i,i+1}(P_{i,i+1}(x, t), t) s_{i,i+1}(x, t) \tilde{\tau}_{i,i+1}(P_{i,i+1}(x, t), t)$$

$$- \frac{1}{2} \alpha_{i,i+1}^2(P_{i,i+1}(x, t), t) s_{i,i+1}^2(x, t) \tilde{n}_{i,i+1}(P_{i,i+1}(x, t), t)$$

$$+ \frac{1}{2} \beta_{i,i+1}(P_{i,i+1}(x, t), t) s_{i,i+1}^2(x, t) \tilde{\tau}_{i,i+1}(P_{i,i+1}(x, t), t)$$

and $\tilde{\xi}_{i,i+1} := -\tilde{\xi}_{i,i+1}$ for $t \in [0, T_{\text{strong}}]$, $x \in \mathbb{H}_{i,i+1}(t) \cap B_r(p(t))$, and $i \in \{1, 2, 3\}$. Here, for every $i \in \{1, 2, 3\}$ the map $\alpha_{i,i+1} : \bigcup_{0 < t < T_{\text{strong}}} I_{i,i+1}(t, t) \rightarrow \mathbb{R}$ is the solution to the following ODE on the interface $I_{i,i+1}$ with initial condition on the triple junction $p(t)$

$$\begin{cases}
\alpha_{i,i+1}(p(t), t) = \tilde{\tau}_{i,i+1}(p(t), t) \cdot \frac{d}{dt} p(t) \\
(\tilde{\tau}_{i,i+1}(x, t) \cdot \nabla) \alpha_{i,i+1}(x, t) = H_{i,i+1}^2(x, t),
\end{cases}
$$

(87)

where $H_{i,i+1} = -\Delta s_{i,i+1}$ is the scalar mean curvature with respect to the normal $\tilde{n}_{i,i+1}$ on the interface $I_{i,i+1}$. We extend $H_{i,i+1}$ to a function on $\mathbb{H}_{i,i+1}(t) \cap B_r(p(t))$ by means of

$$H_{i,i+1}(x, t) := -\Delta s_{i,i+1}(y, t)|_{y = P_{i,i+1}(x, t)}.$$

(88)

Note that as a consequence of the mean curvature flow equation (59) and the relation (66) we have on $\mathbb{H}_{i,i+1}(t) \cap B_r(p(t))$

$$-H_{i,i+1}(x, t) = \partial_t s_{i,i+1}(y, t)|_{y = P_{i,i+1}(x, t)} = \partial_t s_{i,i+1}(x, t).$$

(89)

The functions $\beta_{i,i+1} : \bigcup_{0 < t < T_{\text{strong}}} I_{i,i+1}(t, t) \rightarrow \mathbb{R}$ are defined as

$$\begin{align}
\beta_{i,i+1}(x, t) &:= -\alpha_{i,i+1}(x, t) H_{i,i+1}(x, t) - (\tilde{\tau}_{i,i+1}(x, t) \cdot \nabla) H_{i,i+1}(x, t).
\end{align}
$$

(90)

We briefly present the regularity properties of $\tilde{\xi}_{i,i+1}$. 


Lemma 22. Let the assumptions and notation of Proposition 21 be in place. For all phases \( i \in \{1, 2, 3\} \), the auxiliary vector field \( \bar{\xi}_{i,i+1} \) is of class \( C^3_t C^2_x \cap C^1_t C^0_x \) in the space-time domain \( \bigcup_{t \in [0,T_{\text{strong}})} (\mathbb{H}_{i,i+1}(t) \cap B_r(p(t))) \times \{t\} \) with bounded highest order derivatives. More precisely, we have the estimates

\[
|\nabla^k \bar{\xi}_{i,i+1}| \leq C r_c^{-k-2}, \quad k \in \{0, 1, 2\}.
\]

Here, \( r_c > 0 \) is the parameter from Definition 10 of a smoothly evolving network of interfaces.

Proof. The asserted regularity essentially follows from the regularity of the signed distance function and its time derivative. In fact, Lemma 18 applies, so that we obtain \( s \in C^0_t C^4_x \cap C^1_t C^3_x \) with \( \nabla^3 s \in L^\infty_t L^\infty_x \) and \( \nabla^3 \partial_t s \in L^\infty_t L^\infty_x \) in the space-time domain \( H_{i,i+1} := \bigcup_{t \in [0,T_{\text{strong}})} (\mathbb{H}_{i,i+1}(t) \cap B_r(p(t))) \times \{t\} \). Consequently, the projection \( P_{i,i+1} \) is of class \( C^0_t C^3_x \cap C^1_t C^2_x \) in \( H_{i,i+1} \) due to (63) and (66).

For what follows, we perform a slight abuse of notation by writing \( \bar{n}_{i,i+1}(x,t) = \bar{n}_{i,i+1}(P_{i,i+1}(x,t), t) \), \( \alpha_{i,i+1}(x,t) = \alpha_{i,i+1}(P_{i,i+1}(x,t), t) \) and so on. It follows from the previous considerations that \( \bar{n}, \bar{\tau} \in C^0_t C^3_x \cap C^1_t C^2_x \). We immediately infer from the definition (88), the regularity of the signed distance function and the regularity of the nearest point projection that \( \partial_t H_{i,i+1} \in C^0_t C^2_x \) as well as \( \nabla \partial_t H_{i,i+1} \in C^0_t C^1_x \). It follows from the alternative representation (89) that \( H_{i,i+1} \in C^0_t C^3_x \).

In the proof of Lemma 23 below, we will verify that \( H_{i,i+1} \bar{n}_{i,i+1} + \alpha_{i,i+1} \bar{\tau}_{i,i+1} = H_{j,j+1} \bar{n}_{j,j+1} + \alpha_{j,j+1} \bar{\tau}_{j,j+1} \) holds true for all \( i \neq j \) at the triple junction \( p(t) \), see the argument following (109). Multiplying this identity with the tangent \( \bar{\tau}_{i,i+1}(p(t)) \) and defining \( c_{i,j}(t) := \bar{n}_{i,i+1}(p(t)) \bar{n}_{j,j+1}(p(t)) \) and \( d_{i,j}(t) := \bar{n}_{i,i+1}(p(t)) \bar{\tau}_{j,j+1}(p(t)) \) (which are constant in time as they can be expressed in terms of the given matrix of surface tensions) yields \( \alpha_{i,i+1}(p(t)) = H_{j,j+1}(p(t)) d_{i,j} + \alpha_{j,j+1}(p(t)) c_{i,j} \) for all \( i \neq j \). Switching the roles of \( i \) and \( j \) in the previous formula entails \( (1-c_{i,j}) \alpha_{i,i+1}(p(t)) = H_{j,j+1}(p(t)) d_{i,j} + H_{i,i+1}(p(t)) d_{i,j} c_{i,j} \) for all \( i \neq j \). In particular, we may express the tangential component \( \alpha_{i,i+1} \) at the triple point solely in terms of the scalar mean curvatures and the given matrix of surface tensions since \( |c_{i,j}| \leq 1 \) for all \( i \neq j \). The latter condition is a consequence of the fact that the angle between \( \bar{n}_{i,i+1}(p(t)) \) and \( \bar{n}_{j,j+1}(p(t)) \) is in \( (0, \pi) \) as the surface tensions satisfy the triangle inequality.

We can infer from the previous discussion that the initial value of the ODE (87) is of class \( C^1_t \), whereas the right hand side in the equation for \( \alpha_{i,i+1} \) is of class \( C^0_t C^3_x \cap C^1_t C^2_x \). Since the nearest point projection enjoys the same regularity, it follows that \( \alpha_{i,i+1} \in C^0_t C^3_x \cap C^1_t C^2_x \). It is then a direct consequence of the definition (90), the regularity of the nearest point projection as well as the regularity of \( \alpha_{i,i+1} \) and \( H_{i,i+1} \) that \( \alpha_{i,i+1} \in C^0_t C^2_x \cap C^1_t C^0_x \). In particular, the asserted regularity of the auxiliary vector field \( \bar{\xi}_{i,i+1} \) in the space-time domain \( H_{i,i+1} \) now follows immediately from the ansatz (86).

Observe that it follows from Definition 10 of a smoothly evolving interface that

\[
|\nabla^{k+1} s_{\text{dist}}(x,I(t))| \leq C r_c^{-k}, \quad k \in \{1, 2, 3\},
\]

which in particular entails the following bounds for the nearest-point projections due to (84) and the (extensions of the) scalar mean curvatures due to (88)

\[
|\nabla^k P_{i,i+1}| \leq C r_c^{-k}, \quad |\nabla^k H_{i,i+1}| \leq C r_c^{-k-1}, \quad k \in \{0, 1, 2\}.
\]
It thus follows from (87) and (90) that
\begin{equation}
|\nabla^k \alpha_{i,i+1}| \leq C r_c^{-k}, \quad |\nabla^k \beta_{i,i+1}| \leq C r_c^{-k-2}, \quad k \in \{0, 1, 2\}.
\end{equation}

The asserted bounds for the derivatives of the vector fields \( \xi_{i,i+1} \) can now be inferred from (86). This concludes the proof. \( \square \)

Ultimately, the point of the ansatz (86) is to ensure (77) throughout \( B_r(p(t)) \) together with sufficiently high regularity of \( \xi_{i,j} \) at the triple junction. Moreover, the relations (87) and (90) also holding true away from the triple junction turns out to be crucial to obtain the estimates (78) and (79) on the whole space-time domain. The first step towards these goals are the following relations, which in particular yield that—after rotation \( R_{(i,j)} \)—the vector fields are compatible to second order at the triple junction:

Lemma 23. Let the assumptions and notation of Proposition 21 be in place. For each pair \( i, j \in \{1, 2, 3\} \) there exist uniquely determined rotations \( R_{(i,j)} \in SO(2) \), only depending on the admissible matrix of surface tensions \( \sigma_{i,j} \), for the given regular triod \( \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \) with
\begin{equation}
\tilde{n}_{i,i+1}(p(t), t) = R_{(i,j)} \tilde{n}_{j,j+1}(p(t), t)
\end{equation}
for all \( t \in [0, T^{\text{strong}}] \), and
\begin{align*}
R_{(i,j)} R_{(j,i)} &= \text{Id}, \\
R_{(i,i-1)} R_{(i-1,i-1)} R_{(i+1,i)} &= \text{Id}.
\end{align*}
Furthermore, the ansatz (86) satisfies the following second-order compatibility conditions at the triple junction:
\begin{align}
\tilde{\xi}_{i,i+1}(p(t), t) &= R_{(i,j)} \tilde{\xi}_{j,j+1}(p(t), t), \\
\nabla \tilde{\xi}_{i,i+1}(p(t), t) &= \nabla (R_{(i,j)} \tilde{\xi}_{j,j+1})(p(t), t), \\
\nabla^2 \tilde{\xi}_{i,i+1}(p(t), t) &= \nabla^2 (R_{(i,j)} \tilde{\xi}_{j,j+1})(p(t), t)
\end{align}
for all \( t \in [0, T^{\text{strong}}] \).

Proof. It is immediate from the ansatz (86) and (95) that the zero-order condition (98) is satisfied. The two properties (96) and (97) follow from
\begin{align}
R_{(i,j)} R_{(j,i)} \tilde{n}_{i,i+1} &= \tilde{n}_{i,i+1}, \\
R_{(i,i-1)} R_{(i-1,i-1)} R_{(i+1,i)} \tilde{n}_{i,i+1} &= \tilde{n}_{i,i+1},
\end{align}
which follow straightforwardly from iterating (95). Therefore, it is sufficient to prove the remaining two statements (99) and (100) for \( j = i + 1 \) as it then follows automatically for \( j = i - 1 \) by (96) and (97) that
\begin{align*}
\nabla (R_{(i,i-1)} \tilde{\xi}_{i-1,i})(p(t), t) &= R_{(i,i+1)} \nabla (R_{(i+1,i-1)} \tilde{\xi}_{i-1,i})(p(t), t) \\
&= R_{(i,i+1)} \nabla (\tilde{\xi}_{i+1,i-1})(p(t), t) \\
&= \nabla (\tilde{R}_{(i,i+1)} \xi_{i+1,i-1})(p(t), t) = \nabla \tilde{\xi}_{i,i+1}(p(t), t),
\end{align*}
and analogously for the second derivative.

We extend all functions defined on \( I_{i,i+1} \) to \( \mathbb{H}_{i,i+1} \cap B_r(p) \) via the projection \( P_{i,i+1} \). For ease of notation, we also fix the index \( i \) and omit all indices, superscripts,
and arguments for the rest of the proof unless specifically required otherwise. The ansatz (86) then reads
\begin{equation}
\tilde{\xi} = \tilde{n} + \alpha s \tilde{\tau} - \frac{1}{2} \alpha^2 s^2 \tilde{n} + \frac{1}{2} \beta s^2 \tilde{\tau}.
\end{equation}

Using that \( \tilde{n} \) has been extended via \( P_{i,i+1} \), as well as the representation (84) and \( \tilde{n} = \nabla s \), we then get
\begin{equation}
\nabla \tilde{n} = -H (1 + s H) \tilde{\tau} \otimes \tilde{\tau} + O(s^2),
\end{equation}
\begin{equation}
\nabla \tilde{\tau} = H (1 + s H) \tilde{n} \otimes \tilde{\tau} + O(s^2),
\end{equation}
As a result we infer from the choice (87) that
\begin{equation}
\nabla \tilde{\xi} = -H (1 + s H) \tilde{\tau} \otimes \tilde{\tau} + s(\tilde{\tau} \cdot \nabla) \alpha \tilde{\tau} \otimes \tilde{\tau} + (\alpha + \beta s) \tilde{\tau} \otimes \tilde{n}
+ \alpha H s \tilde{n} \otimes \tilde{\tau} - \alpha^2 s \tilde{n} \otimes \tilde{n} + f
\end{equation}
(87) \( \nabla \tilde{\xi} = -H \tilde{\tau} \otimes \tilde{\tau} + (\alpha + \beta s) \tilde{\tau} \otimes \tilde{n} + \alpha H s \tilde{n} \otimes \tilde{\tau} - \alpha^2 s \tilde{n} \otimes \tilde{n} + f \)
where \( f \in O(s^2) \) such that \( \nabla f \in O(|s|) \). At the triple junction this gives
\begin{equation}
\nabla \tilde{\xi}(p) = -H \tilde{\tau} \otimes \tilde{\tau} + \alpha \tilde{\tau} \otimes \tilde{n}.
\end{equation}
Carefully noting that \( \alpha(\nabla \tilde{\tau}) \otimes \tilde{n} = \alpha H \tilde{n} \otimes \tilde{n} \otimes \tilde{\tau} \) on \( \tilde{I} \), \( (\tilde{\tau} \cdot \nabla) \tilde{n} = -H \tilde{\tau} \otimes \tilde{\tau} \) by (104), as well as using (85) and in the final step the choice (87), the second derivative evaluated at the triple junction turns out to be
\begin{equation}
\nabla^2 \tilde{\xi}(p) = \left( (\tilde{\tau} \cdot \nabla) \alpha - H^2 \right) \tilde{\tau} \otimes (\tilde{\tau} \otimes \tilde{n} + \tilde{n} \otimes \tilde{\tau})
+ \tilde{\tau} \otimes \left( ((-\tilde{\tau} \cdot \nabla) H - \alpha H) \tilde{\tau} \otimes \tilde{\tau} + \beta \tilde{n} \otimes \tilde{n} \right)
- \tilde{n} \otimes (H \tilde{\tau} - \alpha \tilde{n}) \otimes (H \tilde{\tau} - \alpha \tilde{n})
\end{equation}
(87) \( \nabla \tilde{\xi} \otimes \left( ((-\tilde{\tau} \cdot \nabla) H - \alpha H) \tilde{\tau} \otimes \tilde{\tau} + \beta \tilde{n} \otimes \tilde{n} \right)
- \tilde{n} \otimes (H \tilde{\tau} - \alpha \tilde{n}) \otimes (H \tilde{\tau} - \alpha \tilde{n}) \)

Now we are in a position to prove the two compatibility conditions (99) and (100). We start with the former. By the identities (107) it is sufficient to check the two equalities
\begin{equation}
-H_{3,1} \tilde{\tau}_{1,2} + \alpha_{1,2} \tilde{n}_{1,2} = -H_{2,3} \tilde{\tau}_{2,3} + \alpha_{2,3} \tilde{n}_{2,3} = -H_{3,1} \tilde{\tau}_{3,1} + \alpha_{3,1} \tilde{n}_{3,1},
\end{equation}
as (95) together with \( J \tilde{\tau} = \tilde{n} \) (cf. (75)) imply \( \tilde{\tau}_{i,i+1}(p(t),t) = R_{i,j}(\tilde{\tau}_{j,j+1}(p(t),t)) \) from which one can then infer with (109) that
\begin{equation}
\nabla (R_{i,j}(\tilde{\tau}_{j,j+1})) = R_{i,j}(\tilde{\tau}_{j,j+1} \otimes \tilde{\tau}_{j,j+1} + \alpha_{j,j+1} \tilde{n}_{j,j+1})
= \tilde{\tau}_{i,i+1} \otimes \left( -H_{j,j+1} \tilde{\tau}_{j,j+1} + \alpha_{j,j+1} \tilde{n}_{j,j+1} \right)
= \nabla \tilde{\xi}_{i,i+1}
\end{equation}
at \((p(t),t)\) as asserted.

The identities (109) follow by using the evolution equation \( \frac{d}{dt} p(t) \tilde{n}_{i,i+1} = H_{i,i+1} \)
at the triple junction to get
\begin{equation}
\frac{d}{dt} p(t) = H_{i,i+1} \tilde{n}_{i,i+1} + \left( \tilde{\tau}_{i,i+1} \cdot \frac{d}{dt} p(t) \right) \tilde{\tau}_{i,i+1}
\end{equation}
for \( i \in \{1,2,3\} \), identifying the term in the parenthesis as \( \alpha_{i,i+1}(p(t)) \) by (87) and multiplying the above equation with the rotation matrix \( J \).
Turning to the second order compatibility condition, for $i \in \{1, 2, 3\}$ we use our precise choice (90) for the coefficient functions at the triple junction so that the second derivatives at the triple junction (108) take the form
\[
\nabla^2 \tilde{\eta}_{i,i+1}(p) = \beta_{i,i+1} \tilde{\eta}_{i,i+1} \otimes \mathbf{Id} 
\]
\[
- \tilde{\eta}_{i,i+1} \otimes (H_{i,i+1} \tilde{\eta}_{i,i+1} - \alpha_{i,i+1} \tilde{\eta}_{i,i+1}) \otimes (H_{i,i+1} \tilde{\eta}_{i,i+1} - \alpha_{i,i+1} \tilde{\eta}_{i,i+1}).
\]
Hence, because of (109), (95) and $\tilde{\eta}_{i,i+1}(p(t), t) = R_{(i,j)} (p(t), t)$ it suffices to show that $\beta_{i,i+1}(p)$ is independent of $i$ in order to prove the second order compatibility condition (100). However, this is a consequence of differentiating in time the Herring angle condition (67). Indeed, one may compute using (104), (87), and (60)
\[
0 = \frac{d}{dt} \sum_{i=1}^{3} \sigma_{i,i+1} \tilde{\eta}_{i,i+1}(p(t), t)
\]
\[
= \sum_{i=1}^{3} \sigma_{i,i+1} \nabla \tilde{\eta}_{i,i+1}(p(t), t) \frac{d}{dt} p(t) + \sum_{i=1}^{3} \sigma_{i,i+1} \partial_t \tilde{\eta}_{i,i+1}(p(t), t)
\]
\[
= - \sum_{i=1}^{3} \sigma_{i,i+1} \tilde{\eta}_{i,i+1}(p(t), t) \alpha_{i,i+1}(p(t), t) H_{i,i+1}(p(t), t)
\]
\[
- \sum_{i=1}^{3} \sigma_{i,i+1} \tilde{\eta}_{i,i+1}(p(t), t) (\tilde{\eta}_{i,i+1}(p(t), t) \cdot \nabla) H_{i,i+1}(p(t), t)
\]
from which the claim is now immediate. This concludes the proof of Lemma 23. □

Recall that apart from the family of vector fields $(\xi_{i,j})_{i \neq j}$, the notion of gradient flow calibrations also requires a suitably defined velocity field $B$. For its construction in the vicinity of a triple junction, we introduce in a first step certain auxiliary symmetric velocity fields $\tilde{B}_{(i,j)} = \tilde{B}_{(i,i)}$. To this end, we start for every $i \in \{1, 2, 3\}$ with an expansion ansatz of the form
\[
\tilde{B}_{(i,i+1)}(x, t) := H_{i,i+1}(P_{i,i+1}(x, t), t) \tilde{\eta}_{i,i+1}(P_{i,i+1}(x, t), t)
\]
\[
+ \alpha_{i,i+1}(P_{i,i+1}(x, t), t) \tilde{\eta}_{i,i+1}(P_{i,i+1}(x, t), t)
\]
\[
+ \beta_{i,i+1}(P_{i,i+1}(x, t), t) \tilde{\eta}_{i,i+1}(P_{i,i+1}(x, t), t)
\]
and $\tilde{B}_{(i+1,i)} := \tilde{B}_{(i,i+1)}$ for $t \in [0, T_{\text{strong}})$ and $x \in \overline{H_{i+1}(t) \cap B_r(p(t))}$.

