An Algorithmic Approach to Stability Verification of Hybrid Systems: A Summary

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Hybrid system

A dynamical system exhibiting a mixed discrete and continuous behavior.
Hybrid system

A dynamical system exhibiting a mixed discrete and continuous behavior.
Hybrid system

\[ \mathcal{H} = (Q, X, \Sigma, \Delta) \]

- \( Q \) finite set of control modes (discrete state space),
- \( X = \mathbb{R}^n \) continuous state space,
- \( \Sigma \subseteq Trans(Q, X) \) set of transitions and
- \( \Delta \subseteq Traj(Q, X) \) set of trajectories.
Stability

- Stability is a fundamental property in control system design and captures robustness of the system with respect to initial states or inputs.

- A system is stable when small perturbations in the input just result in small perturbations of the eventual behaviours.

- Classical notions of stability:
  - Lyapunov stability
  - Asymptotic stability
Lyapunov stability

The equilibrium point 0 is Lyapunov stable if

\[ \forall \epsilon > 0 \ \exists \delta = \delta(\epsilon) > 0 : ||\sigma(0)|| < \delta \Rightarrow ||\sigma(t)|| < \epsilon \ \forall t \geq 0 \]
The equilibrium point 0 is asymptotic stable if it is Lyapunov stable and every execution converges to 0.
State of the art

- Existence of Lyapunov function assures stability.
- Lyapunov function computation:
  - Choose a template: \( L(x) = ax^2 + bx + c \).
  - Look for coefficients \( a, b, c \), such that \( L(x) \) holds some conditions.
  - If \( a, b, c \) do not exist, choose a new template.
- Template choice requires user ingenuity.
- Coefficient failure does not provide insights on the next template choice.
Motivation

- **Automatization** of stability analysis.

- Development of an abstraction refinement framework.
Algorithmic approach
Abstraction
Theoretical foundation

Quantitative Predicate Abstraction

One dimensional hybrid system

continuous simulation
Continuous simulation

Let $R$ be a continuous simulation from a hybrid system $\mathcal{H}$ to a hybrid system $\mathcal{H}_G$. Then:

- $\mathcal{H}_G$ Lyapunov stable $\Rightarrow$ $\mathcal{H}$ Lyapunov stable

- $\mathcal{H}_G$ asymptotically stable $\Rightarrow$ $\mathcal{H}$ asymptotically stable
Quantitative predicate abstraction

- Abstraction based on *predicates*.
- In addition, *weight* computation.
Partition

\[ \mathcal{H} = (Q, X, \Sigma, \Delta) \quad \text{Hybrid system} \]

\[ \mathcal{P} = \{P_1, \cdots, P_k\} \quad \text{Polyhedral partition of } X \text{ such that:} \]

- \[ X = \bigcup_{i=1}^{k} P_i \]
- \[ \text{Int}(P_i) \cap \text{Int}(P_j) = \emptyset \quad \forall i \neq j \]

Regions = \mathcal{P}
Quantitative predicate abstraction

- Modified predicate abstraction resulting in a finite weighted graph, $G$.
- Nodes correspond to the regions of the partition, $\mathcal{P}$.
- Edges represent existence of an execution from one region to other and evolving through a common adjacent region.
- Weight on every edge corresponds to the maximum scaling of possible executions.
Predicate abstraction: constant derivative
Predicate abstraction: constant derivative

\[ \mathcal{H} \]

\[ P_2 \]

\[ u_2 \]

\[ u_1 \]

\[ P_1 \]

\[ u_3 \]

\[ u_4 \]

\[ P_4 \]
Predicate abstraction: constant derivative

\( \mathcal{H} \)

\( P_1 \)

\( P_2 \)

\( P_3 \)

\( P_4 \)

\( u_2 \)

\( u_1 \)

\( u_3 \)

\( u_4 \)

\( \implies \)

\( P_1 \)

\( P_2 \)

\( P_3 \)

\( P_4 \)
Predicate abstraction: constant derivative

\[ \mathcal{H} \]

\[ H = u_1 u_2 u_3 u_4 \]

\[ P_1 \quad P_2 \quad P_3 \quad P_4 \]

\[ \implies P_1 \quad P_2 \quad P_3 \quad P_4 \]
Predicate abstraction: constant derivative

\[ \mathcal{H} \]

\[ P_2 \]

\[ P_3 \]

\[ P_4 \]

\[ u_2 \]

\[ u_1 \]

\[ u_3 \]

\[ u_4 \]

\[ \implies \]

\[ P_1 \]

\[ P_2 \]

\[ P_3 \]

\[ P_4 \]
Predicate abstraction: constant derivative
Predicate abstraction: constant derivative
Predicate abstraction: constant derivative

\[ \mathcal{H} \]

\[ u_2 \]
\[ u_3 \]
\[ u_4 \]

\[ \mathcal{G} \]

\[ \frac{1}{2} \]
\[ 1 \]

\[ P_1 \rightarrow P_2 \rightarrow P_3 \rightarrow P_4 \rightarrow P_1 \]
Reachability relation

\[(s_1, s_2) \in \text{ReachRel}_{P_1, P_2}\] if there exists an execution \(\sigma\):

- \(\sigma(0) = s_1 \in P_1\),

- \(\exists T \geq 0\) with \(\sigma(T) = s_2 \in P_2\) and

- \(\exists P \in \mathcal{P}\) such that \(\forall t \in (0, T), \sigma(t) \in P\).
Reachability relation - polyhedral dynamics

- Polyhedral hybrid system:

\[
\text{ReachRel}_{P_1, P_2} = \{(s_1, s_2) : s_1 \in P_1, s_2 \in P_2, \exists t, \exists u \in \text{dyn}(P) \text{ for some } P \text{ such that } s_2 = s_1 + ut\}
\]
Weight computation

\[ W(P_1, P_2) = \sup_{(s_1, s_2) \in \text{ReachRel}_{P_1, P_2}} \frac{||s_2||}{||s_1||} \]
Model-checking

\[ \mathcal{H} \]

\[ G \]

- Abstract
- Model-Check
- Validate

- Yes → Stable
- No → Unstable

- Yes → Stable
- No → Unstable

- Yes → Unstable
- No → Unstable
Let $G$ be a quantitative abstraction of a hybrid system $H$.

G1 There is no edge $e$ in $G$ with infinite weight.

G2 The product of the weights on every simple cycle $\pi$ of $G$ is less than or equal to 1.

G3 Every node in $G$ is labelled by “conv”.

G4 The product of the weights on every simple cycle $\pi$ of $G$ is strictly less than 1.

Then:

- $H$ is Lyapunov stable if conditions G1 and G2 hold; and
- $H$ is asymptotically stable if conditions G3 and G4 hold.
Model-checking

Every cycle has weight smaller than 1
\[ \Downarrow \]
\( \mathcal{H} \) is stable
\[ \Downarrow \]
STOP

There is a cycle, \( \pi \), with weight greater than 1
\[ \Downarrow \]
\( \pi \) is a counterexample
AVERIST

Software tool

- **Quantitative predicate abstraction** for polyhedral switched systems.
- **Stability analysis** based on the weighted graph.

- Implemented in **Python**.
- Parma Polyhedra Library (**PPL**) to manipulate polyhedral sets.
- **GLPK** solver to compute the weights.
- **NetworkX** Python package to define and analyse graphs.

http://software.imdea.org/projects/averist/index.html
Conclusions

• Summary of an algorithmic approach for stability verification.

• Future directions:
  – Extension to linear and nonlinear dynamics.
  – Compositional techniques for stability analysis.
Thank you!