Hilbert’s Paradox of the Grand Hotel

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After a first international congress of mathematicians in Zurich (1897) it is decided that the experience should be renewed, with the second congress taking place in Paris (1900). The steering committee would like to open the congress to a large audience, and thus no upper bound is placed on the number of attendees. David Hilbert is hoping for some progress towards a complete axiomatization of mathematics—or at least towards one of the famous 23 problems he is about to announce. The practical problem of accommodating researchers seemed rather trivial and is left to administrative faculty staff.

A large hotel is chosen, conveniently located next to the train station, and entirely reserved for the congress. The night before the congress, attendees begin to arrive. The Parisian hotel actually has an infinite supply of rooms, numbered 0, 1, 2, . . . (room number 0 being occupied by the hotel manager).

A first train arrives at 23:00, carrying an infinite number of mathematicians with seats numbered 0, 1, 2, . . . in a special carriage. When they show up at the hotel, the manager quickly assesses the situation and makes the following arrangement. Every mathematician in this train with seat number $i$ shall be staying in room number $i + 1$.

All these guests have been accommodated, but at 23:30 another train arrives with another infinite, numerable number of congress attendees. The hotel manager, very pragmatic, comes up with the following solution. In order to make space for new arrivals, every current guest with room number $i$ should move to room number $2 \cdot i$. Then every congress attendee arriving in this second train at seat $i$ shall be staying in room number $2 \cdot i + 1$. Once again everyone has been accommodated; however other similar trains keep arriving. Yet for each such train the manager can allocate rooms using the same method: existing guests are asked to move to room $2 \cdot i$ when residing in room $i$; newcomers go to room $2 \cdot i + 1$ when they had seat number $i$ in the train; all the guests are settled again and all the rooms are full.

Towards midnight, trains arrive with increasing frequency. Indeed an infinite number of trains arrive, with train number $j$ for $j = 0, 1, \ldots$ arriving at $2^{-j}$ hours before midnight. Guests find it disagreeable to keep changing rooms but comply with the hotel manager instructions, and even take it upon themselves to enforce the rules set by him. Soon every new train arrival is handled in a routine fashion without the manager having to intervene.

Soon after midnight, the manager returns from his cigarette break—he found
the evening quite stressful. Having witnessed a rather large number of guests going through the entrance, he decides to do a round in order to make sure that all guests are settled. No other train is scheduled to arrive; check-in was before 00:00 by hotel rules. But an unpleasant surprise awaits the manager. When knocking on the first door, he hears no response. When knocking on the second door, he does not hear any response either. Actually after doing the whole round of his hotel, coming back “from infinity” he still has not seen any guest.

Without opening any door, he comes to the conclusion that rooms are all empty. Assume there was a guest in room number 1, and let $j$ be such the train number of this guest; he or she had seat number 0. According to the placement method of the manager, they have moved from room number 1 to room number 2 at precisely $2^{-j}$ hours before midnight, and therefore are longer in room number 1. A similar reasoning applies to other rooms. Assume there was a guest in room number $k$. This guest traveled in train number $j$, seat $i$, and therefore at $2^{-j-\log(\frac{k}{2i+1})} - 1$ hours before midnight has moved to another room.

So it would appear that all the rooms are indeed empty! But then, where did the guests go?