**Lemma 24.** Let the assumptions and notation of Proposition 21 be in place. For all phases $i \in \{1, 2, 3\}$, the auxiliary velocity field $\tilde{B}_{(i,i+1)}$ is of class $C^0_t C^2_x$ in the space-time domain $\bigcup_{t \in [0, T_{\text{strong}})} (\overline{H_{i+1}(t) \cap B_r(p(t))} \times \{t\})$ with bounded highest order derivatives. More precisely, we have the estimates
\[
|\nabla^k \tilde{B}_{(i,i+1)}| \leq C r^{-k-2}, \quad k \in \{0, 1, 2\}.
\]
Here, $r_\varepsilon > 0$ is the parameter from Definition 10 of a smoothly evolving network of interfaces.

**Proof.** In the proof of Lemma 22 we have seen $s_{i,i+1} \in C^0_t C^4_x$ and $\tilde{\eta}_{i,i+1}, \tilde{\eta}_{i,i+1} \in C^0_t C^3_x$, as well as $H_{i,i+1} \in C^0_t C^3_x$, $\alpha_{i,i+1} \in C^0_t C^2_x$, and $\beta_{i,i+1} \in C^0_t C^2_x$. The ansatz (112) then implies the desired regularity and the bound (113) follows from the estimates (93) and (94). □
We now have to make sure that our ansatz (112) for the auxiliary velocity fields satisfies first-order compatibility conditions at the triple junction.

**Lemma 25.** Let the assumptions and notation of Proposition 21 be in place. For every $i, j \in \{1, 2, 3\}$, the ansatz (112) then satisfies

\begin{align}
\nabla \vec{B}_{(i,i+1)}(p(t), t) &= \vec{B}_{(j,j+1)}(p(t), t) = \frac{d}{dt} p(t), \\
\nabla \vec{B}_{(i,i+1)}(p(t), t) &= \nabla \vec{B}_{(j,j+1)}(p(t), t)
\end{align}

for all $t \in [0, T_{\text{strong}})$.

**Proof.** We fix again the index $i$ and omit all indices, superscripts, and arguments unless specifically required. At the triple junction, we have

\begin{equation}
\vec{B}(p(t), t) = \frac{d}{dt} p(t)
\end{equation}

by the choice of the tangential coefficient $\alpha$, see (87), and the evolution equation $\frac{d}{dt} p(t) \cdot \vec{n}_{i,i+1} = H_{i,i+1}$. This of course proves (114).

An explicit computation making use of the ansatz (112), the identities (85), (104) and (105) as well as the choices of the coefficients (87) and (90) moreover gives

\begin{equation}
\nabla \vec{B} = \left(-H^2 + (\vec{r} \cdot \nabla \alpha)\right) \vec{r} \otimes \vec{r}
+ \left((\vec{r} \cdot \nabla) H + \alpha H\right) \vec{n} \otimes \vec{r}
+ \beta \vec{r} \otimes \vec{n} + O(|s|)
= \beta (\vec{r} \otimes \vec{n} - \vec{n} \otimes \vec{r}) + O(|s|).
\end{equation}

As we have $(\vec{r} \otimes \vec{n} - \vec{n} \otimes \vec{r}) \vec{n} = \vec{r} = J \vec{n}$ and $(\vec{r} \otimes \vec{n} - \vec{n} \otimes \vec{r}) \vec{r} = -\vec{n} = J \vec{r}$ it follows that $(\vec{r} \otimes \vec{n} - \vec{n} \otimes \vec{r}) = J$, where $J$ was defined in (75). Therefore we get

\begin{equation}
\nabla \vec{B} = \beta J + O(|s|).
\end{equation}

Furthermore, by the computation in (111) and the definition (90) we know that $\beta_{i,i+1}$ is independent of $i \in \{1, 2, 3\}$ at the triple junction. This entails the first-order compatibility condition (115). \hfill \Box

In a preparatory step towards the proof of (78) and (79), we now present the corresponding estimates for the (rotated) auxiliary vector fields $\tilde{\xi}_{i,i+1}$ and the auxiliary velocity fields $\vec{B}_{(i,i+1)}$ on their respective domains of definition.

**Lemma 26.** Let the assumptions and notation of Proposition 21 be in place. Then there exists a constant $C > 0$, depending only on the given regular triod $\vec{X} = (\vec{X}_1, \vec{X}_2, \vec{X}_3)$, such that for every $i, j \in \{1, 2, 3\}$, it holds throughout the space-time domain $\mathcal{U}_{j,j+1} := \bigcup_{t \in [0, T_{\text{strong}})} (\mathbb{H}_{j,j+1}(t) \cap B_r(p(t))) \times \{t\}$ that

\begin{align}
\left| \partial_t R_{(i,j)} \tilde{\xi}_{j,j+1} + \left(\vec{B}_{(j,j+1)} \cdot \nabla\right) R_{(i,j)} \tilde{\xi}_{j,j+1} + \left(\nabla \vec{B}_{(j,j+1)}\right)^T R_{(i,j)} \tilde{\xi}_{j,j+1} \right| &\leq C \text{dist}(., \vec{I}_{j,j+1}), \\
\left| \nabla \cdot R_{(i,j)} \tilde{\xi}_{j,j+1} + \vec{B}_{(j,j+1)} \cdot R_{(i,j)} \tilde{\xi}_{j,j+1} \right| &\leq C \text{dist}^2(., \vec{I}_{j,j+1}).
\end{align}

Moreover, we have for every $j \in \{1, 2, 3\}$

\begin{align}
\left| \partial_t |\tilde{\xi}_{j,j+1}|^2 \right| &\leq C \text{dist}^3(., \vec{I}_{j,j+1}), \\
\left| \partial_t |\tilde{\xi}_{j,j+1}|^2 + \left(\vec{B}_{(j,j+1)} \cdot \nabla\right) |\tilde{\xi}_{j,j+1}|^2 \right| &\leq C \text{dist}^2(., \vec{I}_{j,j+1}).
\end{align}
throughout $U_{j,j+1}$. We also have for all pairs $i, j \in \{1, 2, 3\}$ with $i \neq j$ that

\begin{align}
R_{(i,j)} \tilde{\xi}_{j,j+1} - R_{(i,j-1)} \tilde{\xi}_{j-1,j} & \leq C \text{ dist}^2(\cdot, p(t)), \\
\nabla R_{(i,j)} \tilde{\xi}_{j,j+1} - \nabla R_{(i,j-1)} \tilde{\xi}_{j-1,j} & \leq C \text{ dist}(\cdot, p(t)),
\end{align}

in the intersection $U_{i+1} \cap U_{j,j+1}$ Finally, for all $i, j \in \{1, 2, 3\}$ with $i \neq j$ it holds

\begin{align}
|B_{(i,i+1)} - B_{(j,j+1)}| & \leq C \text{ dist}^2(\cdot, p(t)) \\
|\nabla B_{(i,i+1)} - \nabla B_{(j,j+1)}| & \leq C \text{ dist}(\cdot, p(t))
\end{align}

Proof. The coercivity estimates (121) and (122) follow from a straightforward computation using the definition (86). Indeed, just note that because of (86) it holds

\begin{align}
|\tilde{\xi}_{i,j}|^2 & = \left(1 - \frac{1}{2} \alpha_{i,j}^2 s_{i,j}^2\right)^2 + \left(\alpha_{i,j} s_{i,j} + \frac{1}{2} \beta_{i,j} s_{i,j}^2\right)^2 \\
& = 1 + \alpha_{i,j} \beta_{i,j} s_{i,j}^3 + \frac{1}{4} (\alpha_{i,j}^4 + \beta_{i,j}^2) s_{i,j}^4
\end{align}

from which both (121) and (122) follow immediately.

To prove the estimate (119), let $i \in \{1, 2, 3\}$ be fixed. For what follows, let us slightly abuse notation by writing $\tilde{n}_{i+1}(x, t) = \tilde{n}_{i,j+1}(P_{i,j+1}(x, t), t) = \alpha_{i,j+1}(P_{i,j+1}(x, t), t)$ and so on. Let us also omit all indices, superscripts and arguments unless specifically required. We first consider the case $j = i$. Differentiating our ansatz (86) for $\tilde{\xi}$ in time using equation (59), i.e., $\partial_t s = -H$ on $\tilde{I}$, and its extension (89)

$$
\partial_t \tilde{\xi} = \nabla \partial_t s + \alpha(\partial_t s) \tilde{\xi} + O(|s|) = -\nabla H - \alpha H \tilde{\xi} + O(|s|).
$$

Moreover, we may compute by means of our ansatz (112) for the auxiliary velocity field $\tilde{B}$ and the explicit computation (106) of $\nabla \tilde{\xi}$ that

$$
(\tilde{B} \cdot \nabla) \tilde{\xi} = O(|s|),
$$

and, resulting from our ansatz (86) for $\tilde{\xi}$ and the explicit computation (117) of $\nabla \tilde{B}$ that

$$
(\nabla \tilde{B})^T \tilde{\xi} = ((\tilde{\xi} \cdot \nabla) H + \alpha H) \tilde{\xi} + O(|s|).
$$

As we have $\nabla H = ((\tilde{\xi} \cdot \nabla) H) \tilde{\xi}$, the last three identities imply the desired bound (119) on $H_{i+1} \cap B_r(p)$.

The validity of (119) for $j \neq i$ essentially boils down to a commutator estimate. In the following, let us abbreviate $\tilde{\xi} = \tilde{\xi}_{i,j+1}$, $R = R_{(i,j)}$ as well as $\tilde{B} = \tilde{B}_{(i,j+1)}$. Then, by the definition (86) of $\tilde{\xi}$ and exploiting the first case $j = i$ we deduce

$$
\partial_t R \tilde{\xi} + \tilde{B} \cdot \nabla R \tilde{\xi} + (\nabla \tilde{B})^T R \tilde{\xi}
$$

$$
= R(\partial_t \tilde{\xi} + \tilde{B} \cdot \nabla \tilde{\xi} + (\nabla \tilde{B})^T \tilde{\xi}) + ((\nabla \tilde{B})^T R - R(\nabla \tilde{B})^T) \tilde{\xi}
$$

$$
= O(\text{dist}(\cdot, \tilde{I}_{j,j+1})) + [(\nabla \tilde{B})^T, R] \tilde{\xi}.
$$

By our computation (117) we have $\nabla \tilde{B} = \beta J + O(|s|)$. Using the fact that $[J^T, R] = 0$ on account of both matrices being rotations in the plane we get

$$
[(\nabla \tilde{B})^T, R] = \beta [J^T, R] + O(|s_{j,j+1}|) = O(|s_{j,j+1}|).
$$

Therefore, we obtain (119) also for all $j \neq i$ on $H_{j,j+1} \cap B_r(p)$. 

In the next step we check the validity of (120). We again make use of the abbreviations \( \tilde{\xi} = \tilde{\xi}_{i,j+1} \), \( R = R_{i,j} \) and \( B = B_{j,j+1} \). Due to the computation (106) of \( \nabla \tilde{\xi} \) we may compute on the one side
\[
\nabla \cdot R \tilde{\xi} = -H(R \tilde{\tau} \cdot \tilde{\tau}) + (\alpha + \beta s)(R \tilde{\tau} \cdot \tilde{n}) + \alpha H s(R \tilde{n} \cdot \tilde{\tau}) - \alpha^2 s(R \tilde{n} \cdot \tilde{\tau}) + O(\text{dist}^2(\cdot, I_{j,j+1})).
\]
On the other side, making use of the definitions (86) and (112) of \( \tilde{\xi} \) we obtain
\[
\tilde{B} \cdot R \tilde{\xi} = H \tilde{n} \cdot R \tilde{n} + (\alpha + \beta s)(\tilde{\tau} \cdot R \tilde{n}) + \alpha H s(\tilde{n} \cdot R \tilde{\tau}) + \alpha^2 s(\tilde{\tau} \cdot R \tilde{\tau}) + O(\text{dist}^2(\cdot, \tilde{I}_{j,j+1})).
\]
Taking the last two identities together therefore yields the estimate (120) by the equations (recall that \( J \tilde{\tau} = \tilde{n} \)) and by the definition (75) of \( J \) that \( J^T = J^{-1} = -J \)
\[
R \tilde{\tau} \cdot \tilde{\tau} = R J^{-1} \tilde{n} \cdot \tilde{\tau} = R \tilde{n} \cdot J \tilde{\tau} = R \tilde{n} \cdot \tilde{n},
\]
\[
R \tilde{\tau} \cdot \tilde{n} = R J^{-1} \tilde{n} \cdot \tilde{n} = R \tilde{n} \cdot J \tilde{n} = -R \tilde{n} \cdot \tilde{n}.
\]
We proceed with the verification of the bounds (123) and (124). However, these are straightforward consequences of the compatibility conditions (98), (99) and (100) which allow by adding zero that we can insert in \( |R_{i,j} \tilde{\xi}_{j,j+1} - R_{i,j-1} \tilde{\xi}_{j-1,j}| \) the second-order Taylor expansions based at \( p(t) \) of both \( R_{i,j} \tilde{\xi}_{j,j+1} \) and \( R_{i,j-1} \tilde{\xi}_{j-1,j} \). Together with (91) the bound (123) is now immediate. One can argue similarly for the estimate (124).

It remains to check (125) and (126). These bounds follow from the compatibility conditions (114) and (115), adding and subtracting the second order Taylor expansions based at the triple junction \( p(t) \) of both \( B_{i,i+1} \) and \( B_{j,j+1} \) as well as (113). This concludes the proof of Lemma 26. □

6.2. Global construction by interpolation. Let \( \tilde{\chi} = (\tilde{\chi}_1, \tilde{\chi}_2, \tilde{\chi}_3) \) be a regular triod with triple junction \( p(t) \) moving by mean curvature in the sense of Definition 20 on the space-time domain \( \bigcup_{[0,T_{\text{strong}}]} B_r(p(t)) \times \{ t \} \). As we discussed in the previous subsection, the auxiliary vector fields \( \tilde{\xi}_{i,i+1} \) and the auxiliary velocity field \( \tilde{B}_{i,i+1} \) serve as the definition of the vector fields \( \xi_{i,i+1} \) and the velocity field \( B \) on the wedge \( W_{i,i+1} \), see Figure 10b for the partition of the neighborhood of the triple junction.

The next step is to extend \( \xi_{i,i+1} \) and \( B \) to the entirety of the space-time domain. As we want (68) to hold throughout the ball \( B_r(p(t)) \) we are essentially forced to set \( \xi_{i,i+1} = R_{i,j} \tilde{\xi}_{j,j+1} \) for all \( i, j \in \{ 1, 2, 3 \} \) wherever the latter is defined, and where \( R_{i,j} \) is given in Lemma 23. In order to resolve the problem of the domains of definitions, i.e., the half-spaces \( \tilde{H}_i \) overlapping, we resort to an interpolation procedure on the interpolation wedges \( W_i \) (see again Figure 10b) both for \( \xi_{i,i+1} \) and for \( B \). We similarly deal with the issue of combining the velocity fields \( \tilde{B}_{i,i+1} \) into a single field. To this end, we first define suitable interpolation functions which move and rotate with the triod.

Lemma 27. Let the assumptions and notation of Proposition 21 be in place. Recall in particular from Definition 20 that we defined \( \tilde{\tau}_{i,j}(p(t), t) := J^{-1} \tilde{n}_{i,j}(p(t), t) \) for \( i, j \in \{ 1, 2, 3 \} \) with \( i \neq j \), where \( J \) was defined in (75), and then have fixed by means
of Lemma 19 wedges \( W_i(t) \) and \( W_{i,i+1}(t) \) according to this choice of tangent vectors at the triple junction, see also Figure 10b.

Then there exists a constant \( C > 0 \), depending only on the given regular triod \( \bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \), and interpolation functions

\[
\lambda_i : \bigcup_{t \in [0,T_{strong})} (B_r(p(t)) \cap W_i(t)) \setminus \{p(t)\} \times \{t\} \rightarrow [0,1]
\]

for every \( i \in \{1, 2, 3\} \) which satisfy the following properties:

i) It holds that

\[
\lambda_i(x,t) = 0 \quad \text{for} \quad x \in (\partial W_i(t) \cap \partial W_{i,i+1}(t)) \setminus \{p(t)\}, \tag{128}
\]

\[
\lambda_i(x,t) = 1 \quad \text{for} \quad x \in (\partial W_i(t) \cap \partial W_{i-1,i}(t)) \setminus \{p(t)\}. \tag{129}
\]

ii) We have the estimates

\[
|\partial_t \lambda_i(x,t)| + |\nabla \lambda_i(x,t)| \leq C \frac{1}{|x-p(t)|}, \tag{130}
\]

\[
|\partial_t \nabla \lambda_i(x,t)| + |\nabla^2 \lambda_i(x,t)| \leq C \frac{1}{|x-p(t)|^2} \tag{131}
\]

for all \( t \in [0,T_{strong}) \) and all \( x \in (B_r(p(t)) \cap W_i(t)) \setminus \{p(t)\} \). Furthermore, it holds

\[
\nabla \lambda_i(x,t) = 0, \quad \partial_t \lambda_i(x,t) = 0, \tag{132}
\]

\[
\nabla^2 \lambda_i(x,t) = 0, \quad \nabla \partial_t \lambda_i(x,t) = 0 \tag{133}
\]

for all \( t \in [0,T_{strong}) \) and all \( x \in (B_r(p(t)) \cap W_i(t)) \setminus \{p(t)\} \).

iii) We have a bound on the advective derivative

\[
|\partial_t \lambda_i(x,t) + \left( \frac{d}{dt} p(t) \cdot \nabla \right) \lambda_i(x,t)| \leq C \tag{134}
\]

for all \( t \in [0,T) \) and all \( x \in (B_r(p(t)) \cap W_i(t)) \setminus \{p(t)\} \).

Proof. Due to (69), the interpolation wedge \( W_i(t) \) is the interior of the conical hull spanned by two unit vectors, say \( v_i(t) \) and \( w_i(t) \), whereas \( W_{i,i+1}(t) \) is the interior of the conical hull spanned by unit vectors \( v_i(t) \) and \( w_{i+1}(t) \) due to (70). In particular, we can represent \( \partial W_i(t) \cap \partial W_{i,i+1}(t) = \{\gamma v_i(t) : \gamma \geq 0\} \) and \( \partial W_i(t) \cap \partial W_{i-1,i}(t) = \{\gamma w_i(t) : \gamma \geq 0\} \). As the vectors \( v_i(t) \) and \( w_i(t) \) can be expressed as a (fixed-time) linear combination of the unit-normals \( \bar{n}_{i,j}(p(t),t) \) at the triple junction, we have the bounds \( \frac{d}{dt}v_i(t), \frac{d}{dt}w_i(t) \in C^0([0,T_{strong})). \)

Let \( \theta_i \in (0, \frac{\pi}{2}) \) denote the opening angle of the interpolation wedge \( W_i \), i.e., we have \( \cos(\theta_i) = v_i(t) \cdot w_i(t) \). Note that \( \theta_i \) is time-independent and fully determined by the admissible matrix of surface tensions \( (\sigma_{i,j})_{i,j=1,2,3} \) for the given regular triod \( \bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3) \). Let \( \bar{\lambda} : \mathbb{R} \rightarrow [0,1] \) be any smooth function such that \( \bar{\lambda} \equiv 0 \) on \( (-\infty, \frac{1}{2}) \) and \( \bar{\lambda} \equiv 1 \) on \( [\frac{1}{2}, \infty) \). We define

\[
\lambda_i(x,t) := \bar{\lambda} \left( \frac{1 - v_i(t) \cdot \frac{z-p(t)}{|z-p(t)|}}{1 - \cos \theta_i} \right). \]

Then the properties (128)–(133) are immediate consequences of the definitions and the fact that \( \frac{d}{dt}v_i(t), \frac{d}{dt}w_i(t) \in C^0([0,T_{strong})) \) as observed above. It remains
to check the bound (134) on the advective derivative. To this end, we abbreviate 
\[ \lambda_i(x, t) = \bar{\lambda}_i(v_i(t) \cdot \frac{x - p(t)}{|x - p(t)|}) \] 
and simply compute
\[
\partial_t \lambda_i(x, t) = -\bar{\lambda}_i'(v_i(t) \cdot \frac{x - p(t)}{|x - p(t)|}) \cdot \left( \Id - \frac{x - p(t)}{|x - p(t)|} \otimes \frac{x - p(t)}{|x - p(t)|} \right) \frac{d}{dt} p(t) + \bar{\lambda}_i' \frac{x - p(t)}{|x - p(t)|} \cdot \frac{d}{dt} v_i(t)
\]
where \( \bar{\lambda}_i' \) is evaluated at \( v_i(t) \cdot \frac{x - p(t)}{|x - p(t)|} \). From this, the last remaining claim (134) immediately follows due to the bound \( \frac{d}{dt} v_i(t) \in L_t^\infty \).

