

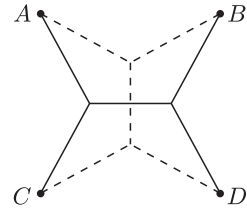
## COMMUNICATIONS OF THE MOSCOW MATHEMATICAL SOCIETY

## On the configuration space of Steiner minimal trees

H. Edelsbrunner and N. P. Strelkova

**1. Introduction.** Let  $S$  be a finite set of points in  $\mathbb{R}^d$ . A *spanning network* of  $S$  is a finite set of rectifiable curves, each connecting two points in  $\mathbb{R}^d$  but not necessarily in  $S$ , whose union is connected and contains  $S$ . In this paper we assume that the curves do not intersect, other than at shared endpoints. We measure the length of the network using the Euclidean metric. A *shortest network* of  $S$  is a spanning network of minimum length. Such a network is known to exist, and it has the following properties [1]: 1) it is a tree whose edges are straight line segments ending at vertices of degree 1, 2, or 3; 2) all degree-1 and degree-2 vertices are points in  $S$ ; 3) the angles between edges meeting at a degree-3 vertex are  $120^\circ$ , while the angle between edges meeting at a degree-2 vertex is at least  $120^\circ$ .

Shortest networks are usually referred to as *Steiner minimal trees*. For a given finite set of points the Steiner minimal tree is not necessarily unique. For example, the four corners of a square have two Steiner minimal trees; see the figure. Note that these two trees have different combinatorial structures if the points are labelled, which they are. Just to mention one difference: the points  $A$  and  $B$  are connected by a path of two edges in the dashed-line tree, and by a path of three edges in the solid-line tree. Suppose that  $S$  consists of four (arbitrary) points in the plane, and we need to find a Steiner minimal tree. If we know how many additional vertices there are in the Steiner minimal tree (that is, vertices that do not belong to  $S$ ) and how they are connected to the four given points and to each other, then we can easily locate them using the  $120^\circ$ -condition. Generalizing this idea to  $n$  points in the plane gives Melzak's algorithm [2]. But a priori we do not know which combinatorial structure gives the optimum, and there is no known way that avoids checking all of them in the worst case. This is the reason why the problem of constructing a Steiner minimal tree in  $\mathbb{R}^d$  is *NP*-hard, even for  $d = 2$ ; see [3].



For a given finite and ordered set of points  $S \subseteq \mathbb{R}^d$ , the collection of Steiner minimal trees of  $S$  is defined, and every tree in this collection has a different combinatorial type [1], a notion that will be made precise later. We are interested in the following question: **for a given combinatorial type, what does the space of ordered sets  $S$  whose Steiner minimal trees are of that type look like, at least topologically?** To make this more specific, we map a set of  $n$  points in  $\mathbb{R}^d$  to a point in  $\mathbb{R}^{dn}$  by listing the coordinates in sequence. For a given combinatorial type, we consider the space  $M \subseteq \mathbb{R}^{dn}$  of points that correspond to ordered sets of  $n$  points in  $\mathbb{R}^d$  for which the Steiner minimal tree is unique and of the given type. Our main result is that  $M$  is connected.

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We note that for  $d \geq 3$  it is an open problem whether the space of configurations  $S \subseteq \mathbb{R}^d$  for which the Steiner minimal tree is not unique has measure zero. For  $d = 2$  it is known to have measure zero, but the proof in [4] is long and complicated. The approach in [5] leads to a shorter proof.

**2. Definitions and results.** Let  $S$  be an ordered set of  $n$  points in  $\mathbb{R}^d$ . We find it convenient to associate to every spanning network  $\Gamma$  of  $S$  a *partially ordered (abstract) graph*  $G = (V \sqcup W, E)$ , with the *edges* in  $E$  representing the curves, the (*boundary*) *vertices* in  $V$  representing the points in  $S$ , and the (*interior*) *vertices* in  $W$  representing the additional endpoints of the curves. Two partially ordered graphs are *equivalent* if there is an order-preserving bijection between their boundary vertices and a bijection between their interior vertices, inducing a bijection between their edges. We say that two Steiner minimal trees  $\Gamma_1$  and  $\Gamma_2$  for points in  $\mathbb{R}^d$ ,  $d \geq 3$ , are of the same *combinatorial type* if the two associated partially ordered abstract graphs are equivalent. In dimension  $d = 2$  we need an additional condition: fixing the cyclic order of the three edges incident to any degree-3 vertex, we require that the bijection between the vertices of the two networks is consistent with these orderings.

As mentioned earlier, we map an ordered set of  $n$  points in  $\mathbb{R}^d$  to a point in  $\mathbb{R}^{dn}$  by listing the coordinates in sequence. The *cell* of a Steiner minimal tree  $\Gamma$ , denoted by  $\text{cell}(\Gamma)$ , is the space of all points  $s \in \mathbb{R}^{dn}$  that correspond to those ordered sets  $S$  for which there exists a Steiner minimal tree of the same type as  $\Gamma$ . The *proper cell* of  $\Gamma$ , denoted by  $\text{pcell}(\Gamma)$ , is the subset of  $\text{cell}(\Gamma)$  that consists of all points  $s \in \mathbb{R}^{dn}$  such that the Steiner minimal tree of  $S$  is unique.

**Theorem 1.** *Let  $\Gamma$  be a Steiner minimal tree of a finite ordered set of points in  $\mathbb{R}^d$ ,  $d \geq 2$ . Then  $\text{pcell}(\Gamma)$  is connected.*

**Theorem 2.** *Let  $\Gamma$  be a Steiner minimal tree of a finite ordered set of points in  $\mathbb{R}^2$  such that all boundary vertices have degree 1. Then  $\text{cell}(\Gamma)$  is connected.*

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