Formal Synthesis of Stabilizing Controllers for Switched Systems

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Switching logic synthesis

Given a set of dynamics and a partition, assign dynamics to each facet such that the resulting switched system is stable.
Switched system

Family of dynamical systems

\[ \mathcal{S} = (\mathcal{P}, \{g_p\}_{p \in \mathcal{P}}) \]

\[ \dot{x}(t) \in g_p(x(t)), \quad p \in \mathcal{P} \]

\[ g_p : \mathbb{R}^n \to 2^{\mathbb{R}^n} \]

Switching strategy

\[ \alpha : \mathcal{F}^+ \to \mathcal{P} \]

\[ f_i, f_j, \ldots, f_l \mapsto p \]

Partition - finite set of valid facets

\[ \mathcal{R} = \{\Omega_1, \Omega_2, \ldots, \Omega_k\} \] closed convex polyhedra

1. \[ \mathbb{R}^n = \bigcup_{\Omega \in \mathcal{R}} \Omega \]
2. \[ \hat{\Omega}_i \neq \emptyset \] for every \( i \)
3. \[ \hat{\Omega}_i \cap \hat{\Omega}_j = \emptyset \] for every \( i \neq j \)

\[ \mathcal{F} = \{f_1, f_2, \ldots, f_k\} \]

maximal closed convex subsets of boundary of \( \Omega \)'s

Switched system \( \mathcal{S}_\alpha = (\mathcal{P}, \{g_p\}_{p \in \mathcal{P}}, \alpha) \)
Stabilization problem

Given a system $S$ and a set of valid facets $\mathcal{F}$, find a switching strategy $\alpha : \mathcal{F}^+ \rightarrow \mathcal{P}$, such that the switched system $S_\alpha$ is stable.

A system is Lyapunov stable with respect to 0 if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every execution $x$ starting from $B_\delta(0)$ implies $x \in B_\varepsilon(0)$. 

\[ x(0) \]

[Diagram: Two overlapping circles with a line indicating a path from $x(0)$ to $0$.]
Overview

- Abstract a game graph $G$ from a family of dynamics $\mathcal{S}$ and a set of valid facets $\mathcal{F}$.
- Induce an energy game graph $G^e$ from $G$.
- Compute an energy winning strategy $\sigma$ from the game graph $G^e$.
- Extract a stabilizing switching strategy $\alpha$ from $\sigma$. 
Abstract Game Graph Construction
Quantitative predicate abstraction

\[ S = (\{1, 2\}, \{A_1, A_2\}) \]

\[ \mathcal{F} = \{f_1, f_2, f_3, f_4\} \]

\[ W((p, f_i), f_j) = \sup \left\{ \frac{||x_j||}{||x_i||} : x_i \in f_i, x_j \in f_j, x_i \xrightarrow{p}{\Omega_{ij}} x_j \right\} \]

\[ \Omega_{ij} \text{ common region of } f_i \text{ and } f_j \]
Auxiliary cycles

divergence

convergence or containment

Abstraction

$\text{convergence or containment}$
Strategy Synthesis
Game graph

Game graph is a weighted graph $G=(V,E,W)$

- $V = V_0 \cup V_1$
- $V_0 \cap V_1 = \emptyset$
- $E \subseteq (V_0 \times V_1) \cup (V_1 \times V_0)$
- $W : E \rightarrow \mathbb{Q}$
- Every node has a successor

A strategy is a function $\sigma : V^* V_0 \rightarrow V_1$, where $V^*$ is the set of finite sequences over $V$ with zero or more elements.
Strategy Example

Weight of the cycle is $\frac{1}{2} \cdot \frac{5}{2} \cdot \frac{1}{2} \cdot \frac{5}{2} > 1$

$S = (\{1, 2\}, \{A_1, A_2\})$

$\dot{x} = A_1 x$

$\dot{x} = A_2 x$
Strategy Example

Weight of the cycle is \( \frac{1}{2} \cdot \frac{3}{10} \cdot \frac{1}{2} \cdot \frac{3}{10} < 1 \)

No cycles with weight greater than 1 implies stability.
A strategy $\sigma$ is a **winning bounded strategy** if there exists $M \in \mathbb{Z}$ such that for every play $\tau$ determined by $\sigma$, $W(\tau) \leq M$.

**Theorem - stabilizable switching strategy**

A winning bounded strategy for the game graph $G(\mathcal{S}, \mathcal{F})$, induces a strategy which solves the stabilization problem for the system $\mathcal{S}$ and the facets $\mathcal{F}$. 
Energy game

A strategy $\sigma$ is a winning energy strategy if there exists $C \in \mathbb{N}$ such that for every play $\tau = v_1 v_2 \ldots$ determined by $\sigma$, $C + \sum_{i=1}^{j} W(v_i, v_{i+1}) \geq 0$.

Theorem - energy strategy

[Brim et al. FMSD’11]

Given a game graph $(V, E, W)$ where $W : E \to \mathbb{Z}$, if there exists a winning energy strategy, then there exists a memoryless winning energy strategy. Further, there is an algorithm which returns the memoryless winning energy strategy.
Energy game

Modification of $G(S, F)$ to an energy game graph

- Reduce multiplicative game graph to addition game graph.
- Weights are required to be integers.
- Winning energy strategy provides plays lower bounded by a value.

**Bounded game graph**

$G = (V, E, W)$

$W(e) = \frac{a_e}{b_e}$

$\text{LCM}_G := \text{least common multiple } \{b_e : e \in E\}$

**Energy game graph**

$G^e = (V, E, W^e)$

$W^e = -\text{LCM}_G \cdot W$

**Theorem - bounded strategy**

$\sigma$ is a winning energy strategy for $G^e$ if and only if $\sigma$ is a winning bounded strategy for $G$. 

$\text{energy game}$

$\text{bounded strategy}$

$\text{winning energy strategy}$

$\text{bounded strategy}$
Reduction to energy game

\[ f_1, f_2, f_3, f_4 \]
Reduction to energy game

Winning energy strategy

\[ \sigma : V_0 \rightarrow V_1 \]

\[ f_1 \mapsto (1, f_1) \]

\[ f_2 \mapsto (2, f_2) \]

\[ f_3 \mapsto (1, f_3) \]

\[ f_4 \mapsto (2, f_4) \]
Conclusion

- An abstraction technique and game based approach for synthesizing a switching logic for stabilization.

- Our approach can be combined with temporal logic properties to obtain stable controllers which satisfy the temporal logic formulas.
Thank you