Equipped with these interpolating functions we are finally in the position to prove the main result of this section.

Proof of Proposition 21. Step 1: Interpolation of the vector fields. We define (not yet normalized) extensions of the normal vector fields \( \tilde{n}_{i,j} \) on \( \bigcup_{t \in [0, T_{\text{strong}}]} B_r(p(t)) \times \{t\} \) as follows:

\[
\xi_{i,i+1}(x, t) := \begin{cases}
R_{i,j}(\tilde{\xi}_{j,j+1}(x, t)) & \text{if } x \in W_{j,j+1}(t), \\
(1 - \lambda_j(x, t))R_{i,j}\tilde{\xi}_{j,j+1}(x, t) + \lambda_j(x, t)R_{i,j-1}\tilde{\xi}_{j-1,j}(x, t) & \text{if } x \in W_j(t),
\end{cases}
\]

and \( \bar{\xi}_{i+1,i} := -\xi_{i,i+1} \) for \( i \in \{1, 2, 3\} \). The velocity field is given by

\[
B(x, t) := \begin{cases}
\tilde{B}_{j,j+1}(x, t) & \text{if } x \in W_{j,j+1}(t), \\
(1 - \lambda_j(x, t))\tilde{B}_{j,j+1}(x, t) + \lambda_j(x, t)\tilde{B}_{j-1,j}(x, t) & \text{if } x \in W_j(t).
\end{cases}
\]

In the subsequent steps of the proof, we first establish all required properties in terms of the vector fields \((\tilde{\xi}_{i,j}, B)\). Only in the penultimate step we will choose the radius \( \tilde{r} = \tilde{r}(\tilde{\chi}) \leq r \) and define unit-length vector fields \( \xi_{i,j} \) by normalization of the vector fields \( \tilde{\xi}_{i,j} \) defined in (135) above. The last step is then devoted to verify the required properties for the normalized vector fields \( \xi_{i,j} \).

Step 2: Regularity of \( \xi_{i,j} \) and \( B \), the estimates (81) and (82), and properties i)–iii). We first remark that the above definitions make sense due to the inclusions \( W_{i,i+1}(t) \subset H_{i,i+1}(t) \) and \( W_i(t) \subset H_{i,i+1}(t) \cap H_{i-1,i}(t) \) of Lemma 19. Indeed, these inclusions are precisely what is needed so that the building blocks \( \tilde{\xi}_{i,i+1} \) and \( \tilde{B}_{i,i+1} \) are only evaluated on their domains of definition.

By the compatibility condition (98) in Lemma 23 for the auxiliary vector fields \( \tilde{\xi}_{j,j+1} \) at the triple junction, as well as the conditions (128) and (129) from Lemma 27 for the interpolation functions, the vector fields \( \xi_{i,j} \) are continuous. Similarly, their first and second derivatives are continuous across the boundaries of the interpolation wedges \( \bigcup_{t \in [0, T_{\text{strong}}]} (B_r(p(t)) \cap \partial W_i(t) \setminus \{p(t)\}) \times \{t\} \) by the properties (132) and (133) of the interpolation functions.

Moreover, all spatial derivatives up to second order are uniformly bounded in \( B_r(p(t)) \setminus \{p(t)\} \) with the asserted estimate given by (81). Indeed, in the wedges \( W_{j,j+1} \) containing the interfaces this follows from the estimates (91) and the definition (135). On the closure of the interpolation wedges \( W_j \), we first compute using
the definition (135)
\[
\nabla \tilde{\xi}_{i,i+1} = (1-\lambda_j)\nabla R_{(i,j)} \tilde{\xi}_{j,j+1} + \lambda_j \nabla R_{(i,j-1)} \tilde{\xi}_{j-1,j} \\
- (R_{(i,j)} \tilde{\xi}_{j,j+1} - R_{(i,j-1)} \tilde{\xi}_{j-1,j}) \nabla \lambda_j,
\]
\[
\nabla^2 \tilde{\xi}_{i,i+1} = (1-\lambda_j)\nabla^2 R_{(i,j)} \tilde{\xi}_{j,j+1} + \lambda_j \nabla^2 R_{(i,j-1)} \tilde{\xi}_{j-1,j} \\
- (\nabla R_{(i,j)} \tilde{\xi}_{j,j+1} - \nabla R_{(i,j-1)} \tilde{\xi}_{j-1,j}) \nabla \lambda_j \\
- (R_{(i,j)} \tilde{\xi}_{j,j+1} - R_{(i,j-1)} \tilde{\xi}_{j-1,j}) \nabla^2 \lambda_j.
\]

Now, the bound (81) follows from the controlled blow-up (130) and (131) of the interpolation functions and the estimates (91), (123) as well as (124) for the auxiliary vector fields \(\tilde{\xi}_{j,j+1}\). In total, this proves \(\tilde{\xi}_{i,j} \in L^\infty_t W^2_x\) together with (81).

The other property \(\tilde{\xi}_{i,j} \in W^4_t L^\infty_x\) follows similarly making use of Lemma 22, Lemma 23 and Lemma 27 and the computation
\[
\partial_t \tilde{\xi}_{i,i+1} = (1-\lambda_j)\partial_t R_{(i,j)} \tilde{\xi}_{j,j+1} + \lambda_j \partial_t R_{(i,j-1)} \tilde{\xi}_{j-1,j} \\
- (R_{(i,j)} \tilde{\xi}_{j,j+1} - R_{(i,j-1)} \tilde{\xi}_{j-1,j}) \partial_t \lambda_j \quad \text{in } W_j.
\]

We proceed with the regularity of the velocity field \(B\). First, by the compatibility condition (114) in Lemma 25 for the auxiliary velocity fields \(\tilde{B}_{j,j+1}\) at the triple junction, as well as the conditions (128) and (129) from Lemma 27 for the interpolation functions, the velocity field \(B\) is continuous. The asserted bound (82) is a consequence of the definition (136), the estimates (113) and (125) for the auxiliary velocity fields, the controlled blow-up (130) of the interpolation functions as well as the computation
\[
\nabla B = (1-\lambda_j)\nabla \tilde{B}_{j,j+1} + \lambda_j \nabla \tilde{B}_{j-1,j} + (\tilde{B}_{j-1,j} - \tilde{B}_{j,j+1}) \nabla \lambda_j \quad \text{in } W_j.
\]

This proves \(B \in L^\infty_t W^2_x\) and the estimate (82).

For every \(i \in \{1, 2, 3\}\), we obtain \(\tilde{\xi}_{i,i+1}(x,t) = \tilde{\xi}_{i,i+1}(x,t) = \tilde{n}_{i,i+1}(x,t)\) for all \(t \in [0, \text{strong}]\) and all \(x \in I_{i,i+1}(t)\) from the inclusion \(I_{i,i+1} \subset W_{i,i+1}(t)\cup\{p(t)\}\) and the ansatz (86), taking care of property i); obviously except for the normalization condition away from the interfaces. The second property \(\tilde{\xi}_{i,j} = -\tilde{\xi}_{j,i}\) for \(i, j \in \{1, 2, 3\}\) with \(i \neq j\) holds by definition. For every \(j \in \{1, 2, 3\}\) we moreover have
\[
\sigma_{1,2}\tilde{\xi}_{1,2} + \sigma_{2,3}\tilde{\xi}_{2,3} + \sigma_{3,1}\tilde{\xi}_{3,1} = (\sigma_{1,2}R_{(1,j)} + \sigma_{2,3}R_{(2,j)} + \sigma_{3,1}R_{(3,j)}) \tilde{\xi}_{j,j+1} = 0
\]
on \(W_{j,j+1}(t)\) by the defining property (95) of the rotations \(R_{(i,j)}\). A similar argument ensures validity of (77) on the interpolation wedges \(W_j(t)\).

Step 3: Proof of the estimate (78) for \((\xi_{i,j}, B)\). By the skew-symmetry \(\tilde{\xi}_{i,j} = -\tilde{\xi}_{j,i}\), we only have to check the remaining properties for \(j = i + 1\). Let \(i \in \{1, 2, 3\}\) be fixed. First, let us remark that the validity of (78) for the vector field \(\tilde{\xi}_{i,i+1}\) on the wedges \(W_{j,j+1}\) follows from the estimate (119) and the definitions (135) and (136). Hence, it remains to prove the bound (78) for the vector field \(\tilde{\xi}_{i,i+1}\) on each interpolation wedge \(W_j, j \in \{1, 2, 3\}\).

To this end, let us fix \(j \in \{1, 2, 3\}\) and recall that \(|x - p(t)| \leq C \text{ dist}(. \tilde{I}_{j,j+1})\) on \(W_j\). For the sake of readability, let us introduce the abbreviations, \(\lambda = \lambda_j, R = R_{(i,j)}, R' = R_{(i,j-1)}, \xi = \tilde{\xi}_{j,j+1}, \xi' = \tilde{\xi}_{j-1,j}, B = \tilde{B}_{j,j+1}\) and \(B' = \tilde{B}_{j-1,j}\).

Using the product rule and the definition (135) of \(\tilde{\xi}_{i,i+1}\) on the interpolation wedge
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\[ W_j, \]
\[ (\partial_t + (B \cdot \nabla) + (\nabla B)^T) \xi_{i,i+1} = (1 - \lambda) (\partial_t + (B \cdot \nabla) + (\nabla B)^T) R\tilde{\xi} + \lambda (\partial_t + (B \cdot \nabla) + (\nabla B)^T) R'\tilde{\xi} + (\partial_t \lambda + (B \cdot \nabla) \lambda) (R'\tilde{\xi} - R\tilde{\xi}). \]

Observe that by the bounds (130) on \( \lambda \) and the second-order compatibility (123), the last right-hand side term is of the desired order \( O(|x - p(t)|) \). We want to manipulate the first two right-hand side terms to make the advection-equations (119) appear. To this end, we write \( B = \tilde{B} + \lambda(\tilde{B}' - \tilde{B}) \) and obtain

\[ (\partial_t + (B \cdot \nabla) + (\nabla B)^T) R\tilde{\xi} = (\partial_t + (\tilde{B} \cdot \nabla) + (\nabla \tilde{B})^T) R\tilde{\xi} - \left( \lambda(\tilde{B}' - \tilde{B}) \cdot \nabla \right) R\tilde{\xi} - \left( (\tilde{B}' - \tilde{B}) \cdot R\tilde{\xi} \right) \nabla \lambda \]

\[ - \lambda \left( \nabla (\tilde{B}' - \tilde{B}) \right)^T R\tilde{\xi} \]

Using the compatibility conditions (123)–(126) alongside with the uniform bounds (91), (113), and (130) shows that the two three right-hand side terms are negligible. Indeed, for example (125) and (91) imply \( |(\lambda(\tilde{B}' - \tilde{B}) \cdot \nabla) R\tilde{\xi}| \leq |\tilde{B}' - \tilde{B}| |\nabla \tilde{\xi}| \leq C|x - p(t)|^2 \). Along the same lines, one shows that the other two terms are \( O(|x - p(t)|) \).

Arguing similarly via \( B = \tilde{B}' + (1 - \lambda)(\tilde{B} - \tilde{B}') \) for the differential operator acting on \( R'\tilde{\xi}' \), we obtain

\[ (\partial_t + (B \cdot \nabla) + (\nabla B)^T) \xi_{i,i+1} = (1 - \lambda) (\partial_t + (\tilde{B} \cdot \nabla) + (\nabla \tilde{B})^T) R\tilde{\xi} + \lambda \left( \partial_t + (\tilde{B}' \cdot \nabla) + (\nabla \tilde{B}')^T \right) R'\tilde{\xi}' + O(|x - p(t)|) \]

throughout \( W_j \). By (119), the two transport equations on the right-hand side are satisfied to first order, which proves (78).

Step 4: Proof of the estimate (79) for \( (\xi_{i,j}, B) \). Let \( i \in \{1, 2, 3\} \) be fixed. Note that because of (135) as well as (120), it only remains to prove (79) for the vector field \( \xi_{i,i+1} \) in the interpolation wedges \( W_j, j \in \{1, 2, 3\} \). We again fix \( j \in \{1, 2, 3\} \) and abbreviate for the sake of readability \( \lambda = \lambda_j, R = R_{(i,j)}, R' = R_{(i,j-1)}, \)

\( \tilde{\xi} = \tilde{\xi}_{j,j+1}, \tilde{\xi}' = \tilde{\xi}_{j-1,j}, \tilde{B} = \tilde{B}_{(j+1,j)} \) as well as \( \tilde{B}' = \tilde{B}_{(j-1,j)} \).

Now, we proceed similarly as in the proof of (78). More precisely, making use of the definition (135) we get

\[ \nabla \cdot \tilde{\xi}_{i,i+1} = (R'\tilde{\xi}' - R\tilde{\xi}) \cdot \nabla + (1 - \lambda) \nabla \cdot R\tilde{\xi} + \lambda \nabla \cdot R'\tilde{\xi}'. \]

By the controlled blow-up (130) of the interpolation functions, the estimate (123) and the approximate mean curvature flow equation (120) it then follows

\[ \nabla \cdot \tilde{\xi}_{i,i+1} = -(1 - \lambda) \tilde{B} \cdot R\tilde{\xi} - \lambda \tilde{B}' \cdot R'\tilde{\xi}' + O(|x - p(t)|). \]

Finally, the estimates (124) and (125) in conjunction with definitions (135) and (136) imply the desired bound

\[ \nabla \cdot \tilde{\xi}_{i,i+1} = -B \cdot \tilde{\xi}_{i,i+1} + O(|x - p(t)|). \]

Step 5: Proof of the estimate

\[ 1 - |\tilde{\xi}_{i,j}|^2 \leq C \text{dist}^2(x, I_{i,j}(t)) \quad \text{for all } (x, t) \in \bigcup_{0,T_{\text{strong}}} B_r(p(t)) \times \{t\}. \]
Let \( i \in \{1, 2, 3\} \) be fixed. The validity of (138) for the vector field \( \tilde{\xi}_{i,i+1} \) in the wedge \( W_{i,i+1} \) follows from (121). Since on the wedges \( W_{j,j+1}, j \neq i \), the vector field \( \tilde{\xi}_{i,i+1} \) is obtained from the vector field \( \tilde{\xi}_{j,j+1} \) by a mere rotation, see (135), we also immediately obtain (138) for all \( x \in W_{j,j+1}(t), j \neq i \). Finally, note that on the interpolation wedges, since each of the two building blocks \( \tilde{\xi}_{j,j+1} \) and \( \tilde{\xi}_{j-1,j} \) satisfy (138), also their convex combination \( \tilde{\xi}_{i+1} \) does.

**Step 6:** Choice of \( \tilde{r} = \tilde{r}(\tilde{\chi}) \leq r \) and definition of normalized vector fields \( \xi_{i,j} \).

By the definition (135) of the vector fields \( \xi_{i,j} \) we have \( |\xi_{i,j}(\cdot,t)| = 1 \) on the union of interfaces \( I(t) \) for all \( t \in [0, T_{\text{strong}}] \). Due to their continuity and the regularity assumptions on the moving triod in Definition 20, we may choose a radius \( \tilde{r} = \tilde{r}(\tilde{\chi}) \leq r \) such that \( B_r(p(t)) \subset \{ |\tilde{\xi}_{i,j}(\cdot,t)| > \frac{1}{2} \} \) for all \( t \in [0, T_{\text{strong}}] \) and all \( i, j \in \{1, 2, 3\} \) with \( i \neq j \). We then define

\[
(139) \quad \xi_{i,j}(x,t) := \frac{\tilde{\xi}_{i,j}(x,t)}{|\tilde{\xi}_{i,j}(x,t)|} \quad \text{for all } (x,t) \in \bigcup_{t \in [0,T_{\text{strong}}]} B_r(p(t)) \times \{t\}
\]

all \( i, j \in \{1, 2, 3\} \) with \( i \neq j \). It remains to verify the asserted properties in terms of the vector fields \( (\xi_{i,j}, B) \) on the restricted space-time domain with radius \( \tilde{r} \).

**Step 7:** Conclusion. Since \( \xi_{i,j}(x,t) = \tilde{\xi}_{i,j}(x,t) \) for all \( t \in [0, T_{\text{strong}}] \) and all \( x \in I_{i,j}(t) \), property (i) is an immediate consequence of definition (139). Obviously, the skew-symmetry relation in property (ii) carries over from \( \tilde{\xi}_{i,j} \) to \( \xi_{i,j} \). Validity of the Herring angle condition (77) in terms of the vector fields \( \xi_{i,j} \) also follows immediately from their definition (139) and the fact that the vector fields \( \tilde{\xi}_{i,j} \) already satisfy (77). Indeed, just recall that the vector fields \( \xi_{1,2}, \xi_{2,3}, \xi_{3,1} \) can be obtained from each of the others by a rotation. The required regularity estimate (81) is satisfied by the choice of the radius \( \tilde{r} \) in the previous step, the definition (139) of \( \xi_{i,j} \) and the fact that the vector fields \( \tilde{\xi}_{i,j} \) are already subject to such an estimate (as established in the second step of this proof). It therefore remains to argue that the estimates (78) and (79) hold true.

Using the product rule, we may compute on \( B_r(p(t)) \) by means of the choice of \( \tilde{r} \) in the previous step

\[
\left( \partial_t + (B \cdot \nabla) + (\nabla B)^T \right) \frac{\tilde{\xi}_{i,j}}{|\tilde{\xi}_{i,j}|} = \frac{1}{|\tilde{\xi}_{i,j}|} \left( \partial_t + (B \cdot \nabla) + (\nabla B)^T \right) \tilde{\xi}_{i,j} - \frac{1}{2|\tilde{\xi}_{i,j}|} \tilde{\xi}_{i,j} \left( \partial_t + (B \cdot \nabla) \right) |\tilde{\xi}_{i,j}|^2
\]

By Step 3, the first right-hand side term is of the order \( O(|x - p(t)|) \). The same rate also holds for the second right-hand side term simply by multiplication with \( \tilde{\xi}_{i,j} \). This proves the estimate (78). We now turn to the proof of (79). Here, we compute on \( B_r(p(t)) \) by means of the choice of \( \tilde{r} \) in the previous step

\[
\nabla \cdot \frac{\tilde{\xi}_{i,j}}{|\tilde{\xi}_{i,j}|} = \frac{\nabla \cdot \tilde{\xi}_{i,j}}{|\tilde{\xi}_{i,j}|} - \frac{(\tilde{\xi}_{i,j} \cdot \nabla)|\tilde{\xi}_{i,j}|^2}{2|\tilde{\xi}_{i,j}|^3}.
\]

It is immediate from the identity (127) and the definition (135) to estimate the second term. Using the approximate mean curvature flow equation (137) for the first term then yields

\[
\nabla \cdot \frac{\tilde{\xi}_{i,j}}{|\tilde{\xi}_{i,j}|} = -B \frac{\tilde{\xi}_{i,j}}{|\tilde{\xi}_{i,j}|} + O(\text{dist}(x, I(t))),
\]
Figure 11. If the angle between two tangent vectors is less than 90°, we trisect it to obtain the desired interpolation wedge, see for example $W_2$. Otherwise, we take the corresponding intersection of the half-spaces, as is done for $W_1$ and $W_3$. The wedges $W_{1,2}$, $W_{2,3}$ and $W_{3,1}$ lie inbetween.

which be definition (139) of $\xi_{i,j}$ is nothing else than (79). This concludes the proof of Proposition 21.

Finally, we provide the elementary-geometric proof for the existence of wedges with the desired properties.

Proof of Lemma 19. By rotation symmetry, we may without loss of generality assume for example $\tau_{1,2} = e_1$. Then the other two vectors $\tau_{2,3}$ and $\tau_{3,1}$ are determined up to a permutation by the requirements

$$\sigma_{1,2}\tau_{1,2} + \sigma_{2,3}\tau_{2,3} + \sigma_{3,1}\tau_{3,1} = 0,$$

$$|\tau_{2,3}|^2 = |\tau_{3,1}|^2 = 1.$$  

Therefore it is sufficient to prove the statement for a single choice of unit vectors $\tau_{1,2}$, $\tau_{2,3}$ and $\tau_{3,1}$. We may furthermore choose $p = 0$.

Using the balance of forces condition (140) we see that there exist $\theta_i \in (0, \pi)$ such that $\cos(\theta_i) = \tau_{i,i+1} \cdot \tau_{i-1,i}$ for $i = 1, 2, 3$. If $\theta_i > \frac{\pi}{3}$ we may simply define $v_i, w_i \in \mathbb{S}^1$ such that $W_i := \{ \gamma_1 v_i + \gamma_2 w_i : \gamma_1, \gamma_2 \in (0, \infty) \} = H_{i,i+1} \cap H_{i-1,i}$. Otherwise, we choose them such that $W_i := \{ \gamma_1 v_i + \gamma_2 w_i : \gamma_1, \gamma_2 \in (0, \infty) \}$ is the middle third of the wedge $\{ \gamma_1 \tau_{i,i+1} + \gamma_2 \tau_{i-1,i} : \gamma_1, \gamma_2 \in (0, \infty) \}$. The desired properties then easily follow.

7. Gradient flow calibrations for a regular network

The aim of this section is to prove Proposition 4: Given a strong solution to multiphase mean curvature flow (in the sense of an evolving network of smooth curves meeting at triple junctions), we construct a gradient flow calibration by gluing the local constructions from the previous two chapters.

In order to define these vector fields, we will distinguish between the two distinct topological features being present in the network of interfaces of a strong solution,
namely triple junctions and smooth two-phase interfaces. The global definitions will be obtained by gluing together suitable local definitions $\xi_{i,j}^k$ and $B^k$ of these vector fields for each topological feature $T_k$ by means of a suitably defined partition of unity. The definition of such a partition of unity and its localization properties are the content of Section 7.1.

We then proceed with the global definitions of the vector fields $\xi_{i,j}$ and $B$ in Section 7.2. The main building blocks are given by the already mentioned local vector fields $\xi_{i,j}^k$ and $B^k$. The construction of these vector fields together with a derivation of their main properties was carried out in Section 5 for the model problem of a smoothly evolving two-phase interface moving by mean curvature; or Section 6 for the model problem of a smoothly evolving triod moving by mean curvature. The main focus of Section 7.2 then consists of the question of the proper definition of the vector fields $\xi_{i,j}^k$ in the case where at least either phase $i$ or $j$ are not locally present at the selected topological feature $T_k$.

Being equipped with the global definitions of the vector fields $\xi_{i,j}$ and $B$, we proceed with the discussion of the compatibility between the local constructions for the different topological features, see Section 7.3. We use these bounds in Section 7.4 to obtain the desired bounds on the time evolution of $\xi_{i,j}$ resp. $|\xi_{i,j}|^2$ as well as the validity of $\nabla \cdot \xi_{i,j} = -B \cdot \xi_{i,j}$ up to an error being controlled by our relative entropy functional.

7.1. Localization of topological features. Let $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_p)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 11 on some time interval $[0, T_{\text{strong}})$. In particular, the family $\bar{\chi}(t)$ is a smoothly evolving regular partition and the family $(\bar{I}_{i,j}(t))_{i \neq j}$ is a smoothly evolving regular network of interfaces in the sense of Definition 10.

We decompose the network of interfaces of the strong solution according to its topological features, i.e., into smooth two-phase interfaces on the one hand and triple junctions on the other hand. Suppose that the strong solution has $K$ of such topological features $T_k$, $1 \leq k \leq K$. We then split $\{1, \ldots, K\} = K^\text{2ph} \cup K^\text{3j}$ with the convention that $K^\text{3j} = \{1, \ldots, |K^\text{3j}|\}$, and where $K^\text{3j}$ has the interpretation of enumerating the triple junctions $p_1, \ldots, p_{|K^\text{3j}|}$ present in the strong solution whereas $K^\text{2ph}$ enumerates the connected components (in space-time) of the smooth two-phase interfaces $\bar{I}_{i,j}$. We next define $T_k := \{p_k\}$, if $k \in K^\text{3j}$, as well as $T_k \subset \bar{I}_{i,j}$ for the corresponding space-time connected component $k \in K^\text{2ph}$ of a two-phase interface $\bar{I}_{i,j}$. We say that the $i$-th phase of the strong solution is present at the topological feature $T_k$ if $\partial\{\bar{\chi}_i = 1\} \cap T_k \neq \emptyset$. Otherwise, we say that the phase is absent at $T_k$.

We now introduce a partition of unity $(\eta, \eta_1, \ldots, \eta_K)$, where each $\eta_k$ localizes in a neighborhood of the corresponding topological feature $T_k$, as follows:

Lemma 28. Let $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_p)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 11, whose network of interfaces decomposes into $K$ topological features $T_k$. Let $r_c$ be the associated constant from the definition of a smoothly evolving network of interfaces, see Definition 10. Then, for every $k \in \{1, \ldots, K\}$ there exists a function

$$\eta_k \in L^\infty([0, T_{\text{strong}}); W^{2, \infty}(\mathbb{R}^2; [0, 1])) \cap W^{1, \infty}([0, T_{\text{strong}}); W^{1, \infty}(\mathbb{R}^2; [0, 1]))$$

such that the family $(\eta_1, \ldots, \eta_K)$ is a partition of unity in the following sense:
i) Let \( \eta := 1 - \sum_{k=1}^K \eta_k \). Then \( \eta(x,t) \in [0,1] \) for all \( (x,t) \in \mathbb{R}^2 \times [0,T_{\text{strong}}] \). Moreover, on the evolving network of interfaces \( \bigcup_{t\in[0,T_{\text{strong}}]} \bigcup_{j\neq j'} \bar{I}_{i,j}(t) \times \{t\} \) it already holds \( \sum_{k=1}^K \eta_k = 1 \).

Furthermore, there exists a constant \( 0 < \tilde{\tau}_c \leq \frac{\bar{T}}{2\tau} \), such that the family \( (\eta_1, \ldots, \eta_K) \) is subject to the following localization properties:

ii) For all \( k \in K^{2\text{ph}} \) we have

\[
\text{supp} \eta_k(\cdot,t) \subset \bar{T}_k(t) + B_{\tilde{r}}(0).
\]

Moreover, \( T_k \) is a regular interface evolving by mean curvature in the space-time domain \( \bigcup_{t\in[0,T_{\text{strong}}]} (\bar{T}_k(t) + B_{2\tilde{r}}(0)) \times \{t\} \) in the precise sense of Definition 16.

iii) For all \( k \in K^{3\text{j}} \) it holds

\[
\text{supp} \eta_k(\cdot,t) \subset B_{\tilde{r}}(\bar{T}_k(t)).
\]

Let \( l_1, l_2, l_3 \in K^{2\text{ph}} \) be the interfaces present at the triple junction \( T_k \). The three interfaces \( (T_{i_1}, T_{i_2}, T_{i_3}) \) form a regular triad with triple junction \( T_k \) evolving by mean curvature in the space-time domain \( \bigcup_{t\in[0,T_{\text{strong}}]} B_{2\tilde{r}}(\bar{T}_k(t)) \times \{t\} \) in the precise sense of Definition 20 (with \( r = 2\tilde{\tau}_c \)).

iv) Let \( k,k' \in K^{3\text{j}} \) be two distinct triple junctions. Then

\[
\text{supp} \eta_k(\cdot,t) \cap \text{supp} \eta_{k'}(\cdot,t) \subset B_{2\tilde{r}}(\bar{T}_k(t)) \cap B_{2\tilde{r}}(\bar{T}_{k'}(t)) = \emptyset.
\]

v) Let \( k \in K^{3\text{j}} \) be a triple junction and let \( l \in K^{2\text{ph}} \) be a two-phase interface. Then \( \text{supp} \eta_k \cap \text{supp} \eta_l \neq \emptyset \) if and only if the interface \( T_l \) has an endpoint at the triple junction \( T_k \). In this case, it holds (with the notation of Definition 20 in place) assuming that \( T_l \subset \bar{I}_{i,j} \)

\[
\text{supp} \eta_k(\cdot,t) \cap \text{supp} \eta_l(\cdot,t) \subset B_{\tilde{r}}(\bar{T}_k(t)) \cap B_{\tilde{r}}(\bar{T}_l(t)) \cap (W_{i,j}(t) \cup W_i(t) \cup W_j(t)).
\]

vi) Let \( l,m \in K^{2\text{ph}} \) be two distinct two-phase interfaces. Then it holds \( \text{supp} \eta_l \cap \text{supp} \eta_m \neq \emptyset \) if and only if both interfaces have an endpoint at the same triple junction \( T_k \), \( k \in K^{3\text{j}} \). In this case, it holds (with the notation of Definition 20 in place)

\[
\text{supp} \eta_l(\cdot,t) \cap \text{supp} \eta_m(\cdot,t) \subset \text{supp} \eta_k(\cdot,t) \cap W_i(t),
\]

where we assume that \( T_l \subset \bar{I}_{i,j} \) and \( T_m \subset \bar{I}_{i,p} \).

Finally, the following coercivity property holds true for the family of localization functions \( (\eta_1, \ldots, \eta_K) \):

vii) Let \( m \in K^{2\text{ph}} \) be a two-phase interface. Then

\[
|\text{dist}(\cdot, T_m(t))|^2 \leq C(1-\eta_m(\cdot,t)) \quad \text{on} \quad \mathbb{R}^2 \setminus \bigcup_{k \in K^{3\text{j}}} B_{\tilde{r}}(\bar{T}_k(t)).
\]

Let \( k \in K^{3\text{j}} \) be such that the two-phase interface \( T_m \subset \bar{I}_{i,j} \) has an endpoint at the triple junction \( T_k \). Then

\[
|\text{dist}(\cdot, T_m(t))|^2 \leq C(1-\eta_m(\cdot,t)-\eta_k(\cdot,t)) \quad \text{on} \quad B_{\tilde{r}}(\bar{T}_k(t)) \cap W_{i,j}(t).
\]

Let \( T_l \subset \bar{I}_{i,p}, l \neq m, \) be another two-phase interface with an endpoint at the triple junction \( T_k \). Then

\[
|\text{dist}(\cdot, T_k(t))|^2 \leq C(1-\eta_m(\cdot,t)-\eta_k(\cdot,t)-\eta_l(\cdot,t)) \quad \text{on} \quad B_{\tilde{r}}(\bar{T}_k(t)) \cap W_i(t).
\]
Proof. For the definition of a partition of unity \((\eta, \eta_1, \ldots, \eta_K)\) with the required localization and coercivity properties we proceed as follows. Let \(\theta\) be a smooth and even cutoff function with \(\theta(r) = 1\) for \(|r| \leq \frac{1}{2}\) and \(\theta \equiv 0\) for \(|r| \geq 1\). Let \(\zeta^{2ph}, \zeta^{2ph}_j : \mathbb{R} \to [0, \infty)\) be another two smooth cutoff functions defined by
\[
\zeta^{2ph}(r) = (1 - r^2)\theta(r)
\]
as well as
\[
\zeta^{2ph}_j(r) = \zeta^{2ph}(\min\{r, 0\}),
\]
where \(\zeta^{2ph}_j\) is a cut-off for \(\eta_j\) in \(K^j\). We first define \(\eta_k\) for triple junctions \(k \in K^j\). To this end, let us assume that the phases \(i, j, p \in \{1, \ldots, P\}\) are present at the triple junction \(T_k\), and the corresponding interfaces are denoted by \(T_{i_k} \subset I_{i,j}, T_{j_k} \subset I_{j,p}\), and \(T_{p_k} \subset I_{p,i}\). Let \(0 < \tilde{r}_c \leq \frac{r_c}{2}\) and \(0 < c_1, c_2 \leq 1\) yet to be determined constants. By choosing \(\tilde{r}_c \leq \frac{r_c}{2}\) sufficiently small, we may assume that the interfaces \((T_{i_k}, T_{j_k}, T_{p_k})\) are indeed a triod with triple junction \(T_k\) evolving by mean curvature in the space-time domain \(\bigcup_{t \in [0, T_{\text{strong}}]} B_{2\tilde{r}_c}(T_k(t)) \times \{t\}\) in the precise sense of Definition 20.

We want to define \(\eta_k\) such that \(\text{supp} \eta_k \subset B_{2\tilde{r}_c}(T_k)\). Recall from Definition 20 that \(B_{2\tilde{r}_c}(T_k)\) decomposes into six wedges. Three of them, namely the wedges \(W_{i,j}, W_{j,p}\), resp. \(W_{p,i}\), contain the interfaces \(T_{i_k}, T_{j_k}\) resp. \(T_{p_k}\). The other three are interpolation wedges denoted by \(W_i\), \(W_j\) resp. \(W_p\). Finally, recall that we identified the set \(W_i \cup W_j \cup W_{i,j}\) as the intersection of certain halfspaces \(\mathbb{H}_i \cap \mathbb{H}_j\). We then define an auxiliary cutoff
\[
\zeta^{2ph}_{i,j}(x, t) := \zeta^{2ph}_{j}(\frac{\text{sdist}(x, \mathbb{H}_j(t))}{c_2\tilde{r}_c}), \quad (x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}),
\]
and analogously the cutoffs \(\zeta^{2ph}_{j,p}\) resp. \(\zeta^{2ph}_{p,i}\). Recall that the convention is that \(\text{sdist}(\cdot, \mathbb{H}_i)\) is negative in the half-space \(\mathbb{H}_i\) and positive outside. Moreover, let us introduce the auxiliary cutoff function
\[
\zeta^{2ph}_{i,j}(x, t) := \zeta^{2ph}(\frac{\text{sdist}(x, I_{i,j}(t))}{c_1\tilde{r}_c}), \quad (x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}),
\]
with an analogous definition for the cutoffs \(\zeta^{2ph}_{j,p}\) resp. \(\zeta^{2ph}_{p,i}\). Hence, \(\zeta^{2ph}_{i,j}\) is a cut-off for the wedge \(\mathbb{H}_i \cap \mathbb{H}_j\). By now, we have everything in place to move on with the definition of \(\eta_k\). We begin by setting
\[
\eta_k(x, t) := \zeta^{2ph}_{i,j}(x, t)\zeta^{2ph}_{i,j}(x, t), \quad t \in [0, T_{\text{strong}}), \quad x \in B_{2\tilde{r}_c}(T_k(t)) \cap W_{i,j}(t),
\]
and analogously on the other wedges \(W_{j,p}\) and \(W_{p,i}\). To define \(\eta_k\) on the interpolation wedges, we make use of the interpolation parameter as built in Lemma 27. To clarify the direction of interpolation, i.e., on which boundary of the interpolation wedge the corresponding interpolation function is equal to one or zero, we make use of the following notational convention. For a triple junction \(T_k\) with the phases \(i, j, p \in \{1, \ldots, P\}\) being present let us define two interpolation functions for each interpolation wedge. For the interpolation wedge \(W_i\), say, we denote by \(\lambda_i^j\) the interpolation function as built in Lemma 27 and which is equal to one on \((\partial W_{i,j} \cap \partial W_i) \setminus T_k\) and which vanishes on \((\partial W_{p,i} \cap \partial W_i) \setminus T_k\). We also define \(\lambda^p_i := 1 - \lambda_i^j\) which interpolates on \(W_i\) in the opposite direction. Analogously, one
introduces the interpolation functions on the other interpolation wedges. Finally, we may then define
\[
\eta_k(x, t) := \lambda^1_t(x, t)\zeta_{i,j}^{3\!j}(x, t)\zeta_{i,j}^{2\!ph}(x, t) + \lambda^p_t(x, t)\zeta_{p,i}^{3\!j}(x, t)\zeta_{p,i}^{2\!ph}(x, t),
\]
with an analogous definition on the other two interpolation wedges \(W_j\) and \(W_p\). Outside of \(B_{\bar{r}_c}(T_k)\), we of course simply set \(\eta_k = 0\). We refer to Figure 12 for an illustration of the construction.

Choosing \(c_1\) and \(c_2\) small enough (depending only on the surface tension matrix \(\sigma \in \mathbb{R}^{P \times P}\)), we indeed obtain (142). The required regularity for \(\eta_k\) follows from the regularity of the strong solution, see Definition 11, the fact that \(\bar{r}_c \leq \frac{\epsilon}{2}\), the property (142) as well as the controlled blowup of the derivatives of the interpolation parameter at the triple junction (130) and (131). Finally, it is just a matter of possibly choosing \(\bar{r}_c \leq \frac{\epsilon}{2}\) even smaller, such that also (143) holds true.

Step 2: \(m \in K^{2\!ph}\). We next define \(\eta_m\) for a smooth two-phase interface \(T_m \subset \bar{I}_{i,j}\). That \(T_m\) is a smoothly evolving interface by mean curvature in the space-time tubular neighborhood \( \bigcup_{t \in [0,T_{\text{strong}})} (T_m(t)+B_{\bar{r}_c}(T_k(t))) \times \{t\} \) in the precise sense of Definition 16 is immediate from the definition of a strong solution, see Definition 11.

If the interface \(T_m \subset \bar{I}_{i,j}\) has no endpoint at a triple junction, we simply set
\[
\eta_m(x, t) := \zeta_{i,j}^{2\!ph}(x, t), \quad (x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}),
\]
where the cutoff \(\zeta_{i,j}^{2\!ph}\) was already defined in (152). If the interface \(T_m \subset \bar{I}_{i,j}\) has an endpoint at exactly one triple junction, say \(T_k\) with phases \(i, j\) and \(p\) being present, we proceed as follows. Away from the triple junction \(T_k\), we still define
\[
\eta_m(x, t) := \zeta_{i,j}^{2\!ph}(x, t), \quad t \in [0, T_{\text{strong}}), \quad x \in \mathbb{R}^2 \setminus B_{\bar{r}_c}(T_k(t)).
\]
Near the triple junction, i.e., on \(B_{\bar{r}_c}(T_k)\), we aim to modify the definition such that \(\eta_m\) is supported within the set \(W_i \cup W_j \cup W_{i,j}\). To this end, we define
\[
\eta_m(x, t) := (1-\zeta_{i,j}^{3\!j}(x, t))\zeta_{i,j}^{2\!ph}(x, t), \quad x \in B_{\bar{r}_c}(T_k(t)) \cap W_{i,j}(t),
\]
where the auxiliary cutoff \(\zeta_{i,j}^{3\!j}\) was introduced in (151). On the interpolation wedges \(W_i\) resp. \(W_j\), we again make use of the interpolation parameter and set
\[
\eta_m(x, t) := \lambda^j_t(x, t)(1-\zeta_{i,j}^{3\!j}(x, t))\zeta_{i,j}^{2\!ph}(x, t), \quad x \in B_{\bar{r}_c}(T_k(t)) \cap W_i(t)
\]
\[
\eta_m(x, t) := \lambda^j_t(x, t)(1-\zeta_{i,j}^{3\!j}(x, t))\zeta_{i,j}^{2\!ph}(x, t), \quad x \in B_{\bar{r}_c}(T_k(t)) \cap W_j(t).
\]
We refer again to Figure 12 for an illustration of the construction. Finally, if the endpoints of the interface \(T_m \subset \bar{I}_{i,j}\) are located at two distinct triple junctions, we simply repeat the preceding procedure at the second triple junction.

The required regularity for \(\eta_m\) follows for the same reasons as for the localization functions \(\eta_k\) around triple junctions \(k \in K^{3\!j}\). Property (141) is a consequence of choosing \(c_1\) and \(c_2\) small enough (depending only on the surface tension matrix \(\sigma \in \mathbb{R}^{P \times P}\)). The statements preceding the relations (144) resp. (145) follow by possibly reducing \(\bar{r}_c \leq \frac{\epsilon}{2}\) even further. Finally, the precise localization properties (144) and (145) are then immediate consequences of our definitions (157) and (158).

Step 3: Partition of Unity. Next, we validate the partition of unity property for the family of localization functions \((\eta_1, \ldots, \eta_K)\). First of all, it is clear from our definitions (153)–(158) that \(\eta_k \in [0, 1]\) for each topological feature \(k \in \{1, \ldots, K\}\). Together with the already established localization properties (141)–(145) and the
Figure 12. The different functions $\eta_k$ in the partition of unity at a single triple junction: The function $\eta_k$ for a single two-phase interface $T_k = \bar{I}_{i,j}$ ending at the triple junction (top left), the function $\eta_k$ for the triple junction $T_k = \{p_k\}$ itself (top right), the sum of all two-phase localization functions at a triple junction (bottom left), and the sum of all localization functions $\sum_k \eta_k = 1 - \eta$ (bottom right). Observe that the sum of all localization functions equals 1 on the interfaces in the strong solution, but decays quadratically away from them.

Step 4: Coercivity estimates. The bound (146) is a consequence of the definition of the quadratic cutoff (149) as well as the definitions (155) resp. (156). The estimate (147) follows from (157), (153) and again the definition (149). Finally, the bound (148) can be inferred from (154), (158), the definition (149) as well as the estimate

$$|\text{dist}(x, T_k)|^2 \leq C\lambda_i^2(x)|\text{dist}(x, \bar{I}_{i,j})|^2 + C\lambda_p^2(x)|\text{dist}(x, \bar{I}_{p,i})|^2,$$

which holds on an interpolation wedge $W_i$ at a triple junction $T_k$ with phases $i, j$ and $p$ being present. This concludes the proof of Lemma 28.

At this point, we have everything in place to move on with the construction of the global vector fields $\xi_{i,j}$ and the global velocity field $B$. Once we defined the latter, we may also prove that the localization functions $(\eta_1, \ldots, \eta_K)$ are subject to an advection equation up to controlled error terms.

7.2. Global construction of the calibration. This section is devoted to the construction of the vector fields $\xi_{i,j}$, extending the unit-normal vector fields of the network of interfaces of a strong solution, as well as the velocity field $B$ along which all constructions are approximately transported during the time evolution.
of the network of interfaces. Given a strong solution \( \bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_p) \) for multiphase mean curvature flow in the sense of Definition 11, recall that we decomposed the associated network of interfaces into \( K \) topological features \( T_1, \ldots, T_K \). More precisely, we defined \( \{1, \ldots, K\} =: \mathcal{K}^{2\text{ph}} \cup \mathcal{K}^{3\text{j}} \), with \( \mathcal{K}^{3\text{j}} \) enumerating the triple junctions \( p_k \) present in the strong solution, whereas \( \mathcal{K}^{2\text{ph}} \) enumerates the space-time connected component of the smooth two-phase interfaces \( \bar{I}_{i,j} \). We then put \( T_k := \{p_k\} \), if \( k \in \mathcal{K}^{3\text{j}} \), or \( T_k \subset \bar{I}_{i,j} \) for the corresponding space-time connected component \( k \in \mathcal{K}^{2\text{ph}} \) of a two-phase interface \( \bar{I}_{i,j} \).

The idea for the construction of the vector fields \( \xi_{i,j} \) is as follows. First, we provide the definition of local vector fields \( \xi_{i,j}^k \) in the support of the associated localization function \( \eta_k \) for each topological feature \( T_k \). If both the phases \( i \) and \( j \) are present at \( T_k \), we define \( \xi_{i,j}^k \) by means of the local constructions provided in Section 5 for the model problem of a smooth manifold resp. Section 6 for the model problem of a triod. This, however, leaves open the question of the definition of the vector fields \( \xi_{i,j} \) for phases absent at \( T_k \). It turns out that the question of the proper definition of the vector fields for phases being absent at \( T_k \) is associated with the conditions of global stability between the phases. In particular, we would like to ensure that at a given topological feature \( T_k \), our relative entropy functional provides a length control for those interfaces not being present at \( T_k \). For this purpose, we rely on the stability conditions provided by (9a) and (9b) for an admissible matrix of surface tension in the sense of Definition 6.

**Lemma 29.** Let \( \sigma \in \mathbb{R}^{P \times P} \) be an admissible matrix of surface tensions in the sense of Definition 6, and let \( \bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_p) \) be a strong solution for multiphase mean curvature flow in the sense of Definition 11. Let \( (\eta, \eta_1, \ldots, \eta_K) \) be a partition of unity as constructed in Lemma 28. Then for every choice of distinct phases \( i, j \in \{1, \ldots, P\} \) and every topological feature \( k \in \{1, \ldots, K\} \) there exist vector fields

\[
\xi_{i,j}^k : \bigcup_{t \in [0,T_{\text{strong}}]} \text{supp} \eta_k(\cdot, t) \times \{t\} \to \mathbb{R}^2,
\]

\[
\xi_i^k : \bigcup_{t \in [0,T_{\text{strong}}]} \text{supp} \eta_k(\cdot, t) \times \{t\} \to \mathbb{R}^2,
\]

where the former are defined by means of the local vector fields constructed in Section 5 resp. Section 6, satisfying the following properties:

i) It holds \( \xi_{i,j}^k, \xi_i^k \in L_t^\infty W_{x,t}^{2,\infty} \cap W_t^{1,\infty} C_\infty^0 \).

ii) We have \( \xi_{i,j}^k = -\xi_{j,i}^k \), \( |\xi_{i,j}^k| \leq 1 \) as well as

\[
(159) \quad \sigma_{i,j,k} \xi_{i,j}^k = \xi_i^k - \xi_j^k.
\]

iii) If the phases \( i \) and \( j \) are both present at the topological feature \( T_k \), then \( \xi_{i,j}^k \) coincides with \( \bar{n}_{i,j} \) on \( \text{supp} \eta_k \cap \bar{I}_{i,j} \).

iv) There exists a constant \( c = c(\sigma) \in (0,1) \) with the property, that if either phase \( i \) or \( j \) is absent at the topological feature \( T_k \), then

\[
(160) \quad |\xi_{i,j}^k| \leq c < 1.
\]

**Proof.** The proof consists of two parts distinguishing between the topological features present in the network of interfaces of the strong solution.

**Step 1: Two-phase interface.** Let \( i, j \in \{1, \ldots, P\} \) with \( i \neq j \) be fixed, and let \( k \in \mathcal{K}^{2\text{ph}} \). We first assume that both the phases \( i \) and \( j \) are present at the two-phase
interface $\mathcal{T}_k$, i.e., $\mathcal{T}_k \subset \bar{I}_{i,j}$. We then define the vector field $\xi_{i,j}^k$ in the support of $\eta_k$ as in Lemma 17. Note that by the localization property $\eta_{ij}$ from Lemma 28 of the partition of unity $(\eta_1, \ldots, \eta_K)$, we are indeed in the setting of Section 5. In particular, $\xi_{i,j}^k = -\xi_{j,i}^k$ and $\xi_{i,j}^k$ coincides with $\bar{u}_{i,j}$ on $\text{supp}\, \eta_k \cap \bar{I}_{i,j}$. Furthermore, let us define the vector fields $\xi_i^k$ resp. $\xi_j^k$ as $\xi_i^k := \sigma_{i,j} \xi_{i,j}^k$, resp. as $\xi_j^k := \sigma_{j,i} \xi_{j,i}^k$. This ensures that the desired formula (159) is indeed satisfied.

Now, let us assume that either phase $i$ or $j$ is absent at the two-phase interface $\mathcal{T}_k$. To be specific, we fix $m, l \in \{1, \ldots, P\}$ with $m \neq l$ such that $\mathcal{T}_k \subset I_{m,l}$. The idea now is to first define vector fields $\xi_i^k$ resp. $\xi_j^k$ and then define $\xi_{i,j}^k$ by means of (159) such that (160) holds true. To this end, we rely on the strict triangle inequality (9a) for the given matrix of surface tensions. Let us define

$$\xi_i^k := \frac{1}{2} (\sigma_{i,l} \xi_{m,l}^k + \sigma_{m,i} \xi_{l,m}^k),$$

and analogously for $\xi_j^k$. Note that this is indeed well-defined since we already provided a definition for the vector fields $\xi_{i,j}^k = -\xi_{j,i}^k$ associated to the phases present at $\mathcal{T}_k$. This definition is also consistent with the previous one because of the convention $\sigma_{i,i} = 0$. We may then compute plugging in the definitions

$$\xi_{i,j}^k := \frac{\xi_i^k - \xi_j^k}{\sigma_{i,j}} = \frac{1}{2} \left( \frac{\sigma_{i,l} - \sigma_{i,j}}{\sigma_{i,j}} \xi_{m,l}^k + \frac{\sigma_{m,i} - \sigma_{m,j}}{\sigma_{i,j}} \xi_{l,m}^k \right).$$

Hence, (160) holds true because we have $\left| \frac{\sigma_{i,l} - \sigma_{i,j}}{\sigma_{i,j}} \right| < 1$ and $\left| \frac{\sigma_{m,i} - \sigma_{m,j}}{\sigma_{i,j}} \right| < 1$ due to the strict triangle inequality (9a).

Step 2: Triple junctions. Let now $k \in \mathcal{K}^{3j}$. Again, we first assume that both the phases $i$ and $j$ are present at the triple junction $\mathcal{T}_k$, i.e., a connected component of the interface $\bar{I}_{i,j}$ has an endpoint at $\mathcal{T}_k$. Note that by the localization property $\eta_{ij}$ from Lemma 28 of the partition of unity $(\eta_1, \ldots, \eta_K)$, we are in the setting of Section 6. We then define the vector field $\xi_{i,j}^k$ in the support of $\eta_k$ by the corresponding vector field from Proposition 21. In particular, $\xi_{i,j}^k = -\xi_{j,i}^k$ and $\xi_{i,j}^k$ coincides with $\bar{u}_{i,j}$ on $\text{supp}\, \eta_k \cap \bar{I}_{i,j}$.

Assume now that $p \in \{1, \ldots, P\}$ is the third phase being present at the triple junction $\mathcal{T}_k$. By construction, we have $\sigma_{i,j} \xi_{i,j}^k + \sigma_{j,p} \xi_{j,p}^k + \sigma_{p,i} \xi_{p,i}^k = 0$ on the support of $\eta_k$. Defining then the vector field $\xi_i^k$ as $\xi_i^k := \frac{1}{3} (\sigma_{i,j} \xi_{i,j}^k + \sigma_{i,p} \xi_{i,p}^k)$, and analogously for $\xi_j^k$ resp. $\xi_p^k$, we indeed obtain (159).

Finally, let us assume that either phase $i$ or $j$ is absent at the triple junction $\mathcal{T}_k$. To be specific, we assume that the phases $l, m, n \in \{1, \ldots, P\}$ are present at $\mathcal{T}_k$. We then employ the stability condition for triple junctions (9b) to construct the vector fields $\xi_{i,j}^k$. This condition allows us to interpret the matrix of surface tensions $\sigma = (\sigma_{ij})_{i,j=1,\ldots,P}$ as pairwise (Euclidean) distances of points in $\mathbb{R}^P$. (Recall that $\sigma_{i,i} = 0$ by convention.) That means we can find points $q_1, \ldots, q_P \in \mathbb{R}^{P-1}$ such that

$$\sigma_{i,j} = \|q_i - q_j\| \text{ for all } i, j = 1, \ldots, P,$$

see [43, Theorem 1]. Note that with $Q$, cf. (9b), also its 3-minor corresponding to the indices $l, m, n$ has full rank, so the triangle $qlm$ is non-degenerate and spans a plane $E = E_{l,m,n}$ in $\mathbb{R}^{P-1}$. We momentarily define the orthogonal projection onto $E$ by $\pi = \pi_{l,m,n}$. 
Now we first construct vectors which define the missing vector fields consistently at the triple point itself. Then their extension to the whole patch around the triple junction will be straightforward. At the triple point, we set
\[
\xi_i^k := R^k \pi_{l,m,n} q_i, \quad \text{for all } i = 1, \ldots, P,
\]
where the orthogonal matrix \( R^k \in O(2) \) is still to be determined (to ensure consistency with the already provided definitions for the phases present at \( T_k \)). In particular, \( \xi_i^k = R^k q_i \) for the phases \( l, m, n \). Then we define \( \xi_{i,j}^k \) in terms of the vectors \( \xi_i^k \) as before in the two-phase case via formula (159).

Note that after projecting an out-of-plane vector onto \( E \) via \( \pi = \pi_{l,m,n} \) it becomes short. Hence, we have
\[
\begin{aligned}
|\xi_{i,j}^k| = 1 & \quad \text{if } i, j \in \{l, m, n\} \\
|\xi_{i,j}^k| \leq c < 1 & \quad \text{otherwise}.
\end{aligned}
\]
Indeed, if \( i, j \in \{l, m, n\} \), both \( q_i \) and \( q_j \) lie in \( E \) and hence the projection leaves them invariant. We handle the second case in two subcases:

If exactly one of the two indices, say, \( j \) corresponds to a phase being present at \( T_k \), then \( \pi q_j = q_j \). Note that with \( Q \), also each 4-minor has full rank, so that \( q_i \) cannot lie in the plane \( E \), hence \( \pi q_i \neq q_i \), which implies the strict inequality in this subcase. If both \( i \) and \( j \) correspond to phases being absent at \( T_k \), we only need to argue that \( q_i - q_j \) does not lie in the linear space \( E - q_i \). Otherwise the 5-minor of \( Q \) corresponding to the indices \( l, m, n, i, j \) were not of full rank, a contradiction. So indeed, (161) holds.

Thanks to the compatibility condition
\[
\sigma_{l,m} \xi_{l,m}^k + \sigma_{m,n} \xi_{m,n}^k + \sigma_{n,l} \xi_{n,l}^k = 0
\]
we can find \( R^k \) such that \( \xi_{l,m}^k = \tilde{n}_{l,m} \); \( \xi_{m,n}^k = \tilde{n}_{m,n} \) as well as \( \xi_{n,l}^k = \tilde{n}_{n,l} \) at the triple junction. In particular, the construction is consistent with the previous constructions for the phases present at \( T_k \). Note also that \( R^k \) is indeed orthogonal since \( |\xi_{l,m}^k| = |\xi_{m,n}^k| = |\xi_{n,l}^k| = 1 \).

To extend the definition of \( \xi_{i,j}^k \) (resp. \( \xi_i^k \)) to the whole support of \( \eta_k \), we first express \( \xi_{i,j}^k \) (resp. \( \xi_i^k \)) at the triple junction as a fixed linear combination of the vector fields of the phases present at the triple junction: there exist time-independent coefficients \( \lambda, \lambda', \lambda'' \) such that
\[
\xi_i^k = \lambda \xi_l^k + \lambda' \xi_m^k + \lambda'' \xi_n^k.
\]
Figure 14. Plot of the length of the vector field $\xi_{i,j}$. Observe that the length is 1 on the interface $\bar{I}_{i,j}$ of the strong solution, but decays quadratically away from it to a value strictly smaller than 1, even on the other interfaces $\bar{I}_{i,p}$ and $\bar{I}_{j,p}$. As a consequence, the integral $\int_{I_{i,j}} 1 - n_{i,j} \cdot \xi_{i,j} \, dH^1$ provides an upper bound for the interface error functional $c \int_{I_{i,j}} \min\{\text{dist}^2(x, \bar{I}_{i,j}), 1\} \, dH^1$.

Since the vector fields on the right hand side are already defined on the whole support of $\eta_k$, we may extend the definition of the vector fields $\xi^k_i$ for the phases absent at the triple junction $T_k$ by means of this fixed linear combination, and then, as already said, define the missing vector fields $\xi^k_{i,j}$ via (159). In particular, the desired properties (159) as well as (160) continue to hold true. This concludes the proof of Lemma 29.

Now we may define the global extensions $\xi_{i,j} = -\xi_{j,i}$ of the unit normal vector fields between the phases $i$ and $j$ in the strong solution by gluing the local definitions by means of the partition of unity $(\eta, \eta_1, \ldots, \eta_K)$ from Lemma 28.

Construction 30. Let $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 11 on some time interval $[0, T_{\text{strong}})$. Let $(\eta, \eta_1, \ldots, \eta_K)$ be a partition of unity as constructed in Lemma 28, and let $\xi_{i,j}$ be the family of vector fields provided by Construction 29. For all $i, j \in \{1, \ldots, P\}$ with $i \neq j$, we then define

$$\xi_{i,j}(x, t) := \sum_{k=1}^{K} \eta_k(x, t) \xi^k_{i,j}(x, t)$$

(162)

for all $x \in \mathbb{R}^2$ and all $t \in [0, T_{\text{strong}})$.

We proceed with the derivation of the coercivity condition provided by the length of the vector fields $\xi_{i,j}$ as defined by Construction 30. For an illustration we refer to Figure 14.

Lemma 31. Let $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 11 on some time interval $[0, T_{\text{strong}})$. Let $(\eta, \eta_1, \ldots, \eta_K)$ be a partition of unity as constructed in Lemma 28, and let $\xi_{i,j}$ be the family of vector fields provided by Construction 30. Then there exists $C = C(\bar{\chi}) > 0$ such that for all $i, j \in \{1, \ldots, P\}$ with $i \neq j$ it holds

$$\text{dist}^2(\cdot, \bar{I}_{i,j}(t)) \wedge 1 \leq C(1 - |\xi_{i,j}(\cdot, t)|).$$

(163)
Proof. Fix \( i, j \in \{1, \ldots, P\} \) with \( i \neq j \). The asserted estimate (163) is trivially fulfilled for \( x \notin \text{supp} \xi_{i,j} \). By the definition (162) we may therefore assume that there exists a topological feature \( k \in \{1, \ldots, K\} \) such that \( x \in \text{supp} \eta_k \).

If either phase \( i \) or phase \( j \) is absent at the topological feature \( T_k \), we argue based on (160) as follows. Writing \( \xi_{i,j} = \eta_k \xi_{i,j}^k + \sum_{l \in \{1, \ldots, K\} \setminus \{k\}} \eta_l \xi_{i,j}^l \) we deduce from (160) together with the triangle inequality that there exists \( c = c(\bar{\chi}) \in (0, 1) \) such that \( |\xi_{i,j}^c| \leq 1 - c \). This in turn immediately implies the estimate (163).

Hence, assume that both the phases \( i \) and \( j \) are present at \( T_k \). The proof of (163) in this case essentially boils down to a direct combination of the localization properties (141)–(145) and the coercivity estimates (146)–(148) of the partition of unity \((\eta, \eta_1, \ldots, \eta_K)\). The precise argument proceeds as follows.

We consider the case of a two-phase interface \( k \in K^{2ph} \). As it will be clear from the subsequent argument, the case of a triple junction \( k \in K^{3} \) follows by symmetry. If \( x \in \text{supp} \eta_k \setminus \bigcup_{l \in K^{3}} \text{supp} \eta_l \) we may infer from (146) together with (142), (144) and (145) that \( \text{dist}^2(x, I_{i,j}) \leq C(1 - \eta_k(x)) \leq C(1 - |\xi_{i,j}(x)|) \) as required. Otherwise, we assume that there exists \( l \in K^3 \) such that \( x \in \text{supp} \eta_k \cap \text{supp} \eta_l \) (which because of (143) is the only such triple junction); in other words, the interface \( T_k \subset I_{i,j} \) has an endpoint at the triple junction \( T_l \). As a consequence of (144), it then suffices to verify (163) on the wedge \( W_{i,j} \) as well as its two adjacent interpolation wedges \( W_i \) resp. \( W_j \). In the case of the wedge \( W_{i,j} \), it follows from (147) together with (144) and (145) that \( \text{dist}^2(x, I_{i,j}) \leq C(1 - \eta_k(x) - \eta_l(x) - \eta_m(x)) \). In the remaining case of one of the interpolation wedges, say \( W_i \), we deduce first from (148) and then followed by (144) and (145) that \( \text{dist}^2(x, I_{i,j}) \leq C(1 - \eta_k(x) - \eta_m(x) - \eta_l(x)) \leq C(1 - |\xi_{i,j}(x)|) \). Here, \( l \in K^{2ph} \) denotes the second two-phase interfaces participating on the interpolation wedge \( W_i \). Since the last two estimates are exactly what is needed, we may thus conclude the proof of Lemma 31.

For a global definition of the velocity field \( B \), we proceed analogously, i.e., we first provide a definition for local velocity fields \( B^k \) for each topological feature \( T_k \) and then glue them together by means of the partition of unity \((\eta, \eta_1, \ldots, \eta_K)\) from Lemma 28. Recall from Section 6 that \( B^k \) necessarily has a tangential component for \( k \in K^{3} \). We therefore want to exploit the freedom in choosing the tangential component for \( B^k \) whenever \( k \in K^{2ph} \), see Section 5.

**Lemma 32.** Let \( \bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P) \) be a strong solution for multiphase mean curvature flow in the sense of Definition 11. Let \((\eta, \eta_1, \ldots, \eta_K)\) be a partition of unity as constructed in Lemma 28. Then for every topological feature \( k \in \{1, \ldots, K\} \) we may define a vector field

\[
B^k : \bigcup_{t \in [0,T_{\text{strong}}]} \text{supp} \eta_k(\cdot, t) \times \{t\} \to \mathbb{R}^2
\]

based on the local velocity fields constructed in Section 5 resp. Section 6 and satisfying the following:

i) It holds \( B^k \in L^\infty_t W^{1,\infty}_x \).

ii) Let \( k \in K^{3j} \) and \( m \in K^{2ph} \), such that the two-phase interface \( T_m = \tilde{I}_{i,j} \) has an endpoint at the triple junction \( T_k \). Then, it holds

\[
B^k = B^m + O(\text{dist}(\cdot, T_m)),
\]

\[
\bar{\tau}_{i,j} \cdot (\bar{\tau}_{i,j} \cdot \nabla) B^k = \bar{\tau}_{i,j} \cdot (\bar{\tau}_{i,j} \cdot \nabla) B^m + O(\text{dist}(\cdot, T_m)),
\]

\[
\bar{\tau}_{i,j} \cdot (\bar{\tau}_{i,j} \cdot \nabla) B^k = O(\text{dist}(\cdot, T_m)),
\]

\[
\bar{\tau}_{i,j} \cdot (\bar{\tau}_{i,j} \cdot \nabla) B^m = O(\text{dist}(\cdot, T_m))
\]
on \text{supp } \eta_k \cap \text{supp } \eta_m.

Proof. Let \( k \in K^{3j} \). We define \( B^k \) on the support of \( \eta_k \) by means of Proposition 21. Note that by the localization property iii) from Lemma 28 of the partition of unity \((\eta_1, \ldots, \eta_K)\), we are indeed in the setting of Section 6, so \( B^k \) is well-defined.

Now, let \( k \in K^{2ph} \). In principle, we then define \( B^k \) on the support of \( \eta_k \) by means of Lemma 17. However, in order to ensure that (164)–(166) hold true, we have to comment on the precise choice of the tangential component for \( B^k \). To this end, let \( \bar{\tau}_{i,j} \) be the localization parameter of Lemma 28. Recall that \( B_{2\bar{\tau}_{i,j}}(p_l) \cap B_{2\bar{\tau}_{i,j}}(p_m) = \emptyset \) by (143) as well as \( \text{supp } \eta_l \subset B_{\bar{\tau}_{i,j}}(p_l) \) by (142) for all triple junctions \( l, m \in K^{3j} \) with \( l \neq m \). Let \( \theta \) be a smooth cutoff function with \( \theta(r) = 1 \) for \( |r| \leq 1 \) and \( \theta \equiv 0 \) for \( |r| \geq 2 \). For a triple junction \( l \in K^{3j} \) and a two-phase interface \( k \in K^{2ph} \) with an endpoint at \( T_k \), we denote by \( \alpha_{k,l} \) the function which is defined on \( B_{2\bar{\tau}_{i,j}}(p_l) \cap \text{supp } \eta_k \) and determined by the equation (87).

If the two-phase interface \( T_k \) has no endpoint at a triple junction, we define \( B^k \) on the support of \( \eta_k \) by means of Lemma 17 with \( \alpha = 0 \). If the two-phase interface \( T_k \) has exactly one of its endpoints at a triple junction \( T_i \), we define \( B^k \) on the support of \( \eta_k \) by means of Lemma 17 with \( \alpha := \theta(\frac{\text{dist}(\cdot,T_i)}{\bar{\tau}_{i,j}}) \alpha_{k,l} \). Finally, if the two-phase interface \( T_k \) has its two endpoints at two distinct triple junctions \( T_i \) and \( T_m \), we set \( \alpha := \theta(\frac{\text{dist}(\cdot,T_i)}{\bar{\tau}_{i,j}}) \alpha_{k,l} + \theta(\frac{\text{dist}(\cdot,T_m)}{\bar{\tau}_{i,j}}) \alpha_{k,m} \) and define \( B^k \) on the support of \( \eta_k \) by means of Lemma 17 with this choice for the tangential component.

From these definitions, it is clear that (164) is satisfied. Moreover, it is an immediate consequence of the equations (87) that (165) resp. (166) hold true. Finally, (167) is satisfied as required on \( \text{supp } \eta_k \cap \text{supp } \eta_m \), since by the localization property (144) we have \( \text{supp } \eta_k \cap \text{supp } \eta_m \subset B_{\bar{\tau}_{i,j}}(T_k) \) entailing that the term corresponding to the gradient hitting the cutoff \( \theta \) vanishes in \( \text{supp } \eta_k \cap \text{supp } \eta_m \). In particular, the estimate (167) then follows along the same lines as (166). This concludes the proof of Lemma 32.

\[\square\]

**Construction 33.** Let \( \tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_P) \) be a strong solution for multiphase mean curvature flow in the sense of Definition 11 on some time interval \([0, T_{\text{strong}}]\). Let \((\eta, \eta_1, \ldots, \eta_K)\) be a partition of unity as constructed in Lemma 28. Let the local vector fields \( B^k \) be given as in Lemma 32. We then define

\[
B(x,t) := \sum_{k=1}^{K} \eta_k (x,t) B^k(x,t)
\]

for all \( x \in \mathbb{R}^2 \) and all \( t \in [0, T_{\text{strong}}] \).

7.3. **Compatibility of the local constructions.** Equipped with the definition of the global velocity field \( B \), we may now prove the required bound on the advective derivative of the localization functions \( \eta_k \) from Lemma 28.
Lemma 34. Let $\tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_\nu)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 11 on some time interval $[0, T_{\text{strong}}]$. Let $(\eta, \eta_1, \ldots, \eta_K)$ be a partition of unity as constructed in Lemma 28. Let the local velocity fields $B^k$ be given as in Lemma 32, and let the global velocity field $B$ be defined as in Definition 33. Then, the localization functions $(\eta, \eta_1, \ldots, \eta_K)$ satisfy

\begin{align}
|\partial_t \eta_m|^2 + |\nabla \eta_m|^2 &\leq C(\text{dist}^2(x, T_m(t)) \wedge 1) + C \eta_m, \tag{169} \\
|\partial_t \eta|^2 + |\nabla \eta|^2 &\leq C(\text{dist}^2(x, T_m(t)) \wedge 1), \tag{170} \\
|\partial_t \eta_m + (B \cdot \nabla) \eta_m| &\leq C(\text{dist}^2(x, T_m(t)) \wedge 1) + C \eta_m, \tag{171} \\
|\partial_t \eta + (B \cdot \nabla) \eta| &\leq C(\text{dist}^2(x, T_m(t)) \wedge 1), \tag{172}
\end{align}

for all $(x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}})$, all $i, j \in \{1, \ldots, P\}$ with $i \neq j$ and all two-phase interfaces $m \in \mathcal{K}^{2ph} \subset \{1, \ldots, K\}$.

Proof. We split the proof in two parts, first establishing the asserted estimates away from triple junctions and second in the vicinity of triple junctions.

Step 1: Estimates away from triple junctions. Let $m \in \mathcal{K}^{2ph}$ refer to a two-phase interface $T_m \subset \tilde{I}_{p,l}$. Because of (152) and (156), i.e., $(\eta_m = \zeta_{2ph}^p \cdot l$, it holds $(\partial_t \eta_m, \nabla \eta_m) = 0$ on $T_m \setminus \bigcup_{k \in \mathcal{K}^{3i}} \text{supp } \eta_k$. Hence, by the regularity of the localization function $\eta_m$, we obtain

$$|\partial_t \eta_m|^2 + |\nabla \eta_m|^2 \leq C \text{dist}^2(x, T_m(t)) \text{ on } \text{supp } \eta_m \setminus \bigcup_{k \in \mathcal{K}^{3i}} \text{supp } \eta_k.$$ 

The bound (170) follows from this, since $1 - \eta = \eta_m$ on $\text{supp } \eta_m \setminus \bigcup_{k \in \mathcal{K}^{3i}} \text{supp } \eta_k$.

Furthermore, on $\text{supp } \eta_m \setminus \bigcup_{k \in \mathcal{K}^{3i}} \text{supp } \eta_k$ we have $B \equiv \eta_m B_m$. Exploiting (59) as well as (152) and $\eta_m = \zeta_{2ph}^p \cdot l$, we deduce that on $\text{supp } \eta_m \setminus \bigcup_{k \in \mathcal{K}^{3i}} \text{supp } \eta_k$

$$\partial_t \eta_m = -(B^m \cdot \nabla) \eta_m = -(B \cdot \nabla) \eta_m - (1 - \eta_m)(B^m \cdot \nabla) \eta_m.$$ 

Hence, the desired estimate

$$|\partial_t \eta_m + (B \cdot \nabla) \eta_m| \leq C \text{dist}^2(x, T_m(t)) \text{ on } \text{supp } \eta_m \setminus \bigcup_{k \in \mathcal{K}^{3i}} \text{supp } \eta_k$$

holds true. The same is then also true for $\eta = 1 - \eta_m$.

Step 2: Estimates in the vicinity of triple junctions. Now, assume that the two-phase interface $T_m$ has an endpoint at the triple junction $T_k = \{p_k\}, k \in \mathcal{K}^{3i}$. By (143) and (144), it suffices to prove (169) and (171) on $B_{\tilde{c}_e}(T_k) \cap (W_{p,l} \cup W_p \cup W_l)$. By (151) resp. (152), we know that $(\partial_t \zeta_{2ph}^p \cdot \nabla \zeta_{2ph}^p) = 0$ on $T_m \cap B_{\tilde{c}_e}(T_k)$ resp. $(\partial_t \zeta_{p,l}^3 \cdot \nabla \zeta_{p,l}^3) = 0$ at the triple junction $T_k$. Since on the wedge $W_{p,l}$ the distance to the triple junction is comparable to the distance of the respective halfspaces in (151), we obtain together with (157) the estimate

$$|\partial_t \eta_m|^2 + |\nabla \eta_m|^2 \leq C \text{dist}^2(\cdot, T_k) \zeta_{p,l}^{2ph} + C \text{dist}^2(\cdot, T_m)(1 - \zeta_{p,l}^3)$$

\hspace{2cm} \leq C(1 - \zeta_{p,l}^3) \zeta_{p,l}^{2ph} + C \text{dist}^2(\cdot, T_m)$$

\hspace{2cm} \leq C \eta_m + C \text{dist}^2(\cdot, T_m) \text{ on } \text{supp } \eta_m \cap B_{\tilde{c}_e}(T_k) \cap W_{p,l}.$$ 

Moreover, since $1 - \eta = \eta_k + \eta_m = \zeta_{p,l}^{2ph}$ on $\text{supp } \eta_m \cap B_{\tilde{c}_e}(T_k) \cap W_{p,l}$ we also get (170).

For the estimate on the advective derivative, we first note that $B = \eta_m B_m + \eta_k B_k$.
on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T) \cap W_{p,l} \). Hence, by (59) we obtain as before
\[
\partial_t \zeta^{2ph}_{p,l} = -(B^n \cdot \nabla) \zeta^{2ph}_{p,l}
\]
\[
= -(B \cdot \nabla) \zeta^{2ph}_{p,l} - (1 - \eta_m - \eta_k)(B^n \cdot \nabla) \zeta^{2ph}_{p,l} - \eta_k ((B^n - B^k) \cdot \nabla) \zeta^{2ph}_{p,l}
\]
on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T_k) \cap W_{p,l} \). In particular, because of \( |\nabla \zeta^{2ph}_{p,l}| < C |\text{dist}(\cdot, T_m)| \) and (164) this entails
\[
|\partial_t \zeta^{2ph}_{p,l} + (B \cdot \nabla) \zeta^{2ph}_{p,l}| \leq C |\text{dist}(\cdot, T_m)|
\]
on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T_k) \cap W_{p,l} \). Since \( 1 - \eta = \eta_p + \eta_m = \zeta^{2ph}_{p,l} \) on this set, see (153) and (157), this already proves (172) on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T_k) \cap W_{p,l} \). To obtain a bound on the advective derivative of \( \eta_m \) itself, we compute \( \partial_t \zeta^{3j}_{p,l} \)

\[
s\text{dist}(x, \mathbb{H}_p(t)) = -(x - p_k(t)) \cdot h_p(t)
\]
and similarly for \( s\text{dist}(x, \mathbb{H}_p(t)) \). Hence, we obtain from \( \frac{d}{dt} p_k(t) = B(p_k(t), t) \) that
\[
\partial_t \zeta^{3j}_{p,l} = -(B \cdot \nabla) \zeta^{3j}_{p,l} - ((B(p_k(t), t) - B) \cdot \nabla) \zeta^{3j}_{p,l}
\]
\[
- \zeta^{3j} \left( s\text{dist}(x, \mathbb{H}_p(t)) \right) \frac{c_2 r_c}{c_2 r_c} \left( s\text{dist}(x, \mathbb{H}_p(t)) \right) (x - p_k(t)) \cdot \frac{d}{dt} h_p(t)
\]
\[
- \zeta^{3j} \left( s\text{dist}(x, \mathbb{H}_p(t)) \right) \frac{c_2 r_c}{c_2 r_c} \left( s\text{dist}(x, \mathbb{H}_p(t)) \right) (x - p_l(t)) \cdot \frac{d}{dt} h_l(t).
\]
Since \( |B(p_k(t), t) - B| \leq C |x - p_k(t)| \), \( \frac{d}{dt} p_k(t), \frac{d}{dt} h_p(t) \in L^\infty_t \) as well as \( (\zeta^{3j})'(0) = 0 \), we thus obtain the estimate
\[
(173)
|\partial_t \zeta^{3j}_{p,l} + (B \cdot \nabla) \zeta^{3j}_{p,l}| \leq C |x - p_k|^2 \leq C(1 - \zeta^{3j}_{p,l})
\]
on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T_k) \cap W_{p,l} \), since on this set the distance to the triple junction is comparable to the distance of the respective halfspaces in (151). Hence, by taking together the bounds for the advective derivative of \( \zeta^{2ph}_{p,l} \) and \( \zeta^{3j}_{p,l} \), we infer by using also (157) that
\[
|\partial_t \eta_m + (B \cdot \nabla) \eta_m| \leq C |\text{dist}(\cdot, T_m)| + C |\eta_m|
\]
holds true on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(p_k) \cap W_{p,l} \) as required.

For the required estimates on the interpolation wedges, we make use of (158), the bounds \( |1 - \zeta^{3j}_{p,l}| \leq C |\text{dist}(\cdot, T_k)| \) and \( |\partial_t \zeta^{2ph}_{p,l}|^2 + |\nabla \zeta^{2ph}_{p,l}|^2 \leq C |\text{dist}(\cdot, T_k)|^2 \) as well as (130) to infer first that
\[
|\partial_t \eta_m|^2 + |\nabla \eta_m|^2
\]
\[
\leq C |\text{dist}(\cdot, T_k)| \leq C |\text{dist}(\cdot, T_m)| \quad \text{on} \quad \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T_k) \cap (W_p \cup W_l).
\]
The last inequality is again due to the fact that on an interpolation wedge, the distance to the triple junction is comparable to the distance of a two-phase interface present there. Moreover, observe that \( 1 - \eta = \lambda^p \zeta^{2ph}_{p,p} + \lambda^n \zeta^{2ph}_{p,n} \) holds true on \( \text{supp} \eta_m \cap B_{\mathcal{T}_k}(T_k) \cap W_p \), where we assume that \( n \in \{1, \ldots, P\} \) is the phase present at the triple junction \( T_k \). Hence, the estimates \( \zeta^{2ph}_{p,l} - \zeta^{2ph}_{p,n} \leq C |\text{dist}(\cdot, T_k)|^2 \) and
then entail (170) on supp $\eta_m \cap B_{\tilde{r}}(T_k) \cap W_p$. The corresponding bound on the other interpolation wedges follows analogously.

For the advective derivative, we first compute
\[
\partial_t \zeta_{p,l}^{2ph} = - (B^m \cdot \nabla) \zeta_{p,l}^{2ph} = - (B \cdot \nabla) \zeta_{p,l}^{2ph} - ((B^m - B^k) \cdot \nabla) \zeta_{p,l}^{2ph} - ((B^k - B) \cdot \nabla) \zeta_{p,l}^{2ph}
\]
on supp $\eta_m \cap B_{\tilde{r}}(T_k) \cap (W_p \cup W_i)$. Hence, it follows from the bounds $|B - B^k| + |B^m - B^k| \leq C|\text{dist}(\cdot, T_k)|$ and $|\nabla \zeta_{p,l}^{2ph}| \leq C|\text{dist}(\cdot, T_k)|$ that
\[
|\partial_t \zeta_{p,l}^{2ph} + (B \cdot \nabla) \zeta_{p,l}^{2ph}| \leq C|\text{dist}(\cdot, T_m)|
on supp \eta_m \cap B_{\tilde{r}}(T_k) \cap (W_p \cup W_i).
\]
From this, we again already obtain (172). Indeed, first recall that $1 - \eta = \lambda^t \zeta_{p,l}^{2ph} + \lambda^p \zeta_{p,l}^{2ph}$ holds true on supp $\eta_m \cap B_{\tilde{r}}(T_k) \cap W_p$, where we assume that $n \in \{1, \ldots, P\}$ is the third phase present at the triple junction $T_k$. Moreover, we have $|\zeta_{p,l}^{2ph} - \zeta_{p,n}^{2ph}| \leq C|\text{dist}(\cdot, T_k)|^2$ on supp $\eta_m \cap B_{\tilde{r}}(T_k) \cap W_p$. Hence, together with (134), we obtain $|\partial_t \eta + (B \cdot \nabla)\eta| \leq C|\text{dist}(\cdot, T_m)|$ on supp $\eta_m \cap B_{\tilde{r}}(T_k) \cap W_p$. The corresponding bound on the other interpolation wedges follows analogously.

Furthermore, it follows from the argument leading to the first inequality in (173) that
\[
|\partial_t \zeta_{p,l}^{\jmath} + (B \cdot \nabla) \zeta_{p,l}^{\jmath}| \leq C|\text{dist}(\cdot, T_m)|
on supp \eta_m \cap B_{\tilde{r}}(T_k) \cap (W_p \cup W_i).
\]
Finally, since $|1 - \zeta_{p,l}^{\jmath} | \leq C|\text{dist}(\cdot, T_k)|^2$, we may deduce from (134), (158) together with the bounds for the advective derivative of $\zeta_{p,l}^{2ph}$ and $\zeta_{p,l}^{\jmath}$ that
\[
|\partial_t \eta_m + (B \cdot \nabla)\eta_m| \leq C|\text{dist}(\cdot, T_m)|
\]
holds true on supp $\eta_m \cap B_{\tilde{r}}(T_k) \cap (W_p \cup W_i)$. This concludes the proof. \hfill $\Box$

We next prove several useful compatibility bounds in the gluing regions for the local constructions associated to the different topological features of a network, i.e., triple junctions and smooth two-phase interfaces. These technical estimates will be needed in order to derive the estimates (4c)–(4e) for the global constructions from the corresponding ones for the local constructions in Lemma 17 and Proposition 21.

Lemma 35. Let $\tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_P)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 11. Let $((\eta, \eta_1, \ldots, \eta_K)$ be a partition of unity as constructed in Lemma 28. Let $\xi_{i,j}^k$ be the local vector fields from Lemma 29 as well as $B^k$ be the local velocity fields from Lemma 32. Let $\xi_{i,j}$ be the global vector fields from Construction 30, and let $B$ be the global velocity field from Construction 33. Then, the local constructions at triple junctions and at smooth two-phase interfaces are compatible in the sense that the following estimates are satisfied

\[
|\eta_k(\xi_{i,j} - \xi_{i,j}^k)| \leq C|\text{dist}(\cdot, \tilde{I}_{i,j}(t)) \wedge 1),
\]
\[
|\eta_k(\xi_{i,j} \cdot (\xi_{i,j} - \xi_{i,j}^k))| \leq C|\text{dist}^2(\cdot, \tilde{I}_{i,j}(t)) \wedge 1),
\]
\[
|\eta_k(B^k - B)| \leq C|\text{dist}(\cdot, \tilde{I}_{i,j}(t)) \wedge 1),
\]
\[
|\eta_k(\xi_{i,j} \cdot (B - B^k) \cdot \nabla) \zeta_{i,j}^k| \leq C|\text{dist}^2(\cdot, \tilde{I}_{i,j}(t)) \wedge 1),
\]
\[
|\eta_k(\xi_{i,j} - \xi_{i,j}^k) \cdot (\nabla B^k - \nabla B) \zeta_{i,j}^k| \leq C|\text{dist}^2(\cdot, \tilde{I}_{i,j}(t)) \wedge 1)
\]
for all \((x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}]\), all \(i, j \in \{1, \ldots, P\}\) with \(i \neq j\) and all topological features \(k \in \{1, \ldots, K\}\). If we restrict to triple junctions, we in addition have the following two slightly refined bounds

\[
\chi_{\text{supp } \eta_k} |\xi_{i,j}^k - \xi_{i,j}|^2 + \chi_{\text{supp } \eta_k} (|\xi_{i,j}^k - \xi_{i,j}|) \leq C(\text{dist}(\cdot, \bar{I}_{i,j}(t)) \wedge 1),
\]

\[
\chi_{\text{supp } \eta_k} |B - B^k| \leq C(\text{dist}(\cdot, \bar{I}_{i,j}(t)) \wedge 1)
\]

being valid for all \((x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}]\), all \(i, j \in \{1, \ldots, P\}\) with \(i \neq j\) and all triple junctions \(k \in K^3 \subset \{1, \ldots, K\}\).

For the proof of Lemma 35, recall that we decomposed \(\{1, \ldots, K\} = K^{2ph} \cup K^3\), with \(K^3\) enumerating the space-time connected components of the smooth two-phase interfaces \(\bar{I}_{i,j}\). We also put \(T_k := \{p_k\}, \) if \(k \in K^3\), or \(T_k \subset \bar{I}_{i,j}\) for the corresponding space-time connected component \(k \in K^{2ph}\) of a two-phase interface \(\bar{I}_{i,j}\).

**Proof of (174), (175) and (179).** Let \(i, j \in \{1, \ldots, P\}\) be fixed with \(i \neq j\). We will focus in the following on deriving (175); the estimates (174) and (179) follow in the process. We start by plugging in the definition (162) of \(\xi_{i,j}\) and exploiting the fact that \(\eta \leq C(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)\)

\[
\eta \xi_{i,j} \cdot (\xi_{i,j} - \xi_{i,j}^k) = -\eta \xi_{i,j} \cdot \xi_{i,j}^k + \sum_{l=1, l \neq k}^K \eta \xi_{i,j} \cdot (\xi_{i,j}^l - \xi_{i,j}^k)
\]

\[
\sum_{m=1}^K \sum_{l=1, l \neq k}^K \eta \xi_{i,j} \cdot (\xi_{i,j}^m - \xi_{i,j}^k) + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

**Step 1: Case \(k \in K^3\).** We consider first the case that \(k \in K^3\). Since \(l \neq k\), we may restrict ourselves to the case that \(l \in K^{2ph}\). Indeed, by the localization property (143) of the partition of unity \((\eta, \eta_1, \ldots, \eta_K)\) we have \(\text{supp } \eta_k \cap \text{supp } \eta_l = \emptyset\) for \(k, l \in K^3\) such that \(l \neq k\). Moreover, we may assume that \(T_k \subset \bar{I}_{i,j}\); since otherwise either phase \(i\) or \(j\) is not present at \(T_k\), i.e., we can conclude directly by means of the estimate

\[
\eta_l \leq C(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

Finally, we may restrict ourselves to the case that the interface \(T_k \subset \bar{I}_{i,j}\) ends at the fixed triple junction \(T_k = \{p_k\}\), since otherwise again \(\text{supp } \eta_k \cap \text{supp } \eta_l = \emptyset\) due to the localization property \(v)\) from Lemma 28 of the partition of unity \((\eta, \eta_1, \ldots, \eta_K)\).

If \(m \in K^3\) it suffices to consider \(m = k\), since otherwise again \(\text{supp } \eta_m \cap \text{supp } \eta_l = \emptyset\) by the localization property (143) of the partition of unity \((\eta, \eta_1, \ldots, \eta_K)\). Moreover, if \(m \in K^{2ph}\) we may restrict ourselves to the case that \(m = l\), i.e., in particular \(T_m \subset \bar{I}_{i,j}\). Indeed, if this would not be the true then either \(T_m\) is not a connected component of \(\bar{I}_{i,j}\) in which case we can argue by the analogous bound to (181). Or \(T_m \subset \bar{I}_{i,j}\), in which case we would necessarily have \(\text{supp } \eta_m \cap \text{supp } \eta_l = \emptyset\). In the following, we will now consider first \(m \in K^{2ph}\), i.e., \(m = l\).

By assumption, the phases \(i\) and \(j\) are present at the triple junction \(T_k\). Let \(p \in \{1, \ldots, P\} \setminus \{i, j\}\) denote the third phase being present at \(T_k\). Due to the localization property (144) of the partition of unity \((\eta, \eta_1, \ldots, \eta_K)\), we know that \(\text{supp } \eta_k \cap \text{supp } \eta_l\) decomposes into three wedges. One of them (denoted in the following by \(W_{i,j}p\)) contains the interface \(T_k \subset \bar{I}_{i,j}\), whereas the other two are the
interpolation wedges (denoted in the following by $W_i$ resp. $W_j$) adjacent to the wedge $W_{i,j}$.

Recall from the proof of Proposition 21 that we defined $\xi_{i,j}^k := |\xi_{i,j}^k|^{-1}\tilde{\xi}_{i,j}^k$, where $\tilde{\xi}_{i,j}^k$ is the local vector field defined by (135), and that $|\tilde{\xi}_{i,j}^k| \geq \frac{1}{2}$ on its domain of definition containing $\text{supp} \eta_k$. Now, because of

$$\xi_{i,j}^k - \tilde{\xi}_{i,j}^k = \frac{1 - |\xi_{i,j}^k|}{|\xi_{i,j}^k|} \xi_{i,j}^k - \frac{1 - |\xi_{i,j}^k|^2}{(1 + |\xi_{i,j}^k||\xi_{i,j}^k|)} \tilde{\xi}_{i,j}^k,$$

we may replace $\xi_{i,j}^k$ by $\tilde{\xi}_{i,j}^k$ at the expense of an error of the order $O(\text{dist}^2(\cdot, I_{i,j}) \wedge 1)$ due to the estimate (138).

On the wedge $W_{i,j}$ we may insert the corresponding local definition for $\xi_{i,j}^k$ resp. $\tilde{\xi}_{i,j}^k$ from (54) resp. (135). This entails $\eta_m^p \cdot (\xi_{i,j}^k - \tilde{\xi}_{i,j}^k) = O(\text{dist}^2(\cdot, T_i))$ on $W_{i,j}$. Hence, we obtain

$$\eta_m^p \cdot (\xi_{i,j}^k - \tilde{\xi}_{i,j}^k) = O(\text{dist}^2(\cdot, T_i) \wedge 1).$$

Note that this bound not only holds true for $m = l$, but also for $m = k$ because then $\xi_{i,j}^m = \xi_{i,j}^l + O(\text{dist}(\cdot, T_i))$ is satisfied by (135). Observe also that (174) holds true, since $\xi_{i,j}^m - \tilde{\xi}_{i,j}^k = O(\text{dist}(\cdot, T_i))$ on the wedge $W_{i,j}$.

On the interpolation wedge $W_i$, we again plug in the local definition for $\tilde{\xi}_{i,j}^k$ from (135), i.e., $\tilde{\xi}_{i,j}^k = (1 - \lambda_l)\tilde{\xi}_{i,j} + \lambda_l R(p_i)\tilde{\xi}_{p,i}$, and compute

$$\xi_{i,j}^l - \tilde{\xi}_{i,j}^k = (1 - \lambda_l)(\xi_{i,j}^l - \tilde{\xi}_{i,j}) + \lambda_l(\xi_{i,j}^l - R(p_i)\tilde{\xi}_{p,i}).$$

Since the argument leading to (182) works without change when $\tilde{\xi}_{i,j}^k$ is being replaced by $\tilde{\xi}_{i,j}$, the term $\xi_{i,j}^m \cdot (\xi_{i,j}^l - \tilde{\xi}_{i,j})$ is bounded by $O(\text{dist}^2(\cdot, I_{i,j} \wedge 1)$. On the other side, by the second-order compatibility (100) of the local vector fields $\tilde{\xi}_{i,j}$ resp. $R(p_i)\tilde{\xi}_{p,i}$ at the triple junction $T_k = \{p_k\}$, we have the bound $|\tilde{\xi}_{i,j} - R(p_i)\tilde{\xi}_{p,i}| \leq C|x - p_k|^2 \leq C(\text{dist}^2(x, I_{i,j}) \wedge 1)$. From this, we then obtain on the interpolation wedge $W_i$ the desired estimate

$$\eta_k \eta_m^p \cdot (\xi_{i,j}^k - \tilde{\xi}_{i,j}^k)^2 \leq C(\text{dist}^2(\cdot, T_i) \wedge 1).$$

The bound on the interpolation wedge $W_j$ follows analogously. This concludes the proof of (174) and (175) for $k \in \mathcal{K}_i^3$. Before we proceed with the argument for $k \in \mathcal{K}_i^{2ph}$, let us make the following observation. In the whole preceding argument for $k \in \mathcal{K}_i^{3}$, we not fully exploited the fact that we have the localization function $\eta_k$ as a prefactor. We actually only used that we have to provide a bound on $\text{supp} \eta_k$. Therefore, what the preceding argument actually proves is the slightly refined estimate (179).

**Step 2:** $k \in \mathcal{K}_i^{2ph}$. As argued before, it is then sufficient to consider the case that $T_k \subset I_{i,j}$ since otherwise we may conclude by the corresponding bound of (181). If now $l \in \mathcal{K}_i^3$, it moreover suffices to restrict to the case that $T_k$ ends at the fixed triple junction $T_l = \{p_l\}$. In other words, the roles of $k$ and $l$ are reversed and we may apply the argument from the previous case (except for (179), but this will not be needed in the sequel anyway). Finally, if $l \in \mathcal{K}_i^{2ph}$ and $l \neq k$, then either $T_k \subset I_{i,j}$ which in turn necessarily entails $\text{supp} \eta_k \cap \text{supp} \eta_l = \emptyset$, or otherwise either phase $i$ or $j$ is absent at $T_l$ meaning we may rely again on the bound (181). This concludes the proof of (174) and (175).
Proof of (176) and (180). Let \( i, j \in \{1, \ldots, P\} \) be fixed with \( i \neq j \). From the definition (168) as well as \( \eta \leq C(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1) \) it follows
\[
\eta_k(B^k - B) = \eta_k B^k = \sum_{l=1, l \neq k}^K \eta_k \eta_l (B^k - B^l)
\]
\[
= \sum_{l=1, l \neq k}^K \eta_k \eta_l (B^k - B^l) + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

Let now \( k \in K^3 \). As argued in the proof of (174), we may restrict ourselves to the case that \( l \in K^{2ph} \) such that \( T_l \subset \bar{I}_{i,j} \), and the interface \( T_l \) ends at the fixed triple junction \( T_k = \{p_k\} \). Again, \( \text{supp} \eta_k \cap \text{supp} \eta_l \) then decomposes into three wedges \( W_{i,j} \) and \( W_j \), see the proof of (174). To fix notation, let us suppose that the phase \( p \in \{1, \ldots, P\} \) is, next to the phases \( i \) and \( j \), present at \( T_k \).

By the precise choice of the tangential component for the velocity field \( B^l \) in \( \text{supp} \eta_k \cap \text{supp} \eta_l \), we have (164). Hence, it follows
\[
\eta_k \eta_l (B^k - B^l) \leq C(\text{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \text{ on } W_{i,j}.
\]

To obtain the corresponding bound on the interpolation wedge \( W_i \), we plug in the local definition of \( B^k \) from (136), i.e., \( B^k = (1 - \lambda_k) \tilde{B}_{(i,j)} + \lambda_k \tilde{B}_{(i,p)} \), and compute
\[
B^k - B^l = (1 - \lambda_k) \tilde{B}_{(i,j)} - B^l + \lambda_k \tilde{B}_{(i,p)} - B^l
\]
\[
= (\tilde{B}_{(i,j)} - B^l) + \lambda_k (\tilde{B}_{(i,p)} - B^l).
\]

Now, since the derivation of the estimate (184) works without change when replacing the velocity field \( B^k \) by \( \tilde{B}_{(i,j)} \), we obtain for the first term in the above decomposition a bound of required order. For the second term, we employ the first-order compatibility (115) of the velocity fields \( \tilde{B}_{(i,j)} \) resp. \( \tilde{B}_{(i,p)} \) at the triple junction \( T_k = \{p_k\} \) to obtain again an estimate of required order. All in all, we may infer that on the interpolation wedge \( W_i \) it holds
\[
\eta_k \eta_l (B^k - B^l)^2 \leq C(\text{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

The corresponding bound on the interpolation wedge \( W_j \) follows along the same lines. Hence, we proved (176) for \( k \in K^3 \). As in the proof of (174) resp. (175), we make the observation that what we actually proved is the estimate (180). Finally, the other case \( k \in K^{2ph} \) follows by an analogous case distinction as at the end of the proof of (174) resp. (175). \( \square \)

Proof of (177). Let \( i, j \in \{1, \ldots, P\} \) be fixed with \( i \neq j \). Plugging in (162) and (168) as well as making use of \( \eta \leq C(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1) \) we may compute
\[
\eta_k \xi_{i,j} \cdot ((B - B^k) \cdot \nabla) \xi^k_{i,j}
\]
\[
= -\eta_k \xi_{i,j} \cdot (B^k \cdot \nabla) \xi^k_{i,j} + \sum_{l=1, l \neq k}^K \eta_k \eta_l \xi_{i,j} \cdot ((B^l - B^k) \cdot \nabla) \xi^k_{i,j}
\]
\[
= \sum_{m=1}^K \sum_{l=1, l \neq k} \eta_k \eta_m \xi_{i,j} \cdot ((B^l - B^k) \cdot \nabla) \xi^k_{i,j} + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

Let \( k \in K^3 \). As it is now routine, we may restrict ourselves to the case that \( l \in K^{2ph} \) such that \( T_l \subset \bar{I}_{i,j} \), and the interface \( T_l \) ends at the fixed triple junction
\( T_k = \{ p_k \} \). Recall that \( \text{supp} \eta_k \cap \text{supp} \eta_l \) then decomposes into three wedges \( W_{i,j} \) and \( W_j \) resp. \( W_j \), see the proof of (174). To fix notation, let us suppose that the phase \( p \in \{1, \ldots, P \} \) is, next to the phases \( i \) and \( j \), present at \( T_k \).

If \( m \in \mathcal{K}^3 \) it suffices to consider \( m = k \), since otherwise \( \text{supp} \eta_m \cap \text{supp} \eta_k = \emptyset \) by the localization property (143) of the partition of unity \( (\eta, \eta_1, \ldots, \eta_K) \). Moreover, if \( m \in \mathcal{K}^{2^h} \) we may restrict ourselves to the case that \( m = l \) as argued in the proof of (174) resp. (175), i.e., in particular \( T_m \subset I_{i,j} \). In the following, we will consider first \( m \in \mathcal{K}^{2^h} \), i.e., \( m = l \).

On the wedge \( W_{i,j} \), we have \( \xi^l_{i,j} = \tilde{u}_{i,j} \). By the precise choice of the tangential component for the velocity field \( B^l \) in \( \text{supp} \eta_k \cap \text{supp} \eta_l \), we have (164). In particular, on the wedge \( W_{i,j} \) it holds \( B^l - B^k = O(\text{dist}(\cdot, T_l))\tilde{\eta}_{i,j} + O(\text{dist}^2(\cdot, T_l)) \) due to the definition of \( B^k \) in (136). Moreover, we have \( \nabla \xi^k_{i,j} = O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l)) \) on the wedge \( W_{i,j} \). However, since \( (\nabla \xi^k_{i,j})^T \tilde{\xi}^k_{i,j} = O(\text{dist}(\cdot, T_l)) \) we get \( \nabla \xi^k_{i,j} = O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l)) \) on the wedge \( W_{i,j} \). Hence, taking all of these bounds together we obtain

\[
\eta_k \eta_m \xi^m_{i,j} : ((B^l - B^k) \cdot \nabla) \xi^k_{i,j} \leq C(\text{dist}^2(\cdot, I_{i,j}) \cup 1)
\]

Note that the bound (187) also holds for \( m \in \mathcal{K}^3 \) (i.e., \( m = k \) as argued above), since on the wedge \( W_{i,j} \) we have \( \xi^k_{i,j} = \xi^l_{i,j} + O(\text{dist}(\cdot, T_l)) \).

On the interpolation wedge \( W_i \), we still have \( \xi^l_{i,j} = \tilde{u}_{i,j} \). For \( B^k - B^l \), we first refer to the decomposition (185). Note that apart from the first-order compatibility (115) of the velocity fields \( \tilde{B}_{(i,j)} \) resp. \( \tilde{B}_{(i,p)} \) at the triple junction \( T_k = \{ p_k \} \), we also know that \( \tilde{B}_{(i,j)} \) resp. \( \tilde{B}_{(i,p)} \) have bounded and continuous second derivatives on \( W_i \). Hence, we may use the first-order compatibility of the velocity fields at the triple junction to estimate \( B^k - B^l = (\tilde{B}_{(i,j)} - \tilde{B}_{(i,j)}^l) + O(\text{dist}^2(\cdot, T_l)) \) on the interpolation wedge \( W_i \). It moreover follows from the definition of the velocity \( \tilde{B}_{(i,j)} \) as well as the argument for the wedge \( W_{i,j} \) that \( \tilde{B}_{(i,j)} = B^l = O(\text{dist}(\cdot, T_l)) \tilde{\eta}_{i,j} + O(\text{dist}^2(\cdot, T_l)) \).

Next, we plug in the local definition for the vector field \( \xi^k_{i,j} \) from (135), i.e.,

\[
\xi^k_{i,j} = (1 - \lambda_i)\tilde{\xi}_{i,j} + \lambda_i R_{(p,i)} \tilde{\xi}_{p,i} + \lambda_i (\tilde{\xi}_{p,i} - \tilde{\xi}_{i,j}) \text{, and compute}
\]

\[
\nabla \xi^k_{i,j} = (R_{(p,i)} \tilde{\xi}_{p,i} - \tilde{\xi}_{i,j}) \otimes \nabla \lambda_i + \nabla \tilde{\xi}_{i,j} + \lambda_i \nabla (R_{(p,i)} \tilde{\xi}_{p,i} - \tilde{\xi}_{i,j}).
\]

Moreover, note that we have \( \nabla (\tilde{\xi}_{i,j}^{-1}) = O(1)(\nabla \tilde{\xi}_{i,j})^T \tilde{\xi}_{i,j} \). It then follows from the second-order compatibility (100) of the vector fields \( \xi_{i,j} \) resp. \( R_{(p,i)} \tilde{\xi}_{p,i} \) at the triple junction \( T_k = \{ p_k \} \), as well as the definition of \( \tilde{\xi}_{i,j} \) together with the argument for the wedge \( W_{i,j} \), that \( \nabla \xi^k_{i,j} = O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l)) \) also holds on the interpolation wedge \( W_i \). Taking all of these bounds together we may infer

\[
\eta_k \eta_m \xi^m_{i,j} : ((B^l - B^k) \cdot \nabla) \xi^k_{i,j} \leq C(\text{dist}^2(\cdot, I_{i,j}) \cup 1)
\]

on the interpolation wedge \( W_i \). Note that the bound (188) also holds for \( m \in \mathcal{K}^3 \) (i.e., \( m = k \) as argued above), since on the interpolation wedge \( W_i \) we again have \( \xi^k_{i,j} = \xi^l_{i,j} + O(\text{dist}(\cdot, T_l)) \). As the argument for the interpolation wedge \( W_j \) proceeds along the same lines, we thus have proved (177) for \( k \in \mathcal{K}^3 \). The argument for \( k \in \mathcal{K}^{2^h} \) proceeds by an analogous case distinction to the one at the end of the proof of (174).
Proof of (178). Let again $i, j \in \{1, \ldots, P\}$ be fixed with $i \neq j$. This time, we compute using (162), (168), $\eta \leq C(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)$ as well as (169) and (174)

$$
\eta_k(\xi_{i,j} - \xi^k_{i,j}) \cdot (\nabla B^k - \nabla B)^T \xi^k_{i,j}
= \eta_k(\xi_{i,j} - \xi^k_{i,j}) \cdot (\nabla B^k)^T \xi^k_{i,j} + \sum_{l=1, l \neq k}^K \eta_l \eta_k(\xi_{i,j} - \xi^k_{i,j}) \cdot (\nabla B^k - \nabla B^l)^T \xi^k_{i,j}
+ \eta_k(\xi_{i,j} - \xi^k_{i,j}) \cdot (\nabla \eta \otimes B)\xi^k_{i,j} - \sum_{l=1}^K \eta_l(\xi_{i,j} - \xi^k_{i,j}) \cdot (\nabla \eta_l \otimes (B^l - B))\xi^k_{i,j}
= \sum_{m=1, m \neq k}^K \sum_{l=1, l \neq k}^K \eta_k \eta_m (\xi^m_{i,j} - \xi^k_{i,j}) \cdot (\nabla B^k - \nabla B^l)^T \xi^k_{i,j}
- \sum_{l=1}^K \eta_l(\xi_{i,j} - \xi^k_{i,j}) \cdot (\nabla \eta_l \otimes (B^l - B))\xi^k_{i,j} + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
$$

We first argue how to estimate the last term in the previous estimate. If $l \in K^3$, we obtain a bound of required order by an application of Hölder’s inequality as well as the estimates (174) and (170). On the other side, if $l \in K^{2ph}$ we may argue by means of Hölder’s inequality as well as the estimates (174), (176) and (179). Hence, we may move on with the bound for the penultimate term.

Let $k \in K^3$; the case $k \in K^{2ph}$ follows by symmetry. We assume , without loss of generality, that $l \in K^{2ph}$ such that $T_l \subset \bar{I}_{i,j}$ and $T_l$ has an endpoint at the triple junction $T_k = \{p_k\}$. We again decompose supp $\eta_k \cap$ supp $\eta_l$ into three wedges $W_{i,j}$ and $W_i$ resp. $W_j$, see the proof of (174). To fix notation, let us suppose that the phase $p \in \{1, \ldots, P\}$ is, next to the phases $i$ and $j$, present at $T_k$. Let $m \in K^{2ph}$ (for $m \in K^3$ with $m \neq k$ we have supp $\eta_k \cap$ supp $\eta_l = \emptyset$ due to the localization property (143) of the partition of unity $(\eta, \eta_1, \ldots, \eta_K)$). We may then also assume that $m = l$, as argued in the proof of (174) resp. (175). Finally, we can replace $\tilde{\xi}^k_{i,j}$ by $\tilde{\xi}^l_{i,j}$ at the expense of an error of the order $O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)$ as in the proof of (174) resp. (175).

By Property $iii)$ of Lemma 29, the Lipschitz continuity of the vector fields $\tilde{\xi}^k_{i,j}$ resp. $\tilde{\xi}^l_{i,j}$ and the definition of $\tilde{\xi}^k_{i,j}$ from (135), we have $\tilde{\xi}^l_{i,j} - \tilde{\xi}^k_{i,j} = O(\text{dist}(\cdot, T_l))\tilde{\eta}_{i,j} + O(\text{dist}^2(\cdot, T_l))$ on the wedge $W_{i,j}$. In particular, $\tilde{\xi}^k_{i,j} = \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l))\tilde{\eta}_{i,j} + O(\text{dist}^2(\cdot, T_l))$. On the other side, by the precise choice of the tangential component for the velocity field $B^l$ in supp $\eta_k \cap$ supp $\eta_l$, we have (165) and (166). Hence, by the local definition of the velocity field $B^k$ from (136) it holds

$$
(\nabla B^k - \nabla B^l)^T = O(1)\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l))\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l))\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j} + O(\text{dist}(\cdot, T_l))\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j}
$$
on the wedge $W_{i,j}$. Note that indeed no zero-order contribution for the $\tilde{\eta}_{i,j} \otimes \tilde{\eta}_{i,j}$ component appears since neither $\nabla B^k$ nor $\nabla B^l$ contain one already on an individual basis. Hence, when contracting with $\tilde{\xi}^k_{i,j}$ and $\tilde{\xi}^l_{i,j} - \tilde{\xi}^k_{i,j}$ we obtain

$$
\eta_k \eta_l \eta_m |\langle \xi^m_{i,j} - \xi^k_{i,j}, (\nabla B^k - \nabla B^l)^T \xi^k_{i,j} \rangle| \leq C(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)
$$
on the wedge $W_{i,j}$.

Thus, we may turn to the bound on the interpolation wedge $W_i$. By substitution of the local definition of $\tilde{\xi}^k_{i,j}$ from (135), we first observe that $\tilde{\xi}^k_{i,j} = \tilde{\xi}^k_{i,j}$.
\[ \tilde{\xi}_{i,j} + \lambda_t (R_{i,p,j} \hat{\xi}_{p,i} - \tilde{\xi}_{i,j}) \]. Hence, it follows from the second-order compatibility of the vector fields \( \tilde{\xi}_{i,j} \) resp. \( R_{i,p,j} \hat{\xi}_{p,i} \) at the triple junction \( T_k = \{ p_k \} \), as well as the definition of \( \tilde{\xi}_{i,j} \) that \( \tilde{\xi}_{i,j}^k = \tilde{n}_{i,j} + O(\text{dist}(\cdot, \overline{T}_1))T_{i,j} + O(\text{dist}^2(\cdot, \overline{T}_1)) \). In particular, by definition of \( \tilde{\xi}_{i,j}^k \) from (54) it holds \( \tilde{\xi}_{i,j}^k - \tilde{\xi}_{i,j}^k = O(\text{dist}(\cdot, \overline{T}_1))T_{i,j} + O(\text{dist}^2(\cdot, \overline{T}_1)) \).

Next, we decompose \( \nabla B^k - \nabla B^l = (\tilde{B}_{i,j} - \tilde{B}_{i,p}) \cap \nabla + (\nabla \tilde{B}_{i,j} - \nabla B^l) + (1 - \lambda)(\nabla \tilde{B}_{i,j} - \nabla \tilde{B}_{i,p}) \).

Using the first-order compatibility of the velocity fields at the triple junction and the uniform bounds on their second spatial derivatives, we estimate \( (\nabla B^k - \nabla B^l)^T = (\nabla \tilde{B}_{i,j} - \nabla B^l)^T + O(|x - p_k|) \). By definition of \( \tilde{B}_{i,j} \) we also obtain as before

\[ (\nabla \tilde{B}_{i,j} - \nabla B^l)^T = O(1)\tilde{n}_{i,j} \cap \tilde{\tau}_{i,j} + O(\text{dist}(\cdot, \overline{T}_1))\tilde{n}_{i,j} \cap \tilde{\tau}_{i,j} + O(\text{dist}(\cdot, \overline{T}_1))\tilde{n}_{i,j} \cap \tilde{\tau}_{i,j} \]

In summary, we may infer that on the interpolation wedge \( W_i \) it holds

\[ |\eta_k \eta_l m| (|\xi_{i,j}^m - \tilde{\xi}_{i,j}^k| \cdot (\nabla B^k - \nabla B^l)^T \tilde{\xi}_{i,j}^k| \leq C(\text{dist}^2(\cdot, \overline{I}_{i,j}) \land 1) \]

This establishes (178) for \( k \in K^{3j} \), and therefore concludes the proof. \( \Box \)

7.4. Approximate transport and mean curvature flow equations: Proof of Proposition 4. We first derive the global (or network) version of the bounds from Lemma 17 resp. Proposition 21, which represent the model problem of a smooth manifold resp. of a regular triod evolving by mean curvature.

Lemma 36. Let \( \tilde{\chi} = (\tilde{\chi}_1, \ldots, \tilde{\chi}_P) \) be a strong solution for multiphase mean curvature flow in the sense of Definition 11. Let \( \xi_{i,j} \) be the global vector fields from Construction 30, and let \( B \) be the global velocity field from Construction 33. Then we have the estimates

\[ |\partial_t \xi_{i,j} + (B \cdot \nabla) \xi_{i,j} + (\nabla B)^T \xi_{i,j}| \leq C(\text{dist}(x, \overline{I}_{i,j}(t)) \land 1), \]

\[ |(\nabla \cdot \xi_{i,j}) + B \cdot \xi_{i,j}| \leq C(\text{dist}(x, \overline{I}_{i,j}(t)) \land 1), \]

\[ |\xi_{i,j} \cdot \partial_t \xi_{i,j} + \xi_{i,j} \cdot (B \cdot \nabla) \xi_{i,j}| \leq C(\text{dist}^2(x, \overline{I}_{i,j}(t)) \land 1) \]

for all \( (x, t) \in \mathbb{R}^2 \times [0, T_{\text{strong}}) \) and all \( i, j \in \{1, \ldots, P \} \) with \( i \neq j \).

For the proof, recall that we decomposed \( \{1, \ldots, K\} = K^{2ph} \cup K^{3j} \), with \( K^{3j} \) enumerating the triple junctions \( p_k \) present in the strong solution, whereas \( K^{2ph} \) enumerates the space-time connected components of the smooth two-phase interfaces \( \overline{I}_{i,j} \). We then defined \( T_k = \{ p_k \} \), if \( k \in K^{3j} \), or \( T_k \subset I_{i,j} \) for the corresponding space-time connected component \( k \in K^{2ph} \) of a two-phase interface \( I_{i,j} \).

Proof of (191). Let \( i, j \in \{1, \ldots, P \} \) with \( i \neq j \) be fixed. If either \( k \in K^{3j} \) such that the interface \( I_{i,j} \) has an endpoint at the triple junction \( T_k \), or \( k \in K^{2ph} \) such that \( T_k \subset I_{i,j} \), then it follows from the local evolution equations (56) resp. (78)

\[ \eta_k \tilde{\xi}_{i,j}^k = -\eta_k (\nabla B^k \cdot \nabla) \xi_{i,j}^k - \eta_k (\nabla B^k)^T \xi_{i,j}^k + O(\text{dist}(\cdot, \overline{I}_{i,j}) \land 1). \]

On the other side, if neither of the above two mentioned cases holds true, we may simply make use of estimate

\[ \eta_k \leq C(\text{dist}(\cdot, \overline{I}_{i,j}) \land 1). \]
Hence, we may compute in total
\[
\partial_t \xi_{i,j} = \sum_{k=1}^{K} \eta_k \partial_t \xi_{i,j}^k + \sum_{k=1}^{K} \xi_{i,j}^k \partial_t \eta_k
\]
\[
= -\sum_{k=1}^{K} \eta_k (B^k \cdot \nabla) \xi_{i,j}^k - \sum_{k=1}^{K} \eta_k (\nabla B^k)^T \xi_{i,j}^k + \sum_{k=1}^{K} \xi_{i,j}^k \partial_t \eta_k + O((\operatorname{dist}(\cdot, I_{i,j}) \wedge 1)).
\]

By inserting \(\xi_{i,j}\) in the second term and \(B\) in the first term, we obtain from the compatibility bounds (174) and (176)
\[
\partial_t \xi_{i,j} = -\sum_{k=1}^{K} \eta_k (B \cdot \nabla) \xi_{i,j}^k - \sum_{k=1}^{K} \eta_k (\nabla B^k)^T \xi_{i,j} + \sum_{k=1}^{K} \xi_{i,j}^k \partial_t \eta_k + O((\operatorname{dist}(\cdot, I_{i,j}) \wedge 1)).
\]

Employing the product rule, we further compute
\[
-\sum_{k=1}^{K} \eta_k (B \cdot \nabla) \xi_{i,j}^k - \sum_{k=1}^{K} \eta_k (\nabla B^k)^T \xi_{i,j} + \sum_{k=1}^{K} \xi_{i,j}^k \partial_t \eta_k
\]
\[
= -(B \cdot \nabla) \xi_{i,j} - (\nabla B)^T \xi_{i,j}
\]
\[
+ \sum_{k=1}^{K} (\xi_{i,j}^k - \xi_{i,j}) (\partial_t \eta_k + (B \cdot \nabla) \eta_k) + \sum_{k=1}^{K} ((B^k - B) \cdot \xi_{i,j}) \nabla \eta_k
\]
\[
- \xi_{i,j} (\partial_t \eta_k + (B \cdot \nabla) \eta_k) - (B \cdot \xi_{i,j}) \nabla \eta + O((\operatorname{dist}(\cdot, I_{i,j}) \wedge 1)).
\]

The terms in the last line are bounded due to (170) and (172). For the sums in the penultimate line, we split them by means of \(\{1, \ldots, K\} = \mathcal{K}^3 \cup \mathcal{K}^{2ph}\). The sum over the two-phase interfaces \(k \in \mathcal{K}^{2ph}\) may be absorbed into the error term \(O((\operatorname{dist}(\cdot, I_{i,j}) \wedge 1))\) by an application of (169), (171), (174) as well as (176). On the other side, the sum over the triple junctions \(k \in \mathcal{K}^3\) is an error term of required order, since we may apply the slightly refined compatibility bounds (179) and (180). This then concludes the proof of (191). \(\square\)

**Proof of (192).** Let \(i, j \in \{1, \ldots, P\}\) with \(i \neq j\) be fixed. Following the argument at the beginning of the proof of (191), i.e., by either making use of the local equations (58) resp. (79) or the bound (194), we may compute
\[
\nabla \cdot \xi_{i,j} = \sum_{k=1}^{K} \eta_k (\nabla \cdot \xi_{i,j}^k) + \sum_{k=1}^{K} (\xi_{i,j}^k \cdot \nabla) \eta_k
\]
\[
= -\sum_{k=1}^{K} \eta_k B^k \cdot \xi_{i,j}^k + \sum_{k=1}^{K} (\xi_{i,j}^k \cdot \nabla) \eta_k + O((\operatorname{dist}(\cdot, I_{i,j}) \wedge 1)).
\]

Next, we rewrite
\[
-\sum_{k=1}^{K} \eta_k B^k \cdot \xi_{i,j}^k + \sum_{k=1}^{K} (\xi_{i,j}^k \cdot \nabla) \eta_k
\]
\[
= -B \cdot \xi_{i,j} - \sum_{k=1}^{K} \eta_k (B^k - B) \cdot \xi_{i,j}^k + \sum_{k=1}^{K} ((\xi_{i,j}^k - \xi_{i,j}) \cdot \nabla) \eta_k - (\xi_{i,j} \cdot \nabla) \eta.
\]

The second term in the latter identity is bounded by means of (176), whereas for the last term one relies on (170). The term with the sum over the topological
features may be treated by splitting \( \{1, \ldots, K\} = \mathcal{K}^{3j} \cup \mathcal{K}^{2ph} \) as at the end of the proof of \((191)\). Hence, we indeed obtain \((192)\).

Proof of \((193)\). Let again \(i,j \in \{1, \ldots, P\} \) with \(i \neq j\) be fixed. Adapting the argument at the beginning of the proof of \((191)\), i.e., either making use of the local equations \((57)\) resp. \((80)\) or exploiting the bound \((194)\), we have

\[
\xi_{i,j} \cdot \partial_t \xi_{i,j} = \sum_{k=1}^{K} \eta_k \xi_{i,j} \cdot \partial_t \xi_{i,j}^k + \sum_{k=1}^{K} \xi_{i,j} \cdot \xi_{i,j}^k \partial_t \eta_k
\]

\[
= -\sum_{k=1}^{K} \eta_k \xi_{i,j}^k \cdot (B^k \cdot \nabla) \xi_{i,j}^k + \sum_{k=1}^{K} \eta_k (\xi_{i,j} - \xi_{i,j}^k) \cdot \partial_t \xi_{i,j}^k
\]

\[
+ \sum_{k=1}^{K} \xi_{i,j} \cdot \xi_{i,j}^k \partial_t \eta_k + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

We proceed by inserting the equation for the time evolution of each local vector field \(\xi_{i,j}^k\). More precisely, by either employing the local equations \((56)\) resp. \((78)\) or making use of \((194)\), we obtain together with \((174)\) the estimate

\[
\xi_{i,j} \cdot \partial_t \xi_{i,j} = -\sum_{k=1}^{K} \eta_k \xi_{i,j} \cdot (B^k \cdot \nabla) \xi_{i,j}^k - \sum_{k=1}^{K} \eta_k (\xi_{i,j} - \xi_{i,j}^k) \cdot (\nabla B^k)^T \xi_{i,j}^k
\]

\[
+ \sum_{k=1}^{K} \xi_{i,j} \cdot \xi_{i,j}^k \partial_t \eta_k + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

Using the compatibility bounds \((178)\) resp. \((174)\) and \(\eta \leq O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)\), we may compute

\[
\sum_{k=1}^{K} \eta_k (\xi_{i,j} - \xi_{i,j}^k) \cdot (\nabla B^k)^T \xi_{i,j}^k
\]

\[
= \sum_{k=1}^{K} \eta_k (\xi_{i,j} - \xi_{i,j}^k) \cdot (\nabla B)^T \xi_{i,j}^k + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)
\]

\[
= (1 - \eta) \xi_{i,j} \cdot (\nabla B)^T \xi_{i,j} - \xi_{i,j} \cdot (\nabla B)^T \xi_{i,j} + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1)
\]

\[
= O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

Hence, together with the compatibility bound \((177)\) we obtain

\[
\xi_{i,j} \cdot \partial_t \xi_{i,j} = -\sum_{k=1}^{K} \eta_k \xi_{i,j} \cdot (B \cdot \nabla) \xi_{i,j}^k + \sum_{k=1}^{K} \xi_{i,j} \cdot \xi_{i,j}^k \partial_t \eta_k + O(\text{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1).
\]

The remaining two sums may be rewritten as follows

\[
-\sum_{k=1}^{K} \eta_k \xi_{i,j} \cdot (B \cdot \nabla) \xi_{i,j}^k + \sum_{k=1}^{K} \xi_{i,j} \cdot \xi_{i,j}^k \partial_t \eta_k
\]

\[
= -\xi_{i,j} \cdot (B \cdot \nabla) \xi_{i,j} + \sum_{k=1}^{K} \xi_{i,j} \cdot (\xi_{i,j}^k - \xi_{i,j}) (\partial_t \eta_k + (B \cdot \nabla) \eta_k)
\]

\[
- |\xi_{i,j}|^2 (\partial_t \eta_k + (B \cdot \nabla) \eta_k).
\]
The last term in the latter identity is of required order because of (172). The term with the sum over the topological features may be treated by distinguishing between two-phase interfaces $k \in K^{2ph}$ and triple junction $k \in K^{3j}$ as at the end of the proof of (191). This concludes the proof of Lemma 36. □

Let us summarize our results from the previous sections to conclude with a proof of the main result, Proposition 4.

Proof of Proposition 4. Let $(\xi_{i,j})_{i \neq j}$ be the family of global vector fields from Definition 30.

Let $i, j \in \{1, \ldots, P\}$ with $i \neq j$ be fixed. The coercivity condition (4b) immediately follows from Lemma 31. The formula (4a) follows from the corresponding local version (159) and the definition (162). Moreover, that $\xi_{i,j}(x, t) = \bar{n}_{i,j}(x, t)$ holds true for $t \in [0, T_{\text{strong}})$ and $x \in \bar{I}_{i,j}(t)$ is a consequence of Property iii) of Lemma 29 and that $(\eta_1, \ldots, \eta_K)$ is a partition of unity on the network of interfaces of the strong solution (see Property i) of Lemma 28).

Finally, let $B$ be the global velocity field from Definition 33. The validity of the equations (4c), (4d) and (4e) is the content of Lemma 36. This concludes the proof of Proposition 4. □

Glossary of notation

\begin{align*}
  d &\geq 2 \quad \text{ambient dimension} \\
  D &\quad \text{open set} \\
  \partial_t v &\quad \text{distributional partial derivative w.r.t. time of } v : D \times [0, T) \to \mathbb{R}^d \\
  \nabla v &\quad \text{distributional partial derivative w.r.t. space, } (\nabla v)_{i,j} = \partial_j v_i \\
  C_{\text{cpt}}^\infty(D) &\quad \text{space of compactly supported and infinitely differentiable functions on } D \\
  u \otimes v &\quad \text{tensor product of } u, v \in \mathbb{R}^d, (u \otimes v)_{i,j} = u_i v_j \\
  A : B &\quad \sum_{i,j} A_{ij} B_{ij}, \text{ scalar product of tensors} \\
  \mathcal{L}^d &\quad d\text{-dimensional Lebesgue measure} \\
  \mathcal{H}^k &\quad k\text{-dimensional Hausdorff measure on } \mathbb{R}^d \text{ for } k \in [0, d] \\
  L^p(\Omega, \mu) &\quad \text{Lebesgue space w.r.t. to a measure } \mu \text{ on } \Omega \subset \mathbb{R}^d \text{ for } p \in [1, \infty] \\
  L^p(D) &\quad \text{Lebesgue space w.r.t. Lebesgue measure} \\
  L^p(D; \mathbb{R}^d) &\quad \text{Lebesgue space for vector valued functions} \\
  L^p([0, T]; X) &\quad \text{Bochner–Lebesgue space for a Banach space } X \text{ and } T \in (0, \infty) \\
  W^{k,p}(D) &\quad \text{Sobolev spaces with } p \in [1, \infty) \text{ and } k \in \mathbb{N} \\
  \| \cdot \|_{L^p_t W^{k,q}_x} &\quad \text{norm of } L^p([0, T]; W^{k,d}(D)) \text{ for } k \in \mathbb{N} \text{ and } q \in (1, \infty) \\
  L^p_t W^{k,q}_x &\quad \text{by abuse of notation, Sobolev space on the space-time domain } \bigcup_{t \in [0, T]} D(t) \times \{t\} \\
  BV(D) &\quad \text{Functions of bounded variation \cite{4} on Lipschitz domain } D \subset \mathbb{R}^d
\end{align*}
\partial^* \Omega \quad \text{reduced boundary of a set of finite perimeter } \Omega \subset D

n = -\nabla \chi_{\Omega} / |\nabla \chi_{\Omega}| \quad \text{outward pointing unit normal vector field along } \partial^* \Omega

sdist(\cdot, \partial \Omega) \quad \text{signed distance function to } \partial \Omega \text{ with } sdist(x, \partial \Omega) \leq 0 \text{ for } x \in \Omega

P \geq 2 \quad \text{number of phases}

\Omega_i \quad \text{region occupied by phase } i = 1, \ldots, P \text{ in weak solutions}

\chi_i \quad \text{characteristic function of } \Omega_i

I_{i,j} \quad \text{interface between phases } \Omega_i \text{ and } \Omega_j

n_{i,j} \quad \text{unit normal vectors along } I_{i,j} \text{ pointing from phase } i \text{ to phase } j

V_i \quad \text{normal velocity of } I_{i,j} \text{ with } V_i > 0 \text{ for expanding } \Omega_i, \text{ see (12b)}

\bar{\Omega}_i, \bar{\chi}_i, \ldots \quad \text{corresponding quantities of the strong solution}

\bar{H}_{i,j} \quad \text{mean curvature vector of } \bar{I}_{i,j}

\bar{H}_{i,j} \quad \text{scalar mean curvature of } \bar{I}_{i,j} \text{ given by } -\nabla \tan \cdot \bar{n}_{i,j} = -\Delta s_{i,j}

s_{i,j} \quad \text{signed distance function to } \bar{I}_{i,j} \text{ with } \nabla s_{i,j} = \bar{n}_{i,j}

J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{counter-clockwise rotation by } 90^\circ

\bar{\tau}_{i,j} \quad \text{tangent vector along } \bar{I}_{i,j} \text{ given by } J^{-1} \bar{n}_{i,j}

O(\cdot) \quad \text{Landau symbol, implicit constant only depends on strong solution}